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Can a variational approach describe pulse splitting in a dispersion managed system?

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When looking for solitons in nonlinear systems, it is often useful to have a simplifying tool. One such tool is the variational method. On the other hand, in the presence of fast oscillations, the wavefunction of the system can split into two distinct parts. This is not describable by the classical variational method. EDWARDS *et al.*, (J. Phys. B **38**(4), 2005, pp. 363–76), introduced a hybrid variational analysis which can describe the dynamics in one selected direction more accurately. However, it remained to be seen how well this method describes the dynamics of solitons, in particular their splitting and subsequent recombining. Here we investigate an application of the hybrid variational analysis to a two dimensional system with dispersion management, where such splitting is known to occur. We conclude that indeed agreement is good. This could encourage wider use of the hybrid method.

Keywords: nonlinear optics, solitons.

1. Introduction

Recently quite a lot of interest has been focused on the propagation of multidimensional solitons in both nonlinear optical systems and Bose–Einstein condensates [1]. A simplified theoretical method to search for these solitons is the variational approximation (VA) [2, 3]. This method reduces partial to ordinary differential equations. When the phase space in question is large, this simplification can save a lot of computational effort or may even be crucial, as we wish to find regions of stability. The pure VA was used in the soliton context by KATYSHEV and MAKHANKOV [4] and then to solitons in optical fibers, *e.g.*, ANDERSON *et al.* [5, 6]. The variational approximation is presently widely applied to problems in nonlinear optics [2, 3, 7], for example, in dispersion managed (DM) optical waveguides [8–15], where it is a natural approach.

2. Illustration of the approach

We illustrate the variational approach using the case of dispersion management following reference [15]. The model is based on the normalized equation describing the evolution of the local amplitude u of an electromagnetic wave propagating along the z direction in a 2D slab:

$$iu_z + \frac{1}{2}u_{xx} + \frac{D(z)}{2}u_{tt} + |u|^2 u = 0 \tag{1}$$

where $u_z \equiv \partial u/\partial z$. The function $D(z)$ is periodic

$$D(z) = \begin{cases} 1 & \text{for} \quad 2k < z \le 2k+1 \\ -1 & \text{for} \quad 2k+1 < z \le 2k+2 \end{cases} \tag{2}$$

with k being zero or a natural number. Equation (2) can be derived from the Lagrangian $L = (1/2)\int_{-\infty}^{+\infty}\left[i(u_z u^* - u_z^* u) - |u_x|^2 - D(z)|u_t|^2 + |u|^4\right]\mathrm{d}x\mathrm{d}t$. The VA is introduced upon using the Gaussian ansatz,

$$u = A(z)\exp\left\{i\theta(z) - \frac{1}{2}\left[\frac{x^2}{W^2(z)} + \frac{t^2}{T^2(z)}\right] + \frac{i}{2}\left[b(z)x^2 + \beta(z)t^2\right]\right\} \tag{3}$$

where A and θ are the amplitude and phase of the soliton, W and T are its transverse and temporal widths, and b and β are the spatial and temporal chirps. Substitution of the ansatz (3) into the Lagrangian and integrating over x and t yields an *effective Lagrangian*,

$$\frac{4}{\pi}L_{\text{eff}} = A^2 WT\left[4\theta' - b'W^2 - \beta' T^2 - W^{-2} - DT^{-2} + A^2 - b^2 W^2 - D(z)\beta^2 T^2\right] \tag{4}$$

where the prime stands for d/dz. The variational equation $\delta L/\delta\theta = 0$, applied to expression (4), yields the energy conservation relation $\mathrm{d}\varepsilon/\mathrm{d}z = 0$, where $\varepsilon \equiv A^2 WT$. This relation can be used to eliminate A^2 in favor of the constant ε. As a consequence, the term of order θ' in the Lagrangian may be dropped:

$$\frac{4L_{\text{eff}}}{\pi\varepsilon} = -b'W^2 - \beta' T^2 - \frac{1}{W^2} - \frac{D(z)}{T^2} + \frac{\varepsilon}{WT} - b^2 W^2 - D(z)\beta^2 T^2 \tag{5}$$

Varying the Lagrangian (5) with respect to the remaining independent variables W, T, b, β, and substituting $b = W'/W$ and $\beta = D^{-1}T'/T$ yields the following closed system of equations:

$$W'' = \frac{1}{W^3} - \frac{\varepsilon}{2W^2 T} \tag{6}$$

$$T'' - \frac{D''}{D}T' = \frac{D^2}{T^3} - \frac{D\varepsilon}{2WT^2} \tag{7}$$

These ordinary differential equations can easily be solved numerically in search for solitons or other regular structures.

