Optica Applicata, Vol. XL, No. 4, 2010

Propagation of electromagnetic waves in weakly anisotropic media: Theory and applications

YURY A. KRAVTSOV, BOHDAN BIEG*

Institute of Physics, Maritime University of Szczecin,
Wały Chrobrego 1-2,70-500 Szczecin, Poland

*Corresponding author: b.bieg@am.szczecin.pl

Quasi-isotropic approximation (QIA) of geometrical optics is outlined, which describes properties of electromagnetic waves in weakly anisotropic media, including weakly anisotropic fibers, liquid crystals and weakly magnetized plasma. QIA equations stem directly from the Maxwell equations and have the form of coupled differential equations of the first order for transverse components of the electromagnetic field. Being applied to magnetized plasma, QIA describes combined action of Faraday and Cotton–Mouton phenomena and serves as a theoretical background of plasma polarimetry in FIR and microwave bands.

The coupled equations of QIA can be reduced to a single equation for complex polarization angle (CPA), which characterizes all the parameters of polarization ellipse. At the same time, the equation for CPA allows obtaining equations for evolution of the traditional parameters of polarization ellipse. Besides, QIA makes it possible to derive in a consistent way Segre's equation for the Stokes vector evolution, which is widely used in microwave and FIR plasma polarimetry.

Keywords: quasi-isotropic approximation, polarization.

1. Introduction

There are two basic approaches for description of electromagnetic waves in weakly anisotropic media: quasi-isotropic approximation (QIA) of the geometrical optics method and Stokes vector formalism (SVF).

Quasi-isotropic approximation of the geometrical optics method, suggested in 1969 [1], describes electromagnetic waves in weakly anisotropic media on the basis of coupled wave equations for the transverse components of the electromagnetic field. QIA was developed in depth in a review paper [2] and in book [3]. QIA equations are presented also in textbooks [4, 5].

Stokes vector formalism, ascending to crystal optics [6, 7], deals with the components of the Stokes vector, which are quadratic in wave field. Compact equation for the Stokes vector evolution in plasma was suggested in 1978 by Segre [8, 9].

This paper outlines the basic principles of QIA and describes recent modifications of QIA. Simultaneously, we derive equations of the SVF directly from QIA equations. We intend to show, on the one hand, similarity and practical equivalence of both approaches and, on the other, their incomplete identity: SVF contains no phase information and therefore the wave field can be reconstructed only partially on the basis SVF.

General form of QIA equations is presented in Section 2, whereas successive Sections 3 through 7 are devoted to various modifications and applications of QIA. Section 3 describes normal electromagnetic modes in magnetized plasma. Section 4 considers double passage of electromagnetic waves through magnetized plasma, using representation of independent normal modes [10]. Section 5 presents a revised theory of the normal modes conversion in the inhomogeneous plasma. This section improves the results presented in paper [11] and in book [3].

An equation for complex polarization angle (CPA), still involved in [1], is presented in Section 6 in the new form suggested recently in [12]. Section 7 derives equations for the evolution of traditional angular parameters of polarization ellipse in a weakly anisotropic plasma.

Based on the results of Refs. [13, 14], we show in Section 8 that Segre's equation for the Stokes vector evolution can be derived directly from QIA equations. Interrelations between QIA and SVF approaches, evidencing their practical equivalence and simultaneously principal distinction between them, are analyzed in Section 9. Finally, Section 10 summarizes the main results of the paper.

2. Quasi-isotropic approximation (QIA)

In weakly anisotropic media the components of the anisotropy tensor $v_{mn} = \varepsilon_{mn} - \varepsilon_0 \sigma_{mn}$ are small as compared with the electric permittivity ε_0 of the isotropic background medium. QIA operates with two small parameters. One of them is the "anisotropic" small parameter $\mu_A = \max|v_{mn}|/\varepsilon_0 \ll 1$, which characterizes weakness of the medium anisotropy. Besides, QIA involves traditional "geometrical" small parameter $\mu_{GO} = 1/k_0 L \ll 1$, where k_0 is a wave number and L is a characteristic scale of the medium inhomogeneity.

According to [1−5], asymptotic solution of the Maxwell equations in the lowest approximation in a combined small parameter $\mu = \max(\mu_{GO}, \mu_A)$ can be presented as

$$\mathbf{E} = A\boldsymbol{\Gamma}\exp(ik\Psi) \qquad (1)$$

where A and Ψ are correspondingly the amplitude and eikonal of the electromagnetic wave in the anisotropic medium and $\boldsymbol{\Gamma}$ is a polarization vector, which is orthogonal to the reference ray, like in an isotropic medium.

Let the unit vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ together with the unit vector $\mathbf{l}$, tangent to the ray, form a basis for Popov's orthogonal coordinate system (ξ_1, ξ_2, σ), associated with a selected (reference) ray [15] (see also chapter 9 in book [16]). The most important feature of Popov's coordinate system is its ability to describe parallel transport of the electrical vector **E** along the reference ray in an isotropic medium. The unit vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ of Popov's orthogonal system obey the equations

$$\frac{d\mathbf{e_i}}{d\sigma} = -\frac{1}{2}(\mathbf{e_i} \cdot \nabla \ln \varepsilon_0)\mathbf{l}, \quad i = 1, 2 \tag{2}$$

It follows from Maxwell equations that the components Γ_1 and Γ_2 of polarization vector $\mathbf{\Gamma} = \Gamma_1 \mathbf{e_1} + \Gamma_2 \mathbf{e_2}$ obey the coupled QIA equations

$$\begin{aligned} \frac{d\Gamma_1}{d\sigma} &= \frac{1}{2} i k_0 \varepsilon_0^{-1/2} \left(v_{11}\Gamma_1 + v_{12}\Gamma_2\right) \\ \frac{d\Gamma_2}{d\sigma} &= \frac{1}{2} i k_0 \varepsilon_0^{-1/2} \left(v_{21}\Gamma_1 + v_{22}\Gamma_2\right) \end{aligned} \tag{3}$$

The coupled Equations (3) describe all possible polarization states of the electromagnetic field, including optical activity, dichroism, gyrotropy and weak absorption. Parameter σ in Eqs. (3) is an arc length along the reference ray, which can be curved and torsioned. In contradistinction to the original form of the QIA equations, written in the frame of natural trihedral coordinate system [1–5], Eqs. (3) do not contain torsion of the ray, because Popov's orthogonal system provides parallel, *i.e.*, torsionless, transport of the electrical intensity vector along the ray.

An isotropic component of permittivity can be chosen as $\varepsilon_0 = 1 - v$. Using expressions for the full electrical permittivity tensor for collisionless magnetized plasma [17] and subtracting isotropic part $\varepsilon_0 \sigma_{mn} = (1 - v)\sigma_{mn}$, the components of the anisotropy tensor v_{mn} can be presented as

$$\begin{cases} v_{11} = -uV\left(\sin^2\alpha_{||}\sin^2\alpha_{\perp} + \cos^2\alpha_{||}\right) \\ v_{12} = iV\sqrt{u}\,\cos\alpha_{||} + uV\sin^2\alpha_{||}\sin\alpha_{\perp}\cos\alpha_{\perp} \\ v_{21} = -iV\sqrt{u}\,\cos\alpha_{||} + uV\sin^2\alpha_{||}\sin\alpha_{\perp}\cos\alpha_{\perp} \\ v_{22} = -uV\left(\sin^2\alpha_{||}\cos^2\alpha_{\perp} + \cos^2\alpha_{||}\right) \end{cases} \tag{4}$$

where $v = \omega_p^2/\omega^2$ and $u = \omega_c^2/\omega^2$ are standard plasma parameters, ω_p and ω_c are plasma frequency and electron cyclotron frequency, correspondingly (absorption terms were taken into account in [14]). The $\alpha_{||}$ and $\alpha_{\perp}$ are polar and azimuthal angles, characterizing orientation of the magnetic field $\mathbf{B_0}$ in the orthogonal basis ($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{l}$), as shown in Fig. 1.

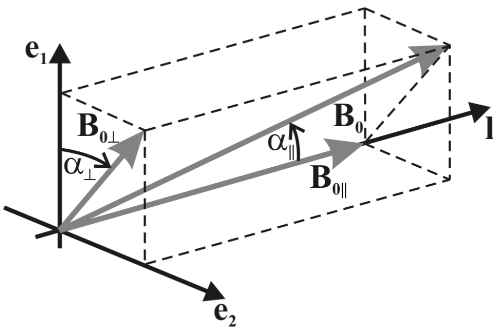

Fig. 1. Orientation of the static magnetic vector $\mathbf{B_0}$ in the orthogonal basis ($\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{l}$).

In what follows, we apply QIA approach for studying the polarization state of the electromagnetic wave in weakly anisotropic magnetized plasma. Substituting anisotropy tensor (4) into QIA equations (3), we can present them in the form:

$$\begin{aligned} \frac{\mathrm{d}\Gamma_1}{\mathrm{d}\sigma} &= -\frac{i}{2}\left(2\Omega_{\perp 0} - \Omega_\perp - \Omega_1\right)\Gamma_1 + \frac{1}{2}\left(i\Omega_2 - \Omega_3\right)\Gamma_2 \\ \frac{\mathrm{d}\Gamma_2}{\mathrm{d}\sigma} &= \frac{1}{2}\left(i\Omega_2 + \Omega_3\right)\Gamma_1 - \frac{i}{2}\left(2\Omega_{\perp 0} - \Omega_\perp + \Omega_1\right)\Gamma_2 \end{aligned} \tag{5}$$

Here, Ω_1, Ω_2, Ω_3 are plasma parameters, introduced by Segre (see review paper [8]):

$$\begin{cases} \Omega_1 = \dfrac{k_0 u V}{2\sqrt{1-v}}\sin^2\alpha_{||}\cos 2\alpha_\perp = \Omega_\perp\cos 2\alpha_\perp = \Omega_0\sin^2\alpha_{||}\cos 2\alpha_\perp \\ \Omega_2 = \dfrac{k_0 u V}{2\sqrt{1-v}}\sin^2\alpha_{||}\sin 2\alpha_\perp = \Omega_\perp\sin 2\alpha_\perp = \Omega_0\sin^2\alpha_{||}\sin 2\alpha_\perp \\ \Omega_3 = \dfrac{k_0\sqrt{u}\,V}{\sqrt{1-v}}\cos\alpha_{||} \end{cases} \tag{6}$$

Parameters Ω_1 and Ω_2 correspond to Cotton–Mouton effect, whereas Ω_3 describes contribution of Faraday effect. Besides, it is convenient to involve two more plasma parameters:

$$\begin{aligned} \Omega_\perp &= \Omega_\perp(\alpha_{||}) = \sqrt{\Omega_1^2 + \Omega_2^2} = \frac{uV}{2\sqrt{1-v}}\sin^2\alpha_{||} = \Omega_{\perp 0}\sin^2\alpha_{||} \\ \Omega_{\perp 0} &= \Omega_\perp\left(\alpha_{||} = \frac{\pi}{2}\right) = \frac{uV}{2\sqrt{1-v}} \end{aligned} \tag{7}$$

Parameter V in Eqs. (4) and (6) is defined as $V = v/(1-u) \approx v + uv$. Besides, e and m are electron charge and mass, respectively, N_e is the electron density, $\mathbf{B_0}$ is a vector of static magnetic field, $\alpha_\perp$ is an angle between the unit vector $\mathbf{e_1}$ of Popov's coordinate system and transverse with respect to the ray component $\mathbf{B_{0\perp}}$ of the magnetic field (Fig. 1). Axes ξ_1 and ξ_2 correspond here to the unit vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ of the curvilinear parallel transport system.

3. Independent normal modes in plasma

The QIA equations (5) allow analyzing polarization evolution along the ray both in homogeneous and in inhomogeneous plasma. A solution to Eqs. (5) for homogeneous plasma can be presented in the form of harmonic wave

$$\begin{aligned} \Gamma_1 &= A_1 \exp(iN\sigma) \\ \Gamma_2 &= A_2 \exp(iN\sigma) \end{aligned} \tag{8}$$

Substituting Eqs. (8) into Eqs. (5), one has

$$\begin{cases} i\left[N + \frac{1}{2}\left(2\Omega_0 - \Omega_\perp - \Omega_1\right)\right] A_1 - \frac{1}{2}\left(i\Omega_2 - \Omega_3\right) A_2 = 0 \\ \frac{1}{2}\left(i\Omega_2 + \Omega_3\right) A_1 - i\left[N + \frac{1}{2}\left(2\Omega_0 - \Omega_\perp + \Omega_1\right)\right] A_2 = 0 \end{cases} \tag{9}$$

Equalling the determinant of this system to zero, we obtain two propagation constants, corresponding to two normal electromagnetic modes in magnetized plasma:

$$\begin{aligned} N_\pm &= -\Omega_0 + \frac{1}{2}\left(\Omega_\perp \pm \sqrt{\Omega_\perp^2 + \Omega_3^2}\right) = -\Omega_0 + \frac{1}{2}(\Omega_\perp \pm \Omega) \\ \Omega &\equiv \sqrt{\Omega_\perp^2 + \Omega_3^2} \end{aligned} \tag{10}$$

Total propagation constants $N_\pm^{\text{tot}}$, uniting QIA constants (10) with common geometrical optics propagation constant $N_0 = k_0\sqrt{\varepsilon_0}$ will be

$$\begin{aligned} N_\pm^{\text{tot}} &= N_0 + N_\pm = N_0 - \Omega_0 + \frac{1}{2}(\Omega_\perp \pm \Omega) \\ N_0 &= k_0 n_0 = k_0\sqrt{\varepsilon_0} \end{aligned} \tag{11}$$

Subscripts ± correspond to signs of the square root in Eqs. (10).

The minus sign in Eqs. (11) refers to the "fast" wave, whose phase velocity $v_{ph-} = \omega/N_{-}^{tot}$ is greater than phase velocity of "slow" wave $v_{ph+} = \omega/N_{+}^{tot}$:

$$v_{ph-} = \frac{\omega}{N_0 - \Omega_0 + \frac{1}{2}(\Omega_{\perp} - \Omega)} > v_{ph+} = \frac{\omega}{N_0 - \Omega_0 + \frac{1}{2}(\Omega_{\perp} + \Omega)} \qquad (12)$$

The electrical vector of the fast wave rotates in direction coinciding with the direction of Larmor rotation of electrons.

From (9) we obtain two equivalent expressions for the complex amplitude ratio $\zeta = A_2/A_1$:

$$\zeta_{\pm} = \frac{-\Omega_1 \pm \Omega}{\Omega_2 + i\Omega_3} = \frac{-\Omega_2 + i\Omega_3}{-\Omega_1 \mp \Omega} \qquad (13)$$

Involving complex eigenvectors of unit length

$$\mathbf{e}_{\pm} = \frac{\mathbf{e_1} + \zeta_{\pm}\mathbf{e_2}}{\sqrt{1 + |\zeta_{\pm}|^2}} \equiv p_{\pm}(\mathbf{e_1} + \zeta_{\pm}\mathbf{e_2})$$

$$p_{\pm} = \frac{1}{\sqrt{1 + |\zeta_{\pm}|^2}} \qquad (14)$$

$$|\mathbf{e}_{\pm}| = 1$$

we may represent the wave field $\mathbf{E}$ as superposition of normal modes:

$$\mathbf{E} = a_{+}\mathbf{e}_{+}\exp(iN_{+}\sigma) + a_{-}\mathbf{e}_{-}\exp(iN_{-}\sigma) \qquad (15)$$

where a_{+} and a_{-} are initial amplitudes of normal modes.

Expression (15), which is valid for homogeneous plasma, can be readily generalized to independent normal modes in a weakly inhomogeneous plasma:

$$\mathbf{E} = a_{+}\mathbf{e}_{+}\exp\left(i\int N_{+}\,\mathrm{d}\sigma\right) + a_{-}\mathbf{e}_{-}\exp\left(i\int N_{-}\,\mathrm{d}\sigma\right) \qquad (16)$$

Here, phases of the normal modes are presented in the form of line integrals along the ray. It is these integrals that are used in plasma polarimetry. The inter-mode phase difference ΔS equals

$$\Delta S_{\text{single passage}} = \int_0^L (N_+ - N_-)\,\mathrm{d}\sigma = \int_0^L \Omega\,\mathrm{d}\sigma \tag{17}$$

where L is a full distance from the source to the receiver. The phase difference (17) characterizes parameters of polarization ellipse (aspect ratio and orientation). Representation (16) is applicable if the scale L_P, characterizing the rate of plasma parameter P changing, is large as compared with the "inter-mode beating length" [4, 5]. As a result, an approximation of independent normal modes is applicable under condition

$$L_P \sim \frac{P}{|\mathrm{d}P/\mathrm{d}\sigma|} \gg l_b \sim \frac{2\pi}{\Omega} \quad \text{or} \quad L_P \Omega \gg 2\pi \tag{18}$$

It follows from Eq. (17) that in the case of Cotton−Mouton effect, which plays the main role when $|\Omega_3| \ll |\Omega_\perp|$, the inter-mode phase difference equals

$$\Delta S_{\text{single passage}}^{\text{Cotton−Mouton}} = \int_0^L \Omega_\perp\,\mathrm{d}\sigma \approx \frac{1}{2}\int_0^L \frac{\omega_p^2 \omega_c^2}{c\,\omega^3}\,\mathrm{d}\sigma \tag{19}$$

Analogously, in the case of Faraday effect, which dominates, when $|\Omega_3| \gg |\Omega_\perp|$, the phase difference (17) describes doubled Faraday rotation angle, presented by the following line integral

$$\Delta S_{\text{single passage}}^{\text{Faraday}} = \int_0^L \Omega_3\,\mathrm{d}\sigma \approx \int_0^L \frac{\omega_p^2 \omega_c}{c\,\omega^2}\,\mathrm{d}\sigma \tag{20}$$

4. Double passage effects in magnetized plasma

Double passage regime is supposed to be used for plasma polarimetry in large thermonuclear projects like ITER and W-7X. Double passage scheme (Fig. 2) uses retro-reflector inside plasma reactor, which allows only one technological window to be explored as compared with two windows in a traditional single passage scheme.

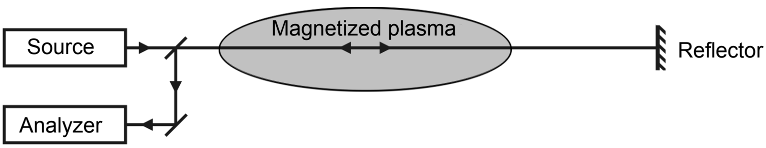

Fig. 2. Double passage scheme of polarimetric measurements.

It was shown in [10] that fast forward wave, incident on the metallic plane, generates the fast backward wave, and correspondingly slow-forward wave produces slow-backward wave. It means that the plane retro-reflector preserves the wave velocity. As a result, the phases of reflected normal waves become twice as much as compared with the single passage regime. The same is true for the phase difference:

$$\Delta S_{\text{double passage}} = 2\int_0^L (N_{f+} - N_{f-})\,\mathrm{d}\sigma_f = 2\int_0^L \Omega\,\mathrm{d}\sigma_f = 2\Delta S_{\text{single passage}} \tag{21}$$

This fact was known earlier for pure Faraday and pure Cotton−Mouton phenomena. Equation (21) generalizes these partial results for general case of arbitrary elliptical polarization.

In the case of Cotton−Mouton effect the phase difference

$$\Delta S^{\text{Cotton−Mouton}}_{\text{double passage}} = 2\int_0^L \Omega_\perp\,\mathrm{d}\sigma_f \approx \int_0^L \frac{\omega_p^2\omega_c^2}{c\,\omega^3}\,\mathrm{d}\sigma_f = 2\Delta S^{\text{Cotton−Mouton}}_{\text{single passage}} \tag{22}$$

can be extracted from polarization ellipse in the same manner as for single passage regime.

Analogously, measurements of Faraday rotation angle in the double passage regime

$$\Delta S^{\text{Faraday}}_{\text{double passage}} = 2\int_0^L \Omega_3\,\mathrm{d}\sigma_f \approx 2\int_0^L \frac{\omega_p^2\omega_c}{c\,\omega^2}\,\mathrm{d}\sigma_f = \Delta S^{\text{Faraday}}_{\text{single passage}} \tag{23}$$

are quite analogous to standard methods, used in the single passage scheme.

The results, characteristic of the plane metallic retro-reflector, hold also true for cubic corner retro-reflector (CCR). Contrary to the metallic plane and CCR, a 2D corner retro-reflector (2DCR) can convert the fast circular wave into slow one and *vice versa*. This kind of conversion can hardly be useful for plasma polarimetry, however, it might be of practical interest in polarimetric measurements, because it eliminates influence of Faraday effect [10].

5. Normal mode conversion near orthogonality point in magnetized plasma

Being written in circular basis $\mathbf{e}_{\mathbf{r},\mathbf{l}} = (\mathbf{e}_1 \pm i\mathbf{e}_2)/\sqrt{2}$, polarization vector becomes $\boldsymbol{\Gamma} = \Gamma_r\mathbf{e}_\mathbf{r} + \Gamma_l\mathbf{e}_\mathbf{l}$, so that QIA equations (5) take the form

$$\frac{d\Gamma_r}{d\sigma} = -\frac{i}{2}\left(\Omega_{\perp 0} + \Omega_3\right)\Gamma_r + \frac{i}{2}\Omega_{\perp 0}e^{-2i\alpha_\perp}\Gamma_l$$
$$\frac{d\Gamma_l}{d\sigma} = \frac{i}{2}\Omega_{\perp 0}e^{2i\alpha_\perp}\Gamma_r - \frac{i}{2}\left(\Omega_{\perp 0} - \Omega_3\right)\Gamma_l \tag{24}$$

Here, we have assumed that $\Omega_1 \pm i\Omega_2 = \Omega_{\perp 0}e^{\pm 2i\alpha_\perp}$.

Equations (24) can be noticeably simplified by involving new variables $\hat{\Gamma}_r$ and $\hat{\Gamma}_l$ such as

$$\hat{\Gamma}_r = \Gamma_r \exp\left(\frac{i\Omega_{\perp 0}\sigma}{2} + i\alpha_\perp\right)$$
$$\hat{\Gamma}_l = \Gamma_l \exp\left(\frac{i\Omega_{\perp 0}\sigma}{2} - i\alpha_\perp\right) \tag{25}$$

In these variables the system of equations (24) becomes

$$\frac{d\hat{\Gamma}_r}{d\sigma} = -\frac{i\Omega_3}{2}\hat{\Gamma}_r + \frac{i\Omega_{\perp 0}}{2}\hat{\Gamma}_l$$
$$\frac{d\hat{\Gamma}_l}{d\sigma} = -\frac{i\Omega_{\perp 0}}{2}\hat{\Gamma}_r + \frac{i\Omega_3}{2}\hat{\Gamma}_l \tag{26}$$

Following [2, 3, 6], we linearize $\cos\alpha_{||}$ in the vicinity of orthogonality point $\alpha_{||} = \pi/2$:

$$\cos\alpha_{||} \cong -\frac{\sigma}{\rho_B} \tag{27}$$

where parameter $\frac{1}{\rho_B} = \left.\frac{d\alpha_{||}}{d\sigma}\right|_{\alpha_{||} = \pi/2}$ characterizes here the rate of the angle $\alpha_{||}$ changes along the ray. Substituting (27) into parameter Ω_3, we have

$$\Omega_3 = 2G\cos\alpha_{||} \approx -\frac{2G\sigma}{\rho_B}$$
$$G = \frac{k_0 v\sqrt{u}}{2\sqrt{1-v}} \tag{28}$$

Involving dimensionless arc length $\xi = \sigma\sqrt{G/\rho}$ and dimensionless parameter $p = Gu\rho_B$, we can rewrite Eqs. (26) as

$$\frac{d\hat{\Gamma}_r}{d\xi} = i\xi\hat{\Gamma}_r + \frac{i\sqrt{p}}{2}\hat{\Gamma}_l$$
$$\frac{d\hat{\Gamma}_l}{d\xi} = \frac{i\sqrt{p}}{2}\hat{\Gamma}_r - i\xi\hat{\Gamma}_l \tag{29}$$

These equations coincide with equations (4.22) in book [3]. The solution of system (29) can be expressed through the functions of the parabolic cylinder $D_n(z)$, similarly to a solution used by LANDAU and LIFSHITS [18] in the problem of electron terms interaction under atomic collisions (Landau–Zener solution).

Assuming the incident wave to be of the right-hand polarization, we use the following initial conditions:

$$|\Gamma_r(-\infty)| = 1$$
$$|\Gamma_l(-\infty)| = 0 \tag{30}$$

Using then asymptotic formulae for parabolic cylinder functions at $z \to \infty$, we find the following intensities $|\Gamma|_r^2$ and $|\Gamma|_l^2$ at $\xi \to \infty$:

$$|\Gamma_r(+\infty)|^2 = |\hat{\Gamma}_r(+\infty)|^2 = e^{-\pi p/4}$$
$$|\Gamma_l(+\infty)|^2 = |\hat{\Gamma}_l(+\infty)|^2 = 1 - e^{-\pi p/4} \tag{31}$$

Accordingly, the coefficient of circular modes conversion $\Gamma_r \to \Gamma_l$ is given by

$$\eta = \frac{|\Gamma_l(\infty)|^2}{|\Gamma_r(\infty)|^2 + |\Gamma_l(\infty)|^2} = \frac{|\hat{\Gamma}_l(\infty)|^2}{|\hat{\Gamma}_r(\infty)|^2 + |\hat{\Gamma}_l(\infty)|^2} = 1 - \exp\left(-\frac{\pi p}{4}\right) \tag{32}$$

Interaction of normal modes takes place mainly within the narrow interval $-\sigma_{\rm int} < \sigma < \sigma_{\rm int}$, where $\sigma_{\rm int} \approx \rho_B\sqrt{u}$. Correspondingly, the interaction interval in dimensionless variable ξ acquires the form $-\xi_{\rm int} < \xi < \xi_{\rm int}$ with $\xi_{\rm int} \approx \sqrt{p}$. Beyond this interval interaction becomes weak enough and normal modes propagate practically independently. This theoretical conclusion can be confirmed by numerical solution of the system (29), performed in Ref. [19]. According to [19], conversion coefficient oscillates before achieving the steady state.

Contrary to traditional plasma diagnostics, dealing with averaging plasma parameters along the ray, Eq. (32) opens up the way for measurements of local plasma parameters, as was first suggested in paper [6]. Having measured the conversion coefficient η, one can determine interaction parameter $p = Gu\rho_B$, which is a combination of local plasma parameters, from the formula

$$p = Gu\rho_B = \frac{4}{\pi}\ln(1-\eta) \tag{33}$$

According to (28) parameter $p = Gu\rho_B$ is proportional to the electron density N_e and to the third power of the magnetic field:

$$p = Gu\rho_B \sim N_e B_0^3 \tag{34}$$

It is this product that can be determined from localized polarimetric measurements. Characteristic scale ρ_B can be determined from numerical modeling of the magnetic field configuration and therefore it can be considered to be known.

6. Complex polarization angle (CPA)

Tangent of CPA is defined as a ratio of complex amplitudes [1–5]: $\tan\gamma = \Gamma_2/\Gamma_1$. It was shown recently [11] that the components γ' and γ'' of CPA $\gamma = \gamma' + i\gamma''$ characterize the angular parameters ψ, χ of the polarization ellipse: real part γ' defines polarization angle, $\gamma' = \chi$ and imaginary part – its ellipticity: $\tanh\gamma'' = \tan\chi = e$.

It follows from QIA equations (5) that complex polarization angle γ in magnetized plasma satisfies the following equation:

$$\dot{\gamma} = \frac{1}{2}k_0 V u^{1/2}\cos\alpha_{||} - \frac{1}{4}ik_0 V u \sin^2\alpha_{||}\sin(2\gamma - 2\alpha_\perp) \tag{35}$$

Here, $\dot{\gamma} = \mathrm{d}\gamma/\mathrm{d}\sigma$ is a derivative along the ray. This equation can be presented also as [11]:

$$\dot{\gamma} = \frac{1}{2}\Omega_3 - \frac{1}{2}i\Omega_1\sin(2\gamma) + \frac{1}{2}i\Omega_2\cos(2\gamma) =$$

$$= \frac{1}{2}\Omega_3 - \frac{1}{2}i\Omega_\perp\sin(2\gamma - 2\alpha_\perp) \tag{36}$$

Equations (35) and (36) have proved to be very helpful for solution of various problems in plasma polarimetry.

7. Equations for evolution of traditional angular parameters of polarization ellipse

It follows from Sec. 6 that $\dot{\gamma}' = \dot{\psi}$, $\dot{\gamma}'' = \dot{\chi}/\cos(2\chi)$, $\tanh(2\gamma'') = \sin(2\chi)$, $\sinh(2\gamma'') = \tan(2\chi)$ and $\cosh(2\gamma'') = 1/\cos(2\chi)$. Substituting these relations into

Equations (36) one can obtain the following system of equations for the traditional angular parameters (ψ, χ) of polarization ellipse:

$$\begin{aligned}\dot{\psi} &= \frac{\Omega_3}{2} + \frac{\Omega_\perp}{2}\cos(2\psi - 2\alpha_\perp)\tan(2\chi)\\ \dot{\chi} &= -\frac{\Omega_\perp}{2}\sin(2\psi - 2\alpha_\perp)\end{aligned} \tag{37}$$

This system has not been known so far. It might be very helpful for plasma polarimetry.

