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APPLICATION OF EXTREME VALUE ANALYSIS IN THE ASSESSMENT OF BUDGET VARIANCE RISK

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Abstract: This paper presents the methods for the evaluation of budget variance risk, i.e. the risk of a difference between the budgeted and actual figures. The postulated approach is based on extreme value analysis (EVA), to offer, among other things, the evaluation of maxima distribution parameters for studied phenomena. The proper recognition of these parameters yields potential for calculation of probabilities for budget variance to pass certain levels established as critical. This methodology can be used to evaluate deviation levels by time period, and to compare them against historical data. The main objective of this paper was to examine the utility of the theory of extreme values in the estimation of budget deviation risks. The study presents the results of probabilistic analyses of data obtained from a budgetary cost control unit of a production company located in eastern Poland, for the period of 2011-2012. The developed method of analysis and assessment of budget deviations is in line with the development of concepts and methods of management accounting.

Keywords: budgetary control, statistical methods of budgetary control, extreme value analysis.

1. Introduction

The evaluation of budget variance is an essential aspect of budget control systems. Companies and organisational units cannot exercise effective planning or budget allocation without the proper recognition of levels and sign of deviations, without a good knowledge of their causes and their potential impact on the company's financial standing and long-term stability. Professional literature provides many examples – both theoretical and practical – of variance analyses based on a wide spectrum of mathematical and statistical models (e.g. see [Welsch et al. 1988, p. 584; Garrison et al. 1990, pp. 551-575; Kes, Kuźmiński 2011; Kes 1999]).

The extension of the variance analysis methodology presented in this paper is based on the use of selected elements of the theory of extreme values for the calculation of deviations between the budgeted and the actual figures. The research involved analyses of empirical data on budget dispersion, with the purpose of estimating probabilistic models of extreme values in their characteristics. The estimated models were then employed for the evaluation of deviation risks, i.e. measures of probability for the extreme values to exceed certain predefined levels.

The choice of the theory of extreme values as the basis for the assessment of budgetary control results from the adopted hypothesis concerning the form of the distribution of budget deviations. Based on previous research, the authors of the study assume that good distributions to modelling budget deviations can be so-called fat-tailed distributions, characterised by significant asymmetry and platykurtic. From a wide family of fat-tailed distributions, the authors chose the Gumbel distribution, which is one of the distributions belonging to the family of extreme value distributions. The estimated distributions of extreme values lend the possibility to calculate the probability of exceeding a certain level by maximum values of deviations. Therefore, according to the authors, it is possible to use this probability as a measure for the assessment of the implementation of budgetary control assumptions. An increase in this probability in a certain period in a given business unit means greater possibilities for the appearance of the maximum budget deviation. Therefore, this situation should be assessed negatively and prompt the search for problems in a malfunctioning budget control system. It seems that the use of a synthetic parameter (in the form of a probability) indicating the quality of control constitutes an important contribution to the methodology employed by management accounting.

For the purpose at hand, the main objective of this paper was to examine the usefulness of the theory of extreme values in the estimation of budget deviation risks. The authors believe that the results obtained in the course of this study should be subject to further validations and tests before they can be used in practical budget control applications.

To meet the above objective, the authors began with a presentation of the theoretical aspects associated with budget control and variance analysis, followed by an overview of the probabilistic models of extreme maxima and methods used for their estimation and significance testing. Empirical evaluations were conducted using data obtained from a budgetary cost control unit of a local production company, in the form of budgeted and actual figures for the period of 2011-2012.

During the conducted research, the authors set out to find the answer to the question: do selected issues of extreme value theory give the opportunity to assess the risk of budgetary deviations? In the research procedure, additional questions were also asked: what kind of distributions possesses extreme values of budget deviations, is the distribution determined by the authors appropriately suited to the surveyed characteristics, and what is the probability level of exceeding the determined deviation value?

The last section of the paper presents the research conclusions and the discussion of the findings.

Due to the application of the described method in one enterprise (case study), the research results are not of a general nature.

2. Theoretical aspects of budget variance analysis and risk evaluation

To present a concise picture of the studied approach, the authors conducted studies of the professional literature in an attempt to identify and present the various theoretical concepts of risk and budget deviations, together with methods for their evaluation and measurement.

Defining risk is not an easy task. In fact, no uniform and homogenous definition of the term can be provided due to the fact that the notion of risk is employed in various contexts and with reference to various aspects of human activities. The term is used in everyday communication and mass media as part of economic and political discourse, and in science (economics, law, psychology, statistical analysis, the theory of probability, and so on). From the viewpoint of the main objective of this study, the most suitable definition of risk should be based on the norms applicable to the widest possible spectrum of human activities the ISO standards seem to fit the description. According to the ISO standard (ISO/TMB, 2009), risk can be defined as "the effect of uncertainty on objectives" or "the chance of something happening that will have an impact upon objectives" [Cooper et al. 2014, p. 389], thus encompassing both the positive and negative impacts, and expressed as a combination of impacts and their associated probabilities.

With respect to risk measurement, valuable suggestions can be found in Kunreuther and Roth, with their definition of risk as a potential or probable loss [Kunreuther, Roth 1998], and in Yen, in his definition of risk as a probability that a system or its element will fail to perform the function for which it was designed [Yen 1988]. Consequently, for the purpose of this study, let us define the risk of budget variance as the probability for its maximum to pass a certain level or threshold value.

Budget variance is a product of a comparative analysis of data on budgeted items against their corresponding performance values obtained by the organisational units under evaluation. Despite the fact that budgeting is an important element of organisational controlling systems, it must be noted that budget deviations can also be addressed and evaluated outside those systems, for example as part of standard cost accounting, management controlling, or any other information system.

A reduction of the controlling functions in budgeting can be observed based on analyses of professional literature on management accounting. These sources present budgeting also in terms of methods for operational management, in their capacity to define the principles for the planning and utilisation of financial resources for effective realisation of objectives. This particular approach accentuates the budget--forming stage of the budgeting process [Anthony 1977; Rayburn 1986; Armstrong (ed.) 1993], as cited in [Łada-Cieślak, 1999, pp. 25-26]). The narrowed down concept of the budgeting processes, i.e. representing only the initial stage of budget formulation, can also be found in: [Atkinson et al. 2006, p. 445; Mallouk et al. 2007, p. 554; Garrison et al. 2008, p. 6; Anthony et al. 2014, p. 332].

For a better understanding of the concept of the budgeting process, it may be useful to note that some sources seem to altogether put away the concept of budgeting as such, despite the fact that the term 'budget' is employed by those sources on a regular basis. For instance, no proper definition of the management instrument under study can be found in the popular management accounting handbook by D. Hansen and M. Mowen [Hansen, Mowen 2005]; the term 'budgeting' is employed, but only in reference to its significance for the planning and controlling functions [Hansen, Mowen 2005, p. 326]. Despite this apparent omission, those authors proceed to identify a 'budgetary control system' [Hansen, Mowen 2005, p. 384], whose task is to compare the actual figures against their respective budgeted limits, measured as deviations for the present level of operational activities. Other fairly manifest omissions of budgeting definitions can be observed in some other sources, such as: [Bragg 2014], and [Horngren et al. 2007].

Based on the above observations, it should be obvious that the thesis on deviations as always being an integral part of the budgeting process will not stand up to tests. On the other hand, most experts on management accounting seem to agree that budgets serve as the basis for the evaluation of actual performance figures in economic operations. This means that they are indispensable for the majority of applications, as measures of performance. Similarly, at least according to Hansen et al., budgeting (taken here as the process of budget formulation) should be perceived as a pillar of the management controlling system in organisations of almost any type [Hansen et al. 2003].

Thus the essence of budget controlling involves the use of information on normative values, determined in the form of budgets, both for the calculation of deviations to be later employed for performance evaluation purposes, and as the basis for corrections and adjustments to the budget or to any of the activities under control. As such, the budget controlling process includes (without limitation) the following processes:

- Calculation of deviations between actual or forecasted figures and their budgeted values,
- Identification of deviation origins,
- Deviation analysis,
- Determination of actors accountable for the manifested deviations,

- Examination of deviation effects and their impact in various areas of company operation,
- Determination of remedial and preventive measures,
- Suggesting changes in company operation,
- Suggesting improvements to the budgeting process,
- Monitoring of changes.

As suggested by the above list of processes, an analysis of the deviations is regarded as one of the elements of the controlling process. Deviation analyses based on mathematical and statistical methods are well-represented in professional literature. Kwang and Slavin postulated methods based on the analytical evaluation of two constituents of total variance, i.e. price variance and quantity variance [Kwang, Slavin 1962]. Further development of these methods placed more emphasis on the examination of intermediary cost, see: [Zannetos 1963; Weber 1963]. Several authors have explored the potential of statistical methods for variance analyses, cf.: [Bierman Jr. et al. 1961; Salman 2008], to name but a few. The wealth of analytical methods in use can be employed for various tasks, such as:

- Classification of deviations (e.g. significant vs. insignificant, favourable vs. unfavourable),
- Examination and evaluation of deviation levels,
- Examination of variance properties (recurrence, trends, configurations),
- Identification of causes for variance and determination of actors accountable for deviations.

The research approach presented in this paper concentrates on the evaluation of levels of deviations produced in the course of budgetary controlling procedures in a production company. For this purpose the authors chose to employ the theory of extreme values developed in the 1920s and 30s. Those methods have already been employed with success in such areas as finance and insurance [Embrechts et al. 1997], the biology, the examination of radioactive emission [Gumbel 1937], the evaluation of fatigue and durability of materials [Weibull 1939a], the evaluation of flood risk [Gumbel 1941; 1944a; 1944b; 1945; 1949; 1954; Rantz and Riggs 1949], seismologic analyses [Nordquist 1945], and rainfall (precipitation) analyses [Potter 1949].

Practical applications of the theory of extreme values go well beyond the limits of the above areas. It seems that this approach may also yield good results in the analyses of budget variance, as evidenced by the results obtained in the course of this study. Any company, when subject to probabilistic analyses of their budget variance, shows specific characteristics that largely determine the levels of deviation risks. These properties determine the probability distribution of budget dispersion extremes (i.e. both minima and maxima). By studying the distribution of extreme values, one may calculate the probabilities for those extreme values to deviate from the budgeted level. This approach offers the potential to measure the behaviour of the system under control. In addition, the probabilities obtained using this method can be employed for the purpose of synthetic comparisons of variance levels year by year, by responsibility centre, or by budgeted item. This approach should be considered as a contribution to the methods used by management accounting. So far, in the field of accounting, there has been no use of the theory of extreme values in assessing the achievements of economic units.

3. Probabilistic instruments and their use in analyses of budget variance risk

3.1. Maxima

Let us assume that y_i are the observations of realizations of a variable *Mm* (formula 2) represent maximum values, i.e.:

$$y_i = \max\left\{x_{i1}, ..., x_{im}\right\}, \qquad i = 1, ..., n, \tag{1}$$

where x_{im} may not necessarily be observable. If x_{im} are readily observable, then the selection of maxima from a given set of m elements can be obtained using a standard method for the determination of extreme values in a data set, also referred to as the block method or Gumbel's method [Thomas, Reiss 2007].

The block maxima method can be employed for predefined time periods (blocks), with maxima of the studied variable calculated separately for each block. The blocks are typically assigned in the following timeframes: annual, biannual, quarterly, monthly, or shorter, depending on the examination purposes and needs.

In some cases, when there is a risk of overlapping dependencies between maxima of two neighbouring blocks, analyses of maxima dispersion should be based on functions of extreme value distribution in a random dependent variable series [Kuźmiński 2013].

At this point, it may be useful to note that observations y_i represent realisations of a random variable M_m described by:

$$M_m = \max\left\{X_1, ..., X_m\right\}. \tag{2}$$

Examinations employed in the course of this research were performed on a one--year block covering data for 12 consecutive months.

3.2. Probabilistic models based on maximum value distribution

In line with the theorem of extreme value distribution, the distributions of maximum values must have one of just three possible forms of "extreme value distribution" [Leadbetter et al. 1983, p. 10].

In addition, if a random variable *X* has F distribution, then the random variable ($\mu + \sigma X$) will have distribution of $F_{\mu,\sigma}(x) = F((x-\mu)/\sigma)$, where μ i $\sigma > 0$ represent the shape parameters of location and scale, respectively [Thomas, Reiss 2007]. By linking the above two statements, we can define a very rich family of extreme value distribution functions, described by the following equations:

$$\text{Gumbel (EV0 or Type I)}: \quad G_{0,\mu,\sigma}(x) = \exp\left(-e^{-(x-\mu)/\sigma}\right), \quad -\infty < x < \infty. \tag{3}$$

$$\text{Frechet (EV1 or Type II)}: \quad G_{1,\mu,\sigma}(x) = \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right), \quad \text{for certain } \alpha > 0, \ x > 0. \tag{4}$$

$$\text{Weibull (EV2 or Type III)}: \quad G_{2,\mu,\sigma}(x) = \exp\left(-\left(-\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right), \quad \text{for certain } \alpha > 0, \ x \le 0. \tag{5}$$

The family of maximum value distribution functions described by equations (3) to (5) represents functions of three separate types. By taking the reparametarization $\gamma = 1/\alpha$ of the distributions of maximum values $G_{i,\alpha}$ (i = 0, 1, 2) as suggested in [Mises von 1936], and by introducing the parameters of scale and location, one may obtain a continuous, unified function, described as:

$$G_{\gamma,\mu,\sigma}(x) = \begin{cases} \exp\left\{-\left[1+\gamma\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1}{\gamma}}\right\} & \text{if} \quad \gamma \neq 0 \\ \exp\left\{-\exp\left(\frac{x-\mu}{\sigma}\right)\right\} & \text{if} \quad \gamma = 0 \end{cases} \tag{6}$$

[England et al. 2004; Kotz, Nadarajah 2005].

In the above representation, the Gumbel distribution function again has the parameter of γ equal to zero. The standard versions in the γ-parameterization, i.e. those with no reference to the parameters of location μ and scale σ, are defined such that:

$$G_{\gamma}(x) \to G_0(x), \qquad \gamma \to 0 \tag{7}$$

[Thomas, Reiss 2007]. By employing the appropriate parameters of location and scale in addition to the γ-parameterization, we have a model described by (6).

For the purpose of this research, the visualisation of deviations between empirical distributions of maximum values of selected characteristics and their theoretical distribution patterns was conducted using the empirical distribution function $\hat{F}_n(x)$.

The empirical distribution function $\hat{F}_n(x)$ of a unidimensional random variable X for given $x_1, \ldots, x_n$ is represented as x_i, less than or equal to zero. Thus,

$$\hat{F}_n(x) = \frac{1}{n} \sum_{i \le n} I(x_i \le x), \tag{8}$$

with indicator function defined as $I(x_i \leq x) = 1$ for $y \leq x$, and equal to zero for the remaining part of the set [Thomas, Reiss 2007, p. 39].

3.3. Methods for the estimation and significance testing of maximum value models

Maximum likelihood estimation, one of the most widely recognised and commonly applied methods for the estimation of distribution function parameters in statistical models, yields effective results only if applied for the estimation of distribution function parameters described by equations (3) to (5). For the Gumbel distribution, the moment estimation method may also be used.

In the case of unified or generalised versions of the extreme values model given by (6), one of three estimation procedures can be applied. The first one is the maximum likelihood estimation, but it requires the values to be computed as solutions to likelihood equations. This method yields local maxima of likelihood equation only if the iterated values of the estimated parameter γ remain in the range of $\gamma > -1$. If the value of γ parameter drops below the threshold of -1, then neither the global nor local maxima of likelihood equations can be established [Smith 1985].

The second method of potential value for this particular model is the minimum distance method. Let d represent the distance between the empirical and the theoretical distribution in the family of distribution functions. Then, the set of $(\gamma_n, \mu_n, \sigma_n)$ can be used as a minimum distance estimator if:

$$d\left(\hat{F}_n, G_{\gamma_n,\mu_n,\sigma_n}\right) = \inf_{\gamma,\mu,\sigma}(\hat{F}_n, G_{\gamma,\mu,\sigma}),$$

where $\hat{F}_n$ represents the function of empirical distribution in an n – element set.

The third method offers the estimation of the γ parameter of the generalised distribution of extreme values using a class of estimators expressed as linear combinations of ratios of spacing (*LRSE*):

$$\hat{r} = \frac{x_{[nq_2]:n} - x_{[nq_1]:n}}{x_{[nq_1]:n} - x_{[nq_0]:n}}, \tag{9}$$

where percentile $q_i = i/(n+1)$ and $q_0 < q_1 < q_2$. It must be noted at this point that the above statistic is independent of both the location and scale parameters within the distribution function. In other words, $\hat{r}$ is constant in the sense postulated by the affine transformation principles [Johnson, Kotz 1970].

Since $\hat{F}_n^{-1}(q_i) = x_{i:n}$, then the relation of

$$x_{i:n} = \hat{F}_n^{-1}\left(\frac{i}{n+1}\right) \approx F^{-1}\left(\frac{i}{n+1}\right), \tag{10}$$

between the empirical sample quantile function and the theoretical function yields the following:

$$\hat{F}_n^{-1}(q_i) \approx F_{\mu,\sigma}^{-1}(q_i) = \mu + \sigma F^{-1}(q_i)\,. \tag{11}$$

Consequently, based on:

$$\hat{r} = \frac{G_\gamma^{-1}(q_2) - G_\gamma^{-1}(q_1)}{G_\gamma^{-1}(q_1) - G_\gamma^{-1}(q_0)} = \left(\frac{-\log q_2}{-\log q_0}\right)^{-\gamma/2}, \tag{12}$$

if q_0, q_1, q_2 satisfies the requirement of $\left(-\log q_1\right)^2 = \left(-\log q_2\right)\left(\log q_0\right)$. In effect, we obtain an estimator for the γ parameter, in the following form:

$$\gamma_n = 2\log(\hat{r}) / \log\left(\log(q_0) / \log(q_1)\right) \tag{13}$$

[Thomas, Reiss 2007].

The parameters of location μ and scale σ for a generalised model $G_{\gamma,\mu,\sigma}$ can be estimated using a variant of the widely popular least squares estimation method.

The hypothesis on concordance between the studied distributions and their respective theoretical distributions of maximum values from a family described by (6) was tested using the Kolmogorov-Smirnov test of equality and the Anderson – Darling test [Magiera 2002].

One other important observation should be emphasised at this point, namely: examinations of extreme values may, in some cases, identify extreme random variables which are best described by the generic normal distribution. For this reason, normal distribution should also be taken into account as part of the distribution adjustment procedure.

4. Research methodology

4.1. Description of the company under examination

The empirical part of this research was based on data obtained from a large Polish production company operating in the segment of sports trophies and memorabilia. Their offered product range includes prize cups, glass trophies, medals, figures and statuettes, a wide assortment of collectables, promotional gadgets, etc. The decision to introduce formal cost budgeting procedures in the company was made in response to the growing organisational complexity and in support of the formally adopted approach of management by objectives (MBO). For this task, over a dozen cost centres were identified, including production, support, and administration. Control reports for individual responsibility centers contained several dozen cost items for which deviations were calculated. Due to the fact that some cost items were not present in the budgets, deviations could not be calculated for them. Therefore, deviations occurring in all the responsibility centres were used for the analysis, which resulted in 161 values in 2011 and 140 in 2012. Based on control reports for all centres of responsibility, 35 individual cost items were identified, and examined for variance in their distribution. Since the company controlling unit was fairly unconvinced by the effectiveness of traditional measures (such as the arithmetic mean, standard deviation, or coefficient of variation), the researchers suggested to adopt an approach based on the theory of extreme values, as a method for the determination of probabilities for extremes to pass a certain predefined threshold value. The observed decrease of probabilities (as measured period by period) suggests a reduction of variance level (or, more specifically, its maxima), which may be interpreted as evidence of the improved control and effectiveness of the individual cost responsibility centres.