In reference [15], a variational analysis based on the above equations led to the conclusion that stable two dimensional solutions in the presence of a periodic dispersion modulation exist, but in the three dimensional case (bulk medium), all the spatiotemporal pulses will spread out or collapse. Later it was demonstrated that three-dimensional spatiotemporal solitary waves *can exist* in self-focusing Kerr media when a combination of dispersion management in the longitudinal direction, and periodic modulation of the refractive index in one of the transverse directions, is applied [16]. In most cases, the variational analysis gave accurate predictions of the stability regions in parameter space. In some cases, however, we noticed pulse splitting in the direction of the modulation. Surely, this kind of dynamics cannot be described by the classical variational approximation, since this approach implicitly assumes that the pulse (wavefunction) remains compact (single piece) and possibly close to Gaussian in shape. In order to get an adequate description to include phenomena like pulse splitting, one needs to treat the dimension in which it can occur in a different way.

Recently, an opportunity to include the possibility of pulse splitting appeared thanks to the method proposed by EDWARDS *et al.* [17]. They called it the hybrid Lagrangian method (HLM). The main idea is to treat all directions *except* the modulated one by a variational ansatz, but derive a one dimensional reduced partial differential equation in the direction along which the modulation occurs. In this paper, we apply the hybrid Lagrangian method to describe pulse splitting in two dimensional dispersion management. Our system is described by Eqs. (1) and (2). We set normalization such that $\int |u(z, x, t)|^2 \mathrm{d}x\mathrm{d}t = E = \pi\varepsilon$, and introduce a hybrid trial function in the form

$$u(z, x, t) = \phi(z, t)\exp\left[\left(\frac{1}{2} - \frac{1}{W^2(z)} + ib(z)\right)x^2\right] \tag{8}$$

and so, $\int |\phi(z, t)|^2 \mathrm{d}t = E/\sqrt{\pi}\, W$. We set the initial conditions

$$W(0) = W_0 \qquad b(0) = 0 \qquad \phi(0, t) = \exp\left[\frac{1}{2}\left(-\frac{1}{T_0^2} + i\beta_0\right)t^2\right] \tag{9}$$

The variational parameters are: $\phi(z, t)$, width $W(z)$ and chirp $b(z)$. Since the parameter ϕ is a function of both z and t, we can only reduce the space of the Lagrange density by integrating over x. We obtain

$$\tilde{L} = \left\{\frac{D}{i}\left(\phi_z\phi^* - \phi_z^*\phi\right) - |\phi_t|^2 + \left[\frac{1}{2}b_zW^2 + \frac{1}{4W^2} + b^2W^2\right]|\phi|^2 + \frac{1}{\sqrt{2}}|\phi|^4\right\}\frac{\pi^{1/2}W}{2} \tag{10}$$

The Euler–Lagrange equations lead to a more compact form if we substitute [17]

$$\phi = E^{1/2}\sqrt{\frac{W(0)}{W(z)}}\,\exp\left[i\left(\frac{1}{W^2}+\frac{\lambda}{2W}\right)\right]\tilde{\phi}$$

$$\lambda = -\frac{\sqrt{\pi}\,E}{2\sqrt{2}}W^2(0)\int_{-\infty}^{\infty}\left|\tilde{\phi}\right|^4 \tag{11}$$

When this is done we obtain the closed system of equations

$$i\tilde{\phi}_z + \frac{D(z)}{2}\tilde{\phi}_{tt} + \frac{E}{\sqrt{2}}\frac{W(0)}{W(z)}\left|\tilde{\phi}\right|^2\tilde{\phi} = 0 \tag{12}$$

$$W'' = \frac{1}{W^3} + \frac{2\lambda}{W^2} \tag{13}$$

These equations, with λ defined in Eq. (11), describe the evolution of our pulse along z. Note that the right hand side of Eq. (13) can describe a potential well, as $\lambda < 0$. In contradistinction to a classical variational approximation, in which the temporal variation would also be modeled by a Gaussian, the hybrid treatment can describe soliton splitting, amalgamation, *etc*.

