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Solitary wave solution to the nonlinear evolution equation in cascaded quadratic media beyond the slowly varying envelope approximations

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We report an exact bright and dark soliton solution to the nonlinear evolution equation derived by MOSES and WISE (Phys. Rev. Lett. **97**, 2006, 073903) for cascaded quadratic media beyond the slowly varying envelope approximations. The integrability aspects of the model are addressed. The travelling wave hypothesis as well as the ansatz method are employed to obtain an exact 1-soliton solution. Both bright and dark soliton solutions are obtained. The corresponding constraint conditions are obtained in order for the soliton solutions to exist.

Keywords: solitons, integrability, exact solution, cascaded quadratic media.

1. Introduction

Since the first experimental realization more than one decade ago, research in a few cycle optical pulses has grown rapidly in recent years [1, 2]. This tremendous boost is due to various reasons, primarily to richness in physics from both fundamental and possible applications of this point of view in many diverse areas such as ultrafast spectroscopy, metrology, medical diagnostics and imaging, optical communications, manipulation of chemical reactions and bond formation, material processing, *etc*. [3, 4]. Moreover, the field of light–matter interactions, high harmonic generation, extreme and single cycle nonlinear optics, and attosecond physics is also greatly influenced by the ultrashort pulses [5–9]. These pulses with duration of a few optical cycles are brief enough to resolve temporal dynamics on an atomic level like chemical reactions, molecular vibrations, and electron motion. They could be used for coherently exciting and controlling the matter on a microscopic level since they are very broadband and can become extremely intense [10]. In this context, in order to describe the dynamics and propagation of a few cycle optical pulses in nonlinear media, many authors looked for an appropriate mathematical model [11–13]. This is mainly owing to the fact that, because of the breakdown of the slowly varying envelope approximation (SVEA), the nonlinear Schrödinger equation (NLSE), which is routinely used as the governing equation for describing pulse propagation in media, is inadequate in a few cycle regimes [2, 11]. In this context, the first widely accepted model has been developed by BRABEC and KRAUSZ [11]. Non-SVEA-based mathematical models are also proposed by some authors [11, 12]. Recently, owing to the efficient manipulation of spectral and temporal properties of few-cycle pulses through cascaded processes in quadratic materials, both theoretical and experimental research is getting tremendous boost [14–17]. And in this context, following the model proposed by Brabec and Krausz, Moses and Wise have derived coupled propagation equations for ultrashort pulses in a degenerate three-wave mixing process in quadratic ($\chi^{(2)}$) media [18]. It may be noted that the Moses–Wise model is restricted to the case of strongly mismatched interaction where the conversion efficiency to second or higher harmonics is negligible. Moses and Wise, based on their model, presented theoretical and experimental evidence of a new quadratic effect, namely the controllable self-steepening (SS) effect. The controllability of the SS effect is very useful in nonlinear propagation of ultrashort pulses as it may be used to cancel the propagation effects of group velocity mismatch. Recently, a modulation instability (MI) analysis of the Moses–Wise model is reported [19]. It is shown that subject to the fulfillment of the MI criterion and the judicious choice of the parameters, MI could be generated in a cascaded-quadratic-cubic medium in both normal and anomalous dispersion regimes. As MI could be considered as a precursor to soliton formation, this clearly motivates us to look at the Moses–Wise model a bit more closely. This work will shine light on the integrability aspects of the model. Moreover we will report an exact 1-soliton solution to the propagation equation by using the travelling wave hypothesis [20] and the ansatz method [21]. Both bright and dark soliton solutions are discussed. It should be noted that both the travelling wave and the ansatz methods are used quite extensively in obtaining solitary wave solutions to various nonlinear evolution equations such as the Gross–Pitaeveskii equation [22], nonlinear Schrödinger equation with time dependent coefficients [23], higher order NLSE [24] and models related to propagation dynamics in proteins chains [25], *etc*.

The pulse propagation equation describing the ultrashort optical pulse propagation in quadratic nonlinear media beyond the slowly varying envelope approximation is given, in dimensionless units, as follows [19]:

$$i\frac{\partial u}{\partial z} + \alpha\frac{\partial^2 u}{\partial \tau^2} + \beta|u|^2 u = i\gamma u^2\frac{\partial u^*}{\partial \tau} + i\delta|u|^2\frac{\partial u}{\partial \tau} \tag{1}$$

where $u(z, \tau)$ represents the wave profile. The coefficients α and β refer to the group velocity dispersion (GVD) and the self-phase modulation (SPM), respectively. Also, z and τ are the spatial and temporal variables, respectively. The coefficient γ is the self-steepening (SS) term that is induced by the group velocity mismatch (GVM), while δ is the so-called controllable self-steepening term that originates from the slowly varying wave approximation [18]. This equation will be now solved by using two approaches, namely the travelling wave solution method and the so-called ansatz method. The first method will reveal a bright 1-soliton solution to Eq. (1) along with constraint conditions that need to hold for the soliton solution to exist. The later method will give both the bright and dark 1-soliton solution to Eq. (1) along with the necessary constraint conditions.

In order to integrate Eq. (1), by using the travelling wave solution method, we assume the following soliton solution with a permanent profile:

$$u(z, \tau) = g(z - v\tau)\exp\left[i(kz - \omega\tau + \theta)\right] \tag{2}$$

where $g(z - v\tau)$ is the amplitude portion while the remainder is the phase portion of the wave; k, ω, θ and v are respectively, the wave number, the frequency, the phase constant and the velocity of the wave. Putting Eq. (2) into Eq. (1) and then decomposing it into real and imaginary parts, we yield

$$\alpha v^2 g'' - (k + \alpha\omega^2)g + (\beta + \omega\gamma - \delta\omega)g^3 = 0 \tag{3}$$

and

$$1 + 2\omega v\alpha + (\gamma + \delta)vg^2 = 0 \tag{4}$$

where $g'' = \mathrm{d}^2 g/\mathrm{d}s^2$ with $s = z - v\tau$. From Eq. (4) we obtain the following relations:

$$v = -\frac{1}{2}\omega\alpha \quad \text{and} \quad \gamma + \delta = 0 \tag{5}$$

These two relations could be considered as the constrains conditions for the solitons to exist where the first one gives the velocity of the soliton while the later relates the GVM induced SS and the controllable SS parameter. Now from Eq. (3), we obtain the following solution:

$$g = \sqrt{\frac{2(k + \alpha\omega^2)}{\beta + \omega\gamma - \delta\omega}}\ \mathrm{sech}\left[s\sqrt{\frac{k + \alpha\omega^2}{\alpha v^2}}\right] \tag{6}$$

Hence, the bright 1-soliton solution of Eq. (1) is given by

$$u(z, \tau) = A\,\mathrm{sech}\left[B(z - v\tau)\right]\exp\left[i(kz - \omega\tau + \theta)\right] \tag{7}$$

where the amplitude of the soliton A and the inverse width B are given by:

$$A = \sqrt{\frac{2(k+\alpha\omega^2)}{\beta+\omega\gamma-\delta\omega}} \tag{7a}$$

$$B = \sqrt{\frac{k+\alpha\omega^2}{\alpha v^2}} \tag{7b}$$

The relations for the amplitude and the width of the soliton introduce the constraints $(k+\alpha\omega^2)(\beta+\omega\gamma-\delta\omega)>0$ and $(k+\alpha\omega^2)\alpha>0$. These constrains along with the ones mentioned earlier must remain valid in order for the bright soliton solution to exist.

We will now integrate Eq. (1) to obtain both the bright and dark 1-soliton solution by using the ansatz method. It is worthwhile to mention that, following the study of MI of the Moses–Wise model, with judicious choice of the GVM and the controllable SS parameter, one may obtain either bright or dark soliton in cascaded quadratic media. We assume,

$$u(z,\tau) = P(z,\tau)\exp\left[i\phi(z,\tau)\right] \tag{8}$$

where the amplitude and the phase component of the soliton are respectively given by $P(z,\tau)$ and $\phi(z,\tau)$, respectively. Here $\phi(z,\tau)=kz-\omega\tau+\theta$ as in the previous sub-section. Substituting (7) in (1) and then decomposing it into real and imaginary parts, we obtain:

$$-(k+\omega^2\alpha)P+(\beta+\omega\gamma-\delta\omega)P^3+\alpha\frac{\partial^2 P}{\partial\tau^2} = 0 \tag{9}$$

and

$$\frac{\partial P}{\partial z}-2\omega\alpha\frac{\partial P}{\partial\tau} = (\gamma+\delta)P^2\frac{\partial P}{\partial\tau} \tag{10}$$

Equation (10) leads to the same velocity of the soliton given by Eq. (5) and the same constraint condition as in (5). We will now further analyse Eq. (9) to obtain the bright and dark 1-soliton solution.

In order to obtain the bright soliton solution, let us assume,

$$P(z,\tau) = A\,\mathrm{sech}^p(\xi) \tag{11}$$

where A is the amplitude of the soliton and p is the unknown exponent. Also $\xi=B(z-v\tau)$, where B is the inverse width of the soliton and v is the velocity of the soliton. Thus, Eq. (9) simplifies to

$$\begin{aligned}&-(k+\omega^2\alpha)\mathrm{sech}^p(\xi)+(\beta+\omega\gamma-\delta\omega)A^2\mathrm{sech}^{3p}(\xi)+\\&\quad+\alpha B^2v^2p\left[p\,\mathrm{sech}^p(\xi)-(p+1)\mathrm{sech}^{p+2}(\xi)\right] = 0\end{aligned} \tag{12}$$

From Eq. (12), by the aid of a balancing principle, equating the exