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Propagation properties of partially coherent beams through turbulent media with coherent modes representation

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The partially coherent beams propagating through turbulent atmosphere have been studied in the past using coherent mode representation. In this research, the propagation of any modes of Hermite–Gaussian beam in a turbulent atmosphere is investigated and analytical formula for the average intensity of these beams is derived. The power in bucket (PIB) for any modes is also examined. The number of modes which exist in a partially coherent beam with known degree of global coherence (ratio of correlation length and the waist width of the Gaussian–Schell model (GSM) beam) is determined and the PIB for partially coherent beams is investigated using coherent mode representation.

Keywords: atmospheric turbulence, partially coherent beam, Hermite–Gaussian beam, power in bucket (PIB).

1. Introduction

Propagation of a laser beam through random media is the topic that has been theoretically and experimentally studied for a long time, as is evident from the number of books and papers written on the subjects [1, 2]. In many practical applications, such as remote sensing, tracking, and long-distance optical communication, *etc.*, the propagation properties of laser beam through atmospheric turbulence are of great importance and atmospheric turbulence has essential effects in such applications. Most research is about the spreading of a laser beam in a turbulent atmosphere [3–6]. Recently, Eyyuboglu and Baykal investigated the properties of cos-Gaussian, cosh-Gaussian, Hermite-sinusoidal-Gaussian and Hermite-cosine-Gaussian laser beams in a turbulent atmosphere [7–9]. Cai and He have studied the spreading properties of an elliptical Gaussian beam in a turbulent atmosphere [10]. In this paper, the average intensity of Hermite–Gaussian modes propagating in turbulent media is investigated and the effects of turbulence on the properties of any arbitrary mode are analyzed.

The number of full coherent modes constituting a specific partially coherent beam has been determined and it is shown that the intensity profile and PIB of a propagated partially coherent beam are equal to the sum of intensity profile and PIB for full coherent modes constituting it. It means that if we expand a specific partially coherent beam into a number of fully coherent modes, then calculating intensity and PIB of the constituting modes and integrating the results will lead to intensity and PIB of that partially coherent beam.

2. Propagation of Hermite–Gaussian mode in turbulent atmosphere

Field distribution of a Hermite–Gaussian laser beam on the source plane is as follows [5]:

$$\phi_{mn}^{(0)}(\rho_x, \rho_y, 0) = B_{mn} H_m\left(\frac{\sqrt{2}}{w_0}\rho_x\right) H_n\left(\frac{\sqrt{2}}{w_0}\rho_y\right) \exp\left(-\frac{\rho_x^2+\rho_y^2}{w_0^2}\right) \tag{1}$$

where H_m and H_n are Hermite polynomials, w_0 is the spot size at the source plane, (ρ_x, ρ_y) is a two-dimensional vector in the source plane and B_{mn} are normalization coefficients represented in the following form [5]:

$$B_{mn} = \frac{1}{w_0\sqrt{\pi 2^{m+n-1} m!\, n!}} \tag{2}$$

Assume that the medium is statistically homogeneous and isotropic, then according to the paraxial form of extended Huygens–Fresnel principle the relation between electrical field before and after propagation is [1, 2]:

$$E(\boldsymbol{\rho}', z) = -\frac{ik\exp(ikz)}{2\pi z}\iint E(\boldsymbol{\rho})\exp\left[ik\frac{|\boldsymbol{\rho}'-\boldsymbol{\rho}|^2}{2z}\right]\exp\left[\psi(\boldsymbol{\rho}', \boldsymbol{\rho}, z)\right]\mathrm{d}^2\rho \tag{3}$$

where $E(\boldsymbol{\rho}, z=0)$ and $E(\boldsymbol{\rho}', z)$ represent the amplitude of incident field and the field after propagating a distance z through the turbulent medium, respectively, and ψ is the phase function that depends on the properties of the medium.

The average intensity at the receiver plane is given by $\langle I(\boldsymbol{\rho}', z)\rangle =$ $= \langle E(\boldsymbol{\rho}', z)\, E^*(\boldsymbol{\rho}', z)\rangle$ where $\langle\ \rangle$ denotes the ensemble average over the medium statistic. From Eq. (3), we obtain

$$\langle I(\boldsymbol{\rho}', z)\rangle = \left(\frac{k}{2\pi z}\right)^2 \iint \mathrm{d}^2\rho_1 \iint \mathrm{d}^2\rho_2 E(\boldsymbol{\rho}_1) E^*(\boldsymbol{\rho}_2)\exp\left[-ik\frac{|\boldsymbol{\rho}'-\boldsymbol{\rho}_1|^2 - |\boldsymbol{\rho}'-\boldsymbol{\rho}_2|^2}{2z}\right]$$
$$\times \left\langle \exp\left[\psi(\boldsymbol{\rho}', \boldsymbol{\rho}_1, z) + \psi^*(\boldsymbol{\rho}', \boldsymbol{\rho}_2, z)\right]\right\rangle \tag{4}$$

The last term in the integrand of Eq. (4) can be expressed as [3–5]

$$\left\langle \exp\left[\psi(\boldsymbol{\rho}', \boldsymbol{\rho}_1, z) + \psi^*(\boldsymbol{\rho}', \boldsymbol{\rho}_2, z)\right]\right\rangle = \exp\left[-0.5 D_\psi(|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|)\right]$$
$$= \exp\left[-\Phi|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|^2\right] \tag{5}$$

where $D_\psi(|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|)$ is the phase structure function in Rytov's representation and $\Phi = (0.545 C_n^2 k^2 z)^{6/5}$ is the coherence length of a spherical wave propagating in the turbulent medium with C_n^2 being the structure constant. In order to obtain an analytical result the quadratic approximation (see, *e.g.*, Eqs. (14) and (15) in [11]) for Rytov's phase structure function is used here since this quadratic approximation has been widely investigated and shown to be reliable [3–5].

To evaluate Eq. (4), it is convenient to introduce the new variable of integration,

$$\mathbf{u} = \frac{\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2}{2}, \quad \boldsymbol{v} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2 \tag{6}$$

and Eq. (4) reduces to

$$\langle I(\boldsymbol{\rho}', z)\rangle = \left(\frac{k}{2\pi z}\right)^2 \iint \mathrm{d}^2 u \iint \mathrm{d}^2 v\, E\left(\mathbf{u} + \frac{\boldsymbol{v}}{2}\right) E^*\left(\mathbf{u} - \frac{\boldsymbol{v}}{2}\right) \exp\left[-i\frac{k}{z}\mathbf{u}\cdot\boldsymbol{v}\right]$$
$$\times \exp\left[i\frac{k}{z}\boldsymbol{\rho}'\cdot\boldsymbol{v}\right]\exp\left[-\Phi v^2\right] \tag{7}$$

This is the basic formula that will be used to study the propagation of Hermite–Gaussian modes through atmospheric turbulence.

Substituting Eq. (1) and Eq. (2) in Eq. (7) and calculating the related integral results:

$$I_{mn}(\boldsymbol{\rho}', z) = \left(\frac{k}{2\pi z}\right)^2 |B_{mn}|^2 \frac{\pi}{2} w_0^2 2^{m+n}$$
$$\times \sum_{l=0}^{n}\sum_{j'=1}^{l}\sum_{l'=0}^{m}\sum_{j'=1}^{l'}(-1)^{l+l'}\frac{(m!n!)^2}{(m-l)!(n-l')!l!l!l'!l'!}$$
$$\times \eta^{l+l'}\exp\left[-\frac{k^2}{4z^2\zeta^2}\left(\rho_x'^2 + \rho_y'^2\right)\right]\frac{\pi}{2^{2l+2l'}\zeta^{2(l+l')+2}}B_{j,2n}B_{j',2n}$$
$$\times\left(-\frac{ik\rho_x'}{2\zeta z}\right)^{2l-2j}\left(-\frac{ik\rho_y'}{2\zeta z}\right)^{2l'-2j'} \tag{8}$$

where

$$B_{m,\,n+1} = nB_{m-1,\,n-1} + B_{m,\,n}, \quad B_{1,\,n} = 1, \quad B_{n,\,n} = 0, \quad B_{1,\,1} = 1 \tag{9}$$

$$\eta = \frac{1}{w_0^2} + \frac{k^2 w_0^2}{4z^2} \tag{10}$$

$$\zeta^2 = \Phi + \frac{1}{2w_0^2} + \frac{k^2 w_0^2}{8z^2} \tag{11}$$

The propagation properties of Hermite–Gaussian modes in a turbulent atmosphere can be investigated using Eqs. (8)–(11). Figure 1 shows the behavior of the normalized

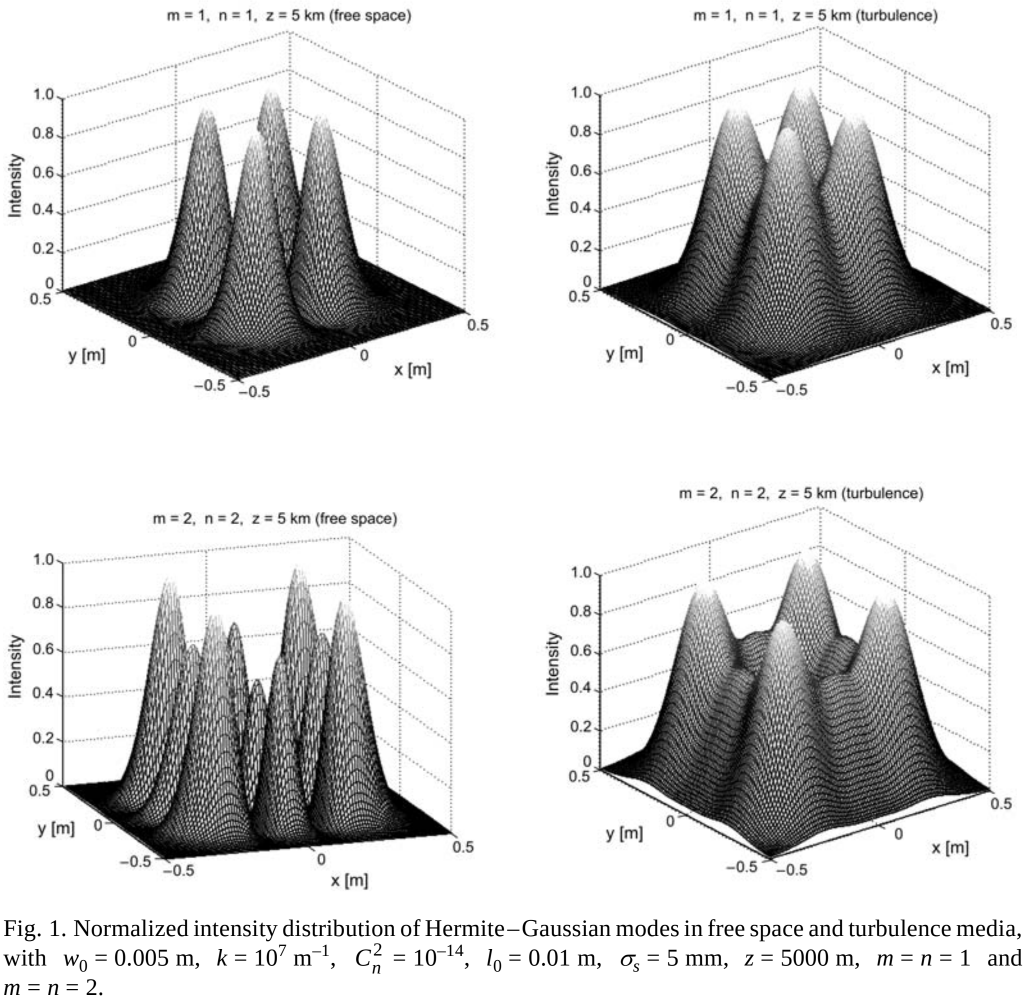

Fig. 1. Normalized intensity distribution of Hermite–Gaussian modes in free space and turbulence media, with $w_0 = 0.005$ m, $k = 10^7$ m^{-1}, $C_n^2 = 10^{-14}$, $l_0 = 0.01$ m, $\sigma_s = 5$ mm, $z = 5000$ m, $m = n = 1$ and $m = n = 2$.

intensity distribution of Hermite–Gaussian modes in atmospheric turbulence and in free space, with $w_0 = 0.005$ m, $k = 10^7$ m^{-1}, $C_n^2 = 10^{-14}$, $m = n = 1$ and $m = n = 2$ and $z = 5$ km. As it is shown in Fig.(1) the peaks of intensity of Hermite–Gaussian modes in atmospheric turbulence and in free space. As is shown in Fig. 1, the peaks of intensity of Hermite–Gaussian modes in atmospheric turbulence are more spread out and overlapped in comparison to those in free space.