4.2. Source data

The research involved examinations of cost deviations; these can be measured in relative or absolute terms. For the purpose of this study, the authors adopted the former, with deviations calculated from the following formula:

$$O = \frac{B-A}{B} \cdot 100\% \,, \tag{13}$$

where: O – relative deviation level, B – budgeted value, A – actual value.

In addition, budget variance may be presented cumulatively (e.g. from the beginning of a year up to the period for which the deviation is calculated) or in non-cumulative form (e.g. for a given month). For the purpose of this study, the authors adopted the latter formula.

Due to the possibility for deviations to adopt positive or negative signs, selection of maxima from such a set would – in most cases – yield only the positive deviations. For this reason, the source data was segmented into three sets:

1) underestimations – represented as absolute values of negative deviations,
2) overestimations – represented by positive deviations,
3) over and underestimations – comprising of both the above sets.

The estimation of probability distribution parameters was conducted separately for each of the three sets.

Monthly data on budget and performance were obtained in the form of SAP system reports for the years of 2011 and 2012[1]. Analyses were conducted using standard spreadsheet software, and involved calculations of absolute deviations for each of the sets (underestimations, overestimations, and combined), separately for each cost item, responsibility centre, and period.

[1] The data are not the latest. This is connected with the end of the cooperation with the examined unit and the authors in 2015.

4.3. The research problem

Based on available literature and practical experience, the authors formulated the following research problem: is it viable to adopt selected elements of the theory of extreme values for the estimation of budget deviance risks? For this purpose, the authors formulated the following auxiliary problems:

What is the distribution of extreme values of budget variance in each of the sets under study (underestimations, overestimations, and combined)?

Is the observed distribution well-adjusted to the characteristics under study?

What is the probability for each of those deviations to pass a predefined threshold value?

To tackle the research problem at hand, the authors designed a research procedure involving five stages:

1) formulation of hypotheses,
2) collection and preparation of source data,
3) verification of hypotheses,
4) calculation of probabilities for minima and maxima to pass the predefined budget variance threshold,
5) examination of results.

A detailed presentation of the research procedure is provided in the next section of this article.

5. Modelling of probabilistic measures of budget variance risk

Two hypotheses were formulated in the early stage of the research procedure:

H1. Extreme values of budget variances observed in the 'underestimations' set have the Gumbel distribution, with good fitting between their theoretical and empirical distributions.

H2. Extreme values of budget variances observed in the 'overestimations' set have the Gumbel distribution, with good fitting between their theoretical and empirical distributions.

Based on the source data obtained in the form of monthly cost reports on both the budgeted and the performance values, the percentage values of deviations between the two were calculated. The procedure was applied individually for each cost item reported by each of the selected responsibility centres.

Using the block method, annual maxima (both underestimations and overestimations) were identified for each cost item separately for 2011 and 2012.

Since 2011, there were surveyed (in terms of revaluations and underestimations) 161 series of deviations (12 items each) occurring in various centers of responsibility, and in 2012 the survey included 140 series, the formula (1) takes the following form:

for 2011: $$y_i = \max\{x_{i1}, ..., x_{i12}\}, \qquad i = 1, ..., 161,$$

for 2012: $$y_i = \max\{x_{i1}, ..., x_{i12}\}, \qquad i = 1, ..., 140.$$

The identified sets of over and underestimation maxima for each of the two annual periods under study represent realisations of a random variable defined by (2), with $m = 12$.

Based on the use of the estimation methods presented in the previous section, the parameters of maxima distribution of budget under and overestimation maxima were valued for each of the two annual periods under study to determine their adjustment with their respective empirical distributions. The values of estimators in parameter distributions are presented in Table 1.

Table 1. Values of estimators for distributions adjusted to the distribution of over and underestimation maxima in 2011 and 2012

Period	Data set	Distribution	Parameter estimation values
2011	underestimations	$G_{\gamma, \mu, \sigma}$	$\gamma = 1.197$ $\mu = 0.778$ $\sigma = 1.262$
2011	overestimations	normal	$\mu = 0.813$ $\sigma = 0.438$
2012	underestimations	$G_{\gamma, \mu, \sigma}$	$\gamma = 1.166$ $\mu = 0.622$ $\sigma = 0.967$
2012	overestimations	normal	$\mu = 0.817$ $\sigma = 0.418$

Source: own research.

Figure 1 presents a comparison of the plots describing the empirical and the optimum-adjusted theoretical distribution functions derived from underestimation maxima. The corresponding comparison of function plots for the overestimation maxima is presented in Figure 2.

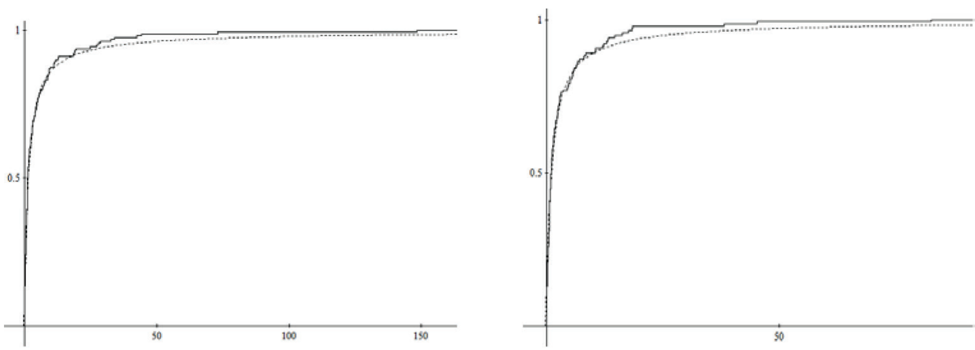

Fig. 1. Plots of empirical and adjusted theoretical distribution functions of the underestimation maxima. Left side – 2011, right side 2012 (axis X – underestimation values, axis Y – values of the distribution functions under study)

Source: own research.

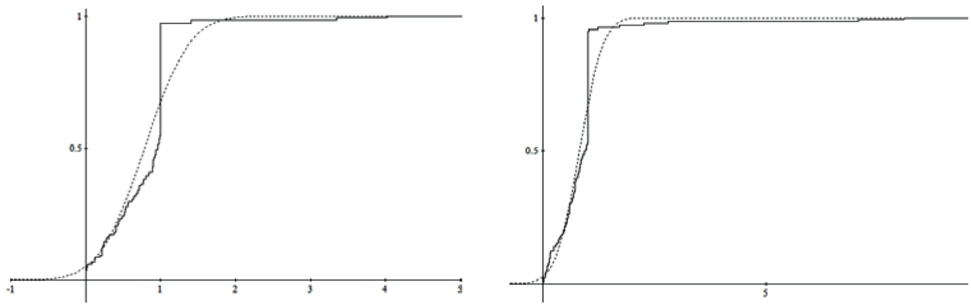

Fig. 2. Plots of empirical and adjusted theoretical distribution functions of the overestimation maxima. Left side – 2011, right side 2012 (axis *X* – overestimation values, axis *Y* – values of the distribution functions under study)

Source: own research.

Visual inspection alone provides the basis for the conclusion that the theoretical distributions selected for the mapping of empirical distribution seem clearly more adjusted for the maxima of underestimations.

To confirm the above results of visual inspection, for each of the four cases under examination, a series of two statistical goodness-of-fit tests was conducted, namely: the Kolmogorov-Smirnov (K-S), and the Anderson-Darling test (A-D). The results of the two tests, in the form of *p-values*, are presented in Table 2.

Table 2. The *p-value* results, based on the Kolmogorov-Smirnov and Anderson-Darling goodness-of-fit tests

Period	Data set	Distribution	K-S Test	A-D Test
2011	underestimations	$G_{\gamma, \mu, \sigma}$	$p_v = 0.224$	$p_v = 0.26$
2011	overestimations	normal	$p_v = 0.000$	$p_v = 0.000$
2012	underestimations	$G_{\gamma, \mu, \sigma}$	$p_v = 0.479$	$p_v = 0.635$
2012	overestimations	normal	$p_v = 0.000$	$p_v = 0.000$

Source: own research.

In the verification of hypotheses H1 and H2, the authors adopted a threshold *p-value* of 0.05, as the minimum requirement to be met by the observations included in the analysis. Based on the above, the H1 hypothesis was validated, and the H2 hypothesis was falsified.

The results of both tests seem to corroborate the conclusions made by a simple visual inspection of the plots. In addition, they seem to provide unambiguous arguments for the rejection of normal distribution as a best-fit for the empirical distribution of budget overestimation maxima. For this reason, the postulated functions cannot be adopted for the calculation of budget overestimation risk.

With regard to the theoretical distribution functions postulated as best fit for the empirical distribution of underestimation maxima, the results of both validation tests seem to prove their good adjustment; this conclusion seems to validate their applicability in the estimation of budget underestimation risk.

Since the procedure failed to provide a suitably adjusted theoretical distribution to represent the empirical distribution of budget overestimation risk, the authors chose to examine another approach, based on the entire set of budget variance maxima (i.e. both under- and overestimations, combined). To this effect, the H3 hypothesis was formulated, as follows: the extreme budget variance values for the combined set of under and overestimations have a Gumbel distribution, and there is a goodness-of-fit adjustment between the theoretical and the empirical distribution functions for the data under study.

By combining the maxima in both the under and overestimation sets, the authors obtained two sets of annual data, described by equation (1):

for the year 2011: $y_i = \max\{x_{i1}, ..., x_{i12}\}, \quad i = 1, ..., 322,$

for the year 2012: $y_i = \max\{x_{i1}, ..., x_{i12}\}, \quad i = 1, ..., 280.$

The examination procedure was similar to that used in the evaluation of separate under and overestimation sets. Based on the produced data sets, the authors estimated the parameters of theoretical distribution presented in Table 3.

Table 3. Values of estimators for distributions adjusted to the distribution of combined over and underestimation maxima in 2011 and 2012

Period	Data set	Distribution	Parameter estimation values
2011	under- and overestimations, combined	$G_{\gamma, \mu, \sigma}$	$\gamma = 0.909$ $\mu = 0.439$ $\sigma = 0.75$
2012	under- and overestimations, combined	$G_{\gamma, \mu, \sigma}$	$\gamma = 0.528$ $\mu = 0.442$ $\sigma = 0.849$

Source: own research.

Figure 3 presents comparisons between the postulated theoretical and the empirical distribution functions. Visual inspection of both sets of plots suggests good adjustment between the two functions. This observation is further corroborated by the results of goodness-of-fit tests, presented in Table 4. Thus the H3 hypothesis can be validated, proving that the postulated approach may be employed to good effect in the assessment of budget variance risk.

Based on the results presented in Tables 1 and 3, and equation (6), probabilistic models were construed in the form of adjusted theoretical distribution functions. These are presented in Table 5.

The final section of this chapter presents the practical application of the above models of theoretical adjusted distribution functions (Table 5) in the estimation of budget variance risk.

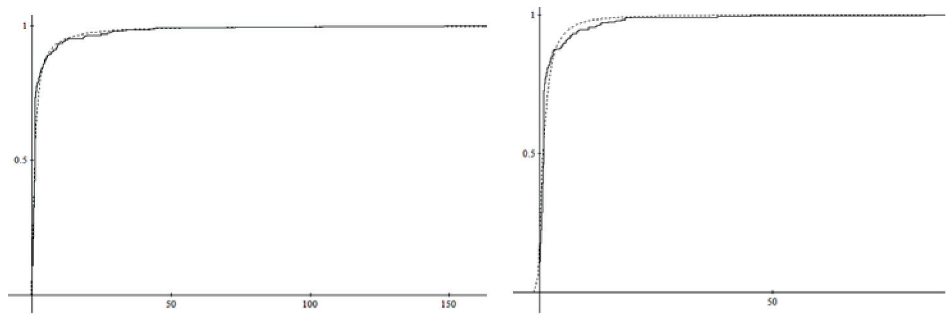

Fig. 3. Plots of empirical and adjusted theoretical distribution functions of the combined sets of under and overestimation maxima. Left side – 2011, right side 2012 (axis *X* – overestimation values, axis *Y* – values of the distribution functions under study)

Source: own research.

Table 4. The *p-value* results, based on the Kolmogorov-Smirnov and Anderson-Darling goodness-of-fit tests for the combined sets of under and overestimation maxima

Period	Data set	Distribution	K-S Test	A-D Test
2011	under and overestimations, combined	$G_{\gamma,\mu,\sigma}$	$p_v = 0.178$	$p_v = 0.196$
2012	under and overestimations, combined	$G_{\gamma,\mu,\sigma}$	$p_v = 0.325$	$p_v = 0.456$

Source: own research.

Table 5. Probabilistic models for the maxima of the characteristics under study, for 2011 and 2012

Period	Probabilistic model
2011 underestimations	$G_{\gamma,\mu,\sigma}(x) = \exp\left\{-\left[1+1.197\left((x-0.778)/1.262\right)\right]^{-\frac{1}{1.197}}\right\}$
2012 underestimations	$G_{\gamma,\mu,\sigma}(x) = \exp\left\{-\left[1+1.166\left((x-0.622)/0.967\right)\right]^{-\frac{1}{1.166}}\right\}$
2011 under and overestimations, combined	$G_{\gamma,\mu,\sigma}(x) = \exp\left\{-\left[1+0.909\left((x-0.439)/0.75\right)\right]^{-\frac{1}{0.909}}\right\}$
2012 under and overestimations, combined	$G_{\gamma,\mu,\sigma}(x) = \exp\left\{-\left[1+0.528\left((x-0.442)/0.849\right)\right]^{-\frac{1}{0.528}}\right\}$

Source: own research.

Budget deviation risk was measured in terms of probabilities for maximum annual budget deviations to pass a certain pre-established threshold (*u*). The *u* value was defined as 0.1, representing a 10%[2] deviation from the budgeted value. The

[2] Based on interviews with representatives of the controlling unit, a 10% deviation from the budgeted value was established as a critical threshold in the identification of significant budget departures.

calculations were performed for two separate periods, i.e. for 2011 and 2012. In addition for each of the annual periods, probabilistic models of maximum deviations were formulated, separately for the underestimation set and for the combined set of under and overestimations. The results of the above calculations of risk, in the form of probabilities for the random variable M_{12} to exceed the value of $u = 0.1$, are presented in Table 6.

Table 6. Measures of budget deviation risk

Period	Risk measure $P(M_{12} > u)$
2011 underestimations	0.906
2012 underestimations	0.904
2011 under and overestimations, combined	0.833
2012 under and overestimations, combined	0.792

Source: own research.

The above results show that in this case, the risk of significant budget deviations was very high in both of the annual periods under examination. For 2012 the risk was marginally lower, both in the underestimation set and in the combined set of under and overestimations.

In conclusion, the authors suggest to conduct risk assessment procedures based on a subjective qualification of risk probabilities. The measures adopted for the purpose of this study are presented in Table 7.

Table 7. Measures of budget deviation risk

Probability	Risk assessment
<0-0.25)	Very low
<0.25-0.5)	Low
<0.5-0.7)	Average
<0.8-0.9)	High
<0.9-1.0)	Very high

Source: own research.

Taking into account the results presented in Table 7, the risk of extreme deviations in both the annual periods under study was very high for the underestimations set. With regard to the combined set of under and overestimations, the associated risk for both periods was assessed as high. Of note here is the measurable year-by-year decrease of budget deviation risk by –0.041, representing a risk change of approximately 5% compared to 2011 data.

6. Results and discussion

The main objective of this study, namely the determination of the potential applicability of the theory of extreme values for the assessment of budget deviation risk, was met, as evidenced by the wealth of results presented herein. Based on the findings, the authors recommend the Gumbel distribution as best adjusted to describe both the underestimation set, and the combined set of under and overestimations, at least for both periods under study. The authors were unable to identify any theoretical distribution to fit the empirical data for the estimation of maxima in the overestimation set. Distribution functions of the estimated distributions were then employed in the assessment of measures of budget deviation risk against a predetermined threshold of 10%. Based on the adopted scale of risk measures (cf. Table 7), it was established that the risk of exceeding the critical threshold was very high for the underestimation set in 2011 and 2012, and high for the combined set of under and overestimations in the periods under study. Over the period under examination, there was a marginal decreasing trend in risk probability, observed both in the underestimation set and the combined set.

Taking into consideration the nature of budget deviations and of their extremes, the above findings may be the evidence of the positive changes in budget variance in the entity under study. This change may be attributed to various factors, such as the better understanding of the processes subject to the controlling procedures, improved planning, or improved budget control. For the purpose of this study, it was assumed that the decreasing trend of the extreme value deviations has a positive effect on the deviation for which the associated extreme value was calculated. This assumption was validated based on the calculated probabilities of deviations past the critical threshold, both in the underestimations set, and in the combined set of under and overestimations for both annual periods under examination. Marginal year-by--year decreases of these probabilities may be evidence of a slight decrease of risk of budget variance extremes (this applies both to under and overestimations).

In view of the above, the authors recommend the use of the postulated budget deviation risk assessment method in the evaluation of controlling effectiveness, both across various responsibility centres, or against historical data.

According to the authors, the measurement of the probability of exceeding budget variances is part of the trend devoted to the creation of new methods supporting budget control. Using this approach gives managers the opportunity to look at variance levels in a synthetic way. Thanks to this, it is possible to assess the effects of budgeting over time or between different entities. On the basis of a numerical example, the authors demonstrated the applicability of this method, which means that practitioners can use the deviation assessment method. The authors also hope that the synthetic measurement of deviations will be useful for researchers in the field of management accounting. Using the approach presented in this article, it will be possible to examine what impact the characteristics such as the type of activity, budgeting methods, organizational culture, etc. have on the level of variance.

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PROBABILISTYCZNA ANALIZA WARTOŚCI EKSTREMALNYCH W OCENIE RYZYKA WYSTĄPIENIA ODCHYLEŃ BUDŻETOWYCH

Streszczenie: W niniejszym artykule przedstawiono metody oceny ryzyka wystąpienia różnicy między danymi budżetowymi a rzeczywistymi. Postulowane podejście wykorzystuje teorię wartości ekstremalnych, która pozwala m.in. na ocenę parametrów rozkładu badanych zjawisk. Właściwe rozpoznanie tych parametrów daje możliwość obliczenia prawdopodobieństwa przekroczenia wariancji budżetowej o wartość ustaloną jako krytyczna. Metoda ta umożliwia ocenę poziomu odchyleń w badanym okresie, a także przeprowadzanie porównań w czasie tego poziomu w jednostce gospodarczej. Głównym celem tego artykułu było zbadanie użyteczności teorii wartości ekstremalnych w ocenie ryzyka powstania odchyleń budżetowych. W opracowaniu przedstawiono wyniki analizy probabilistycznej przeprowadzonej dla danych pochodzących z systemu kontroli budżetowej kosztów przedsiębiorstwa produkcyjnego w latach 2011-2012. Opracowana metoda analizy i oceny odchyleń budżetowych jest zgodna z rozwojem koncepcji i metod rachunkowości zarządczej.

Słowa kluczowe: kontrola budżetowa, statystyczne metody kontroli budżetowej, analiza wartości ekstremalnych.