8. Derivation of equation for Stokes vector from QIA

For monochromatic wave field the components of the full (four components) Stokes vector $\mathbf{s} = (s_0, s_1, s_2, s_3)$ are connected with the components of the polarization vector by relations [20]:

$$\begin{aligned} s_0 &= |\Gamma_1|^2 + |\Gamma_2|^2\\ s_1 &= |\Gamma_1|^2 - |\Gamma_2|^2\\ s_2 &= \Gamma_1\Gamma_2^* + \Gamma_1^*\Gamma_2\\ s_3 &= i\left(\Gamma_1^*\Gamma_2 - \Gamma_1\Gamma_2^*\right)\end{aligned} \tag{38}$$

According to Eqs. (38), the derivatives $\dot{s}_k = \mathrm{d}s_k/\mathrm{d}\sigma$ with respect to arc length σ are as follows:

$$\begin{cases} \dot{s}_0 = \dot{\Gamma}_1\Gamma_1^* + \Gamma_1\dot{\Gamma}_1^* + \dot{\Gamma}_2\Gamma_2^* + \Gamma_2\dot{\Gamma}_2^*\\ \dot{s}_1 = \dot{\Gamma}_1\Gamma_1^* + \Gamma_1\dot{\Gamma}_1^* - \dot{\Gamma}_2\Gamma_2^* - \Gamma_2\dot{\Gamma}_2^*\\ \dot{s}_2 = \dot{\Gamma}_1\Gamma_2^* + \Gamma_1\dot{\Gamma}_2^* + \dot{\Gamma}_1^*\Gamma_2 + \Gamma_1^*\dot{\Gamma}_2\\ \dot{s}_3 = i\left(\dot{\Gamma}_1^*\Gamma_2 + \Gamma_1^*\dot{\Gamma}_2 - \dot{\Gamma}_1\Gamma_2^* - \Gamma_1\dot{\Gamma}_2^*\right)\end{cases} \tag{39}$$

In the case of collisionless plasma the component s_0 preserves its value: $s_0 = \text{const}$. In this case, it is convenient to deal with a reduced (three component) Stokes vector [20]:

$$
\begin{aligned}
\mathbf{s} &= (s_1, s_2, s_3) \\
s_1 &= \cos(2\chi)\cos(2\psi) \\
s_2 &= \cos(2\chi)\sin(2\psi) \\
s_3 &= \sin(2\chi)
\end{aligned}
\tag{40}
$$

Here, ψ is the azimuth angle ($0 \le \psi \le \pi$) and $\tan\chi = b/a$ is the ellipticity, or aspect ratio, of polarization ellipse. Using derivatives $\Gamma_{1,2}$ from QIA equations (5), papers [13, 14] have proved that three-component Stokes vector $\mathbf{s} = (s_1, s_2, s_3)$ satisfies the equation:

$$
\dot{\mathbf{s}} = \mathbf{\Omega} \times \mathbf{s}
\tag{41}
$$

This equation forms a basis for Stokes vector formalism and is widely used in plasma polarimetry. It was originally derived in [8, 9] by analogy with the equations of crystal optics [6, 7]. Our derivation of Eq. (41) from QIA equations (5) demonstrates deep unity of two methods, though they are quite different in form: Stokes vector (38) is quadratic in wave field, whereas QIA equations are linear in wave field. From general point of view, Eq. (41) may be considered as Heisenberg form of QIA equations, the latter playing the role of Schrödinger equation.

9. Comparison of QIA and SVF approaches

Though Eq. (41) for Stokes vector stems directly from QIA equations (5), these equations cannot be considered as completely identical ones. To illustrate this, let us present polarization vector $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2) = (|\Gamma_1| e^{i\delta_1}, |\Gamma_2| e^{i\delta_2})$ as

$$
\mathbf{\Gamma} = e^{i(\delta_1 + \delta_2)/2} \left(|\Gamma_1| e^{i(\delta_1 - \delta_2)/2}, |\Gamma_2| e^{i(-\delta_1 + \delta_2)/2} \right)
\tag{42}
$$

and express four field parameters (two modules $|\Gamma_1|$, $|\Gamma_2|$ and two phases δ_1, δ_2) through the components s_0, s_1, s_2, s_3 of the Stokes vector $\mathbf{s}$. It follows from Eqs. (38) that

$$
\begin{aligned}
|\Gamma_1|^2 &= \frac{1}{2}(s_0 + s_1) \\
|\Gamma_2|^2 &= \frac{1}{2}(s_0 - s_1) \\
\tan(\delta_1 - \delta_2) &= -\frac{s_3}{s_2}
\end{aligned}
\tag{43}
$$

According to Eqs. (43), only three field parameters: two modules $|\Gamma_1|$, $|\Gamma_2|$ and phase difference $(\delta_1 - \delta_2)$, can be extracted from Stokes vector. It means that three equations (41) for three components of the Stokes vector are not completely equivalent to the QIA equations (5): the phase half sum $(\delta_1 + \delta_2)/2$ cannot principally be extracted from Stokes vector, so that polarization vector $\mathbf{\Gamma} = \Gamma_1 \mathbf{e_1} + \Gamma_2 \mathbf{e_2}$ can be restored from the Stokes vector only within the total phase uncertainty. The reason is that four components s_0, s_1, s_2 and s_3 and of the Stokes vector are connected by a relation $s_0^2 = s_1^2 + s_2^2 + s_3^2$ and cannot be considered as independent values.

However, the total phase $(\delta_1 + \delta_2)/2$ does not influence the shape of polarization ellipse: it determines only the starting point on polarization ellipse. Therefore, one can speak of practical equivalence of QIA and SVF.

10. Conclusions

Quasi-isotropic approximation of the geometrical optics method provides adequate description for evolution of the polarization vector in inhomogeneous weakly anisotropic plasma. QIA equations allow deriving an equation for the complex polarization angle as well as equations for angular parameters of polarization ellipse. Besides QIA equations allow us to substantiate Stokes vector formalism and to derive the equation for the Stokes vector evolution in a consequent way.

Acknowledgements – This work, supported by the European Communities under the contract of association between EURATOM and IPPLM (project P-12), was carried out within the framework of the European Fusion Development Agreement. This work is also supported by the Polish Ministry of Science and High Education (grant No. 202 249535).

References

[1] KRAVTSOV YU.A., *Quasi-isotropic geometrical optics approximation*, Soviet Physics – Doklady **13**(11), 1969, pp. 1125–1127.

[2] KRAVTSOV YU.A., NAIDA O.N., FUKI A.A., *Waves in weakly anisotropic 3D inhomogeneous media: Quasi-isotropic approximation of geometrical optics*, Physics–Uspekhi **39**(2), 1996, pp. 129–134.

[3] FUKI A.A., KRAVTSOV YU.A., NAIDA O.N., *Geometrical Optics of Weakly Anisotropic Media*, Gordon and Breach, London, New York, 1997.

[4] KRAVTSOV YU.A., ORLOV YU.I., *Geometrical Optics of Inhomogeneous Media*, Springer Verlag, Berlin, Heidelberg, 1990.

[5] KRAVTSOV YU.A., *Geometrical Optics in Engineering Physics*, Alpha Science, London, 2005.

[6] RAMACHANDRAN G.N., RAMASESHAN S., *Crystal Optics*, [In] *Encyclopedia of Physics*, Vol. 25/1, Springer, Berlin, 1961.

[7] AZZAM R.M.A., *Propagation of partially polarized light through anisotropic media with or without depolarization: A differential 4 × 4 matrix calculus*, Journal of the Optical Society of America **68**(12), 1978, pp. 1756–1767.

[8] SEGRE S.E., *A review of plasma polarimetry – theory and methods*, Plasma Physics and Controlled Fusion **41**(2), 1999, pp. R57–R100.

[9] SEGRE S.E., *New formalism for the analysis of polarization evolution for radiation in a weakly nonuniform, fully anisotropic medium: A magnetized plasma*, Journal of the Optical Society of America A **18**(10), 2001, pp. 2601–2606.
[10] KRAVTSOV YU.A., NAIDA O.N., *Linear transformation of electromagnetic waves in three--dimensional inhomogeneous magneto-active plasma*, Soviet Physics – JETP **44**(1), 1976, pp. 122–126.
[11] KRAVTSOV YU.A., BIEG B., *Double passage of electromagnetic waves through magnetized plasma: approximation of independent normal waves*, Central European Journal of Physics (in press).
[12] CZYZ Z.H., BIEG B., KRAVTSOV YU.A., *Complex polarization angle: Relation to traditional polarization parameters and application to microwave plasma polarimetry*, Physics Letters A **368**(1–2), 2007, pp. 101–107.
[13] BLIOKH K.YU., FROLOV D.YU., KRAVTSOV YU.A., *Non-Abelian evolution of electromagnetic waves in a weakly anisotropic inhomogeneous medium*, Physical Review A **75**(5), 2007, p. 053821.
[14] KRAVTSOV YU.A., BIEG B., BLIOKH K.YU., *Stokes-vector evolution in a weakly anisotropic inhomogeneous medium*, Journal of the Optical Society of America A **24**(10), 2007, pp. 3388–3396.
[15] POPOV M.M., *Vestnik Leningradskogo Universiteta (Bulletin of the Leningrad University)*, Vol. 22, 1969, pp. 44–54 (in Russian).
[16] BABICH V.M., BULDYREV V.S., *Short-Wavelength Diffraction Theory: Asymptotic Methods*, Springer Verlag, Berlin, 1990.
[17] GINZBURG V.I., *Propagation of Electromagnetic Waves in Plasma*, Gordon and Breach, New York, 1970.
[18] LANDAU L.D., LIFSHITS E.M., *Quantum Mechanics: Non-Relativistic Theory*, Pergamon Press, Oxford, 1977.
[19] KRAVTSOV YU.A., BIEG B., *Polarization transformations in the vicinity of orthogonality point in the inhomogeneous magnetized plasma*, Proceedings of SPIE **7141**, 2008, p. 71410K.
[20] BORN M., WOLF E., *Principles of Optics*, 7th Edition, University Press, Cambridge, 1999.

Received September 29, 2009