3. Explanation of the partially coherent mode in terms of fully coherent mode

The initial field of a partially coherent beam takes the typical form of Gaussian–Schell model (GSM) beam, whose cross-spectral density function $W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z)$ at the source $z = 0$ is [13, 14]

$$W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, z = 0) = A\exp\left[-\frac{|\boldsymbol{\rho}_1|^2 + |\boldsymbol{\rho}_2|^2}{4\sigma_s^2}\right]\exp\left[-\frac{|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|^2}{2\sigma_\mu^2}\right] \tag{12}$$

where A is a constant, $\boldsymbol{\rho}_1 = (\rho_{1x}, \rho_{1y})$ and $\boldsymbol{\rho}_2 = (\rho_{2x}, \rho_{2y})$ denote two different points at the source plane and σ_s and σ_μ are the waist width and correlation length of the GSM beam, respectively.

It supposes that the same conditions for turbulent atmosphere as mentioned in the pervious section, the propagation equation of intensity of GSM beams turns out to be [5]

$$I(\rho', z) = \frac{A}{\Delta^2(z)}\exp\left(-\frac{\rho'^2}{2k\sigma_s^2\Delta^2(z)}\right) \tag{13}$$

where

$$\Delta^2(z) = 1 + \frac{1}{k^2 w_0^2}\left(\frac{1}{4w_0^2} + \frac{1}{\sigma_\mu^2}\right)z^2 + \frac{\Phi}{\sigma_s^2}z^3 \tag{14}$$

The cross-spectral density function of such a source assumed, for simplicity, to be rectangular, may be represented in the form [5]:

$$W^{(0)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \omega) = \sum_m\sum_n \beta_{mn}(\omega)\phi_{mn}^{(0)}(\boldsymbol{\rho}_1, \omega)\phi_{mn}^{(0)}(\boldsymbol{\rho}_2, \omega) \tag{15}$$

where ϕ_{mn} represent the different coherent modes such as Hermite–Gaussian polynomials. This expansion is called coherent mode representation. Expansion coefficient of Eq. (15) is as follows [5]:

$$\frac{\beta_{mn}(\omega)}{\beta_{00}(\omega)} = \left[\frac{1}{\frac{l_g^2}{2} + 1 + l_g\sqrt{\left(\frac{l_g}{2}\right)^2 + 1}}\right]^{m+n} \tag{16}$$

where $l_g = \sigma_\mu/\sigma_s$ is the degree of global coherence. Evidently, in the limit of the complete coherence ($l_g \to \infty$) and in the incoherence limit ($l_g \to 0$). This implies that an infinite number of modes are needed to represent a spatially non-coherent source.

To determine the number of modes in a special partially coherent beam, first using Eq. (13) we calculate the intensity profile of that partially coherent beam and then name it profile one.

Then using Eqs. (8)–(11) each propagated mode $\phi_{mn}^{(0)}$ is calculated and replaced in right-hand side of Eq. (15). We choose the modes whose respective expansion

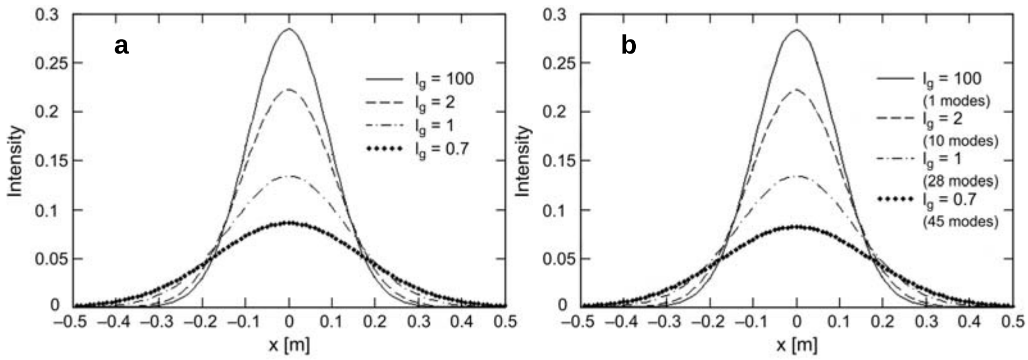

Fig. 2. Intensity distribution of partial coherence beam (**a**), intensity distribution of partial coherence beam by using coherent modes representation (**b**) with $w_0 = 0.01$ m, $k = 10^7$ m^{-1}, $C_n^2 = 10^{-14}$, $l_0 = 0.01$ m, $\sigma_s = 5$ mm, $z = 5000$ m.

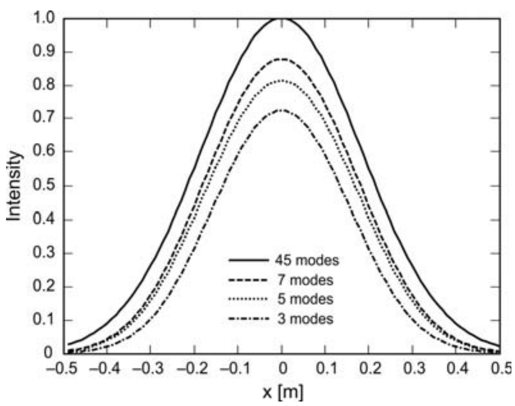

Fig. 3. Intensity profile of partially coherent beams (45 modes) and the behavior of the intensity distribution for limited number of modes.

T a b l e. Number of modes with full coherence which exist in partially coherent beam.

l_g	Fully coherent modes
100	ϕ_{00}
2	ϕ_{00}, ϕ_{10}, ϕ_{01}, ϕ_{11}, ϕ_{20}, ϕ_{02}, ϕ_{21}, ϕ_{12}, ϕ_{30}, ϕ_{03}
1	ϕ_{00}, ϕ_{10}, ϕ_{01}, ϕ_{11}, ϕ_{20}, ϕ_{02}, ϕ_{21}, ϕ_{12}, ϕ_{22}, ϕ_{30}, ϕ_{03}, ϕ_{31}, ϕ_{13}, ϕ_{32}, ϕ_{23}, ϕ_{33}, ϕ_{40}, ϕ_{04}, ϕ_{41}, ϕ_{14}, ϕ_{42}, ϕ_{24}, ϕ_{50}, ϕ_{05}, ϕ_{51}, ϕ_{15}, ϕ_{60}, ϕ_{06}
0.7	ϕ_{00}, ϕ_{10}, ϕ_{01}, ϕ_{11}, ϕ_{20}, ϕ_{02}, ϕ_{21}, ϕ_{12}, ϕ_{22}, ϕ_{30}, ϕ_{03}, ϕ_{31}, ϕ_{13}, ϕ_{32}, ϕ_{23}, ϕ_{33}, ϕ_{40}, ϕ_{04}, ϕ_{41}, ϕ_{14}, ϕ_{42}, ϕ_{24}, ϕ_{43}, ϕ_{34}, ϕ_{44}, ϕ_{50}, ϕ_{05}, ϕ_{51}, ϕ_{15}, ϕ_{52}, ϕ_{25}, ϕ_{53}, ϕ_{35}, ϕ_{60}, ϕ_{06}, ϕ_{61}, ϕ_{16}, ϕ_{62}, ϕ_{26}, ϕ_{70}, ϕ_{07}, ϕ_{71}, ϕ_{17}, ϕ_{80}, ϕ_{08}

coefficients are greater than 0.001 ($\beta_{mn} > 0.001$) and calculate corresponding intensity profile and name it profile two.

Finally, comparing profiles one and two, one can find the number of fully coherent modes constituting supposed partially coherent beam.

In Figure 2**a**, intensity profiles of various partially coherent beams are plotted using Eq. (13). The behavior of the intensity distribution using coherent modes representation and Eqs. (8)–(11) is illustrated in Fig. 2**b**.

Comparing Figs. 2**a** and 2**b**, we have determined corresponding number of full coherent modes for any l_g, as mentioned before; the result of this determination are shown in the Table.

It is worth mentioning that, as is evident from Fig. 2, the number of modes involved, increases with decreasing l_g.

One goal of this paper is to find the number of full coherent modes required for construction of any arbitrary partially coherent beam. In practice, it is very difficult to combine more than several modes which will lead us to a remarkable error, as is clearly shown in Fig. 3.

Figure 3 shows the intensity profile of partially coherent beams (45 modes) and the behavior of the intensity distribution using coherent modes representation for limited number of modes. Although it seems that combining 45 modes is impracticable but deceasing the number of modes will lead to a remarkable error, which is clearly shown in this figure.