We looked at cases for which the full numerics gave soliton splitting-recombining dynamics. Figure 1 shows a comparison of the splitting and recombining of the pulse as described by two approaches; a full numerical approach and the hybrid method. The phenomenon pictured in Fig. 1 occurs periodically (with the imposed period of the dispersion management). As we see, the hybrid description provides an excellent

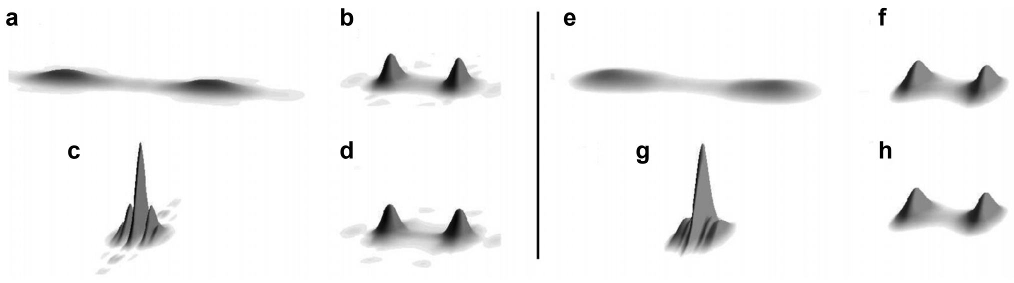

Fig. 1. Example of a stable solution with pulse splitting as found from full two dimensional numerical simulation (**a**–**d**) compared to those found by the hybrid variational approach (**e**–**h**). The subsequent frames show pulse intensity as taken at the beginning, quarter period, half and three quarters of the dispersion management period. The parameters are: $T_0 = 1$, $W_0 = 1$, $E = 2\pi$, and $\beta_0 = 0$.

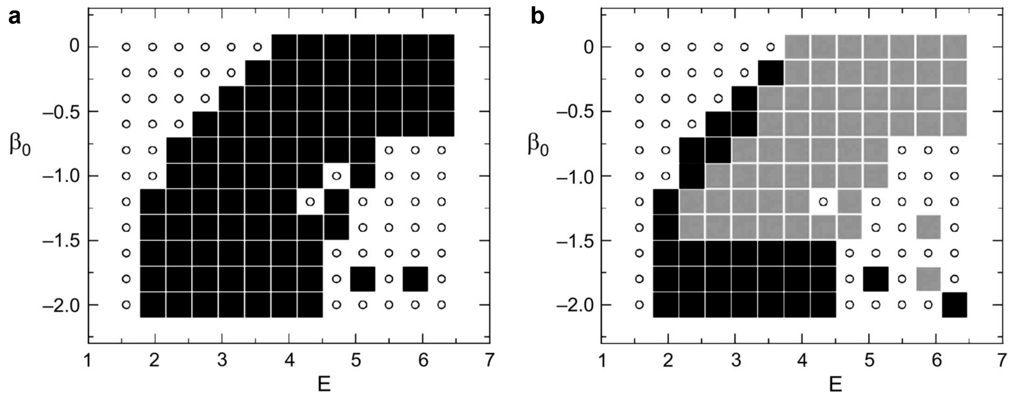

Fig. 2. Comparison of the stability regions in parameter space obtained from the classical variational method (**a**), and the hybrid model (**b**). The black region corresponds to stable one peak solutions, the grey region corresponds to stable pulse splitting-recombining solutions, and circles to unstable configurations (spreading). The parameters are: $T_0 = 1.35$, $W_0 = 1.35$.

fit. In Figure 2, a comparison of the stability regions in $E - \beta_0$ parameter space is presented. Figure 2**a** pictures the results obtained from the classical variational method, and Fig. 2**b** those obtained from the hybrid variational approach. The stability regions are very similar, but in the case of the hybrid method we get a much fuller picture.

3. Conclusions

In conclusion, our analysis confirms that indeed the hybrid method provides a tool for looking at pulse phenomena (*e.g.*, pulse splitting) that are missed by the very nature of the classical variational analysis.

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