4. Power in bucket

As suggested by Siegman, PIB is a measure of laser power focus ability in the far field [14]. It indicates the amount of beam power within a given bucket. The PIB is defined as [15]

$$\mathrm{PIB} = \frac{\int_0^{2\pi}\int_0^{\alpha} I(\boldsymbol{\rho}, z)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta}{\int_0^{2\pi}\int_0^{\infty} I(\boldsymbol{\rho}, z)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta} \tag{17}$$

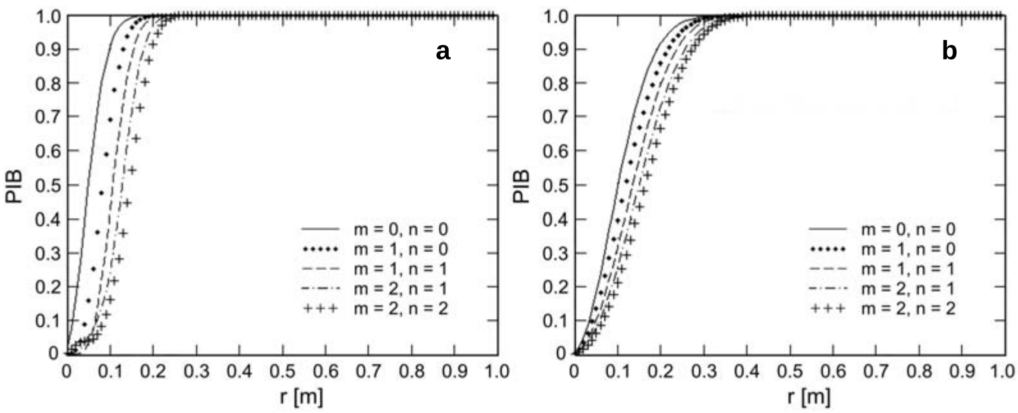

Fig. 4. PIB for Hermite–Gaussian modes in free space (**a**), PIB for Hermite–Gaussian modes in turbulent media (**b**) with w_0 = 0.01 m, $k = 10^7$ m^{-1}, $C_n^2 = 10^{-14}$, l_0 = 0.01 m, σ_s = 5 mm, z = 5000 m.

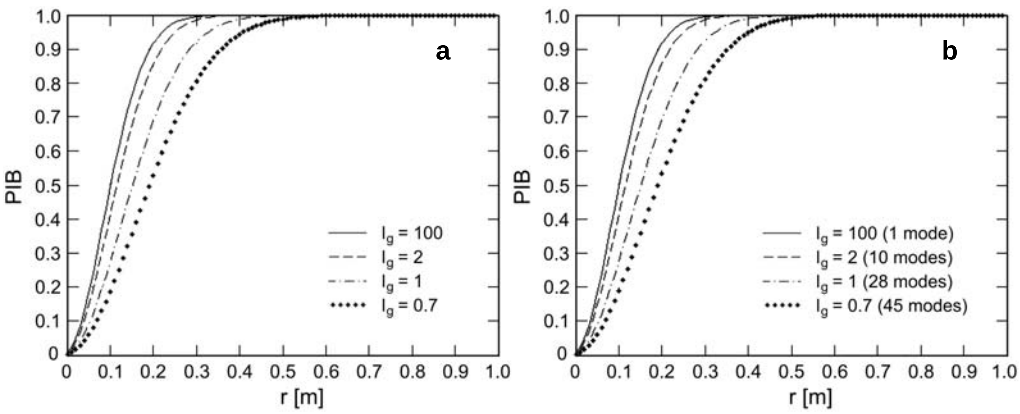

Fig. 5. PIB for partially coherent beam in turbulent media (**a**), PIB for partially coherent beam in turbulent media by using the sum of PIB of modes which exist in the partially coherent beam (**b**) with w_0 = 0.01 m, $k = 10^7$ m^{-1}, $C_n^2 = 10^{-14}$, l_0 = 0.01 m, σ_s = 5 mm, z = 5000 m.

where α is the radius of the bucket. The PIB for any Hermite–Gaussian mode was calculated using Eqs. (8) and (17).

Figures 4**a** and 4**b** show the behavior of PIB for Hermite–Gaussian modes in free space and turbulent atmosphere respectively. These figures show that PIB for higher order modes is more extended. It is also evident that PIB in turbulence for any arbitrary mode is more broadened in comparison with PIB in free space.

Figure 5**a** shows the behavior of PIB for partially coherent beam calculated from Eqs. (13), (17) and Fig. 5**b** shows the behavior of PIB calculated by the method of coherent mode representation. The accurate conformity of these two figures demonstrates that our procedure of determining the number of fully coherent beams involved in a specific partially coherent beam (specific l_g) is quite reliable.

5. Conclusions

In this article, the analytical formula for average intensity of Hermite–Gaussian modes propagating in turbulent media was investigated. We have noticed that peaks of Hermite–Gaussian beams propagating through turbulent atmosphere are distributed in plane and are overlapped. It has been found that the number of full coherent modes constituting a specific partially coherent beam can be determined by calculating intensity profile and PIB of the partially coherent beam by two distinct methods and comparing the results of them. It has also been found that PIB for Hermite–Gaussian mode propagating through atmospheric turbulence is more broadened in comparison with that in free space, and calculating PIB of the constituting modes of a partially coherent beam and summing up the results, will lead to PIB of that partially coherent beam.

This procedure is implemented in this article by comparing the intensity profiles and validated by calculation of corresponding PIBs.

References

[1] FANTE R.L., *Wave propagation in random media: a system approach*, [In] *Progress in Optics*, E. Wolf [Ed.], Elsevier, Amsterdam, 1985, Vol. 22, pp. 341–98.

[2] ANDREWS L.C., PHILLIPS R.L., *Laser Beam Propagation through Random Media*, SPIE Press, Bellingham, Washington 1998.

[3] WANG S.C.H, PLONUS M.A., *Optical beam propagation for a partially coherent source in the turbulent atmosphere,* Journal of the Optical Society of America **69**(9), 1979, pp. 1297–304.

[4] LEADER J.C., *Atmospheric propagation of partially coherent radiation*, Journal of the Optical Society of America **68**(2), 1978, pp. 175–85.

[5] SHIRAI T., DOGARIU A., WOLF E., *Mode analysis of spreading of partially coherent beams propagating through atmospheric turbulence*, Journal of the Optical Society of America A: Optics, Image Science and Vision **20**(6), 2003, pp. 1094–102.

[6] YANGJIAN CAI, SAILING HE, *Propagation of a partially coherent twisted anisotropic Gaussian Schell-model beam in a turbulent atmosphere*, Applied Physics Letters **89**(4), 2006, p. 041117.

[7] EYYUBOGLU H.T., BAYKAL Y., *Analysis of reciprocity of cos-Gaussian and cosh-Gaussian laser beams in turbulent atmosphere*, Optics Express **12**(20), 2004, pp. 4659–74.

[8] EYYUBOGLU H.T., BAYKAL Y., *Average intensity and spreading of cosh-Gaussian laser beams in the turbulent atmosphere,* Applied Optics **44**(6), 2005, pp. 976–83.

[9] BAYKAL Y., *Correlation and structure functions of Hermite-sinusoidal-Gaussian laser beams in a turbulent atmosphere*, Journal of the Optical Society of America A: Optics, Image Science and Vision **21**(7), 2004, pp. 1290–9.

[10] YANGJIAN CAI, SAILING HE, *Average intensity and spreading of an elliptical Gaussian beam propagating in a turbulent atmosphere,* Optics Letters **31**(5), 2006, pp. 568–70.

[11] EYYUBOGLU H.T., ARPALI C., BAYKAL Y.K., *Flat topped beams and their characteristics in turbulent media*, Optics Express **14**(10), 2006, pp. 4196–207.

[12] MANDEL L.M., WOLF E., *Optics Coherence and Quantum Optics*, Cambridge University Press, Cambridge, England 1995.

[13] FRIBERG A.T., SUDOL R.J., *Propagation parameters of Gaussian Schell-model beams*, Optics Communications **41**(6), 1982, pp. 383–7.

[14] SIEGMAN A.E., *How to (maybe) measure laser beam quality*, OSA TOPS, Vol. 17, 1998, p. 184.

[15] HAI XING YAN, SHU SHAN LI, DE LIANG ZHANG, SHE CHEN, *Numerical simulation of an adaptive optics system with laser propagation in the atmosphere*, Applied Optics **39**(18), 2000, pp. 3023–31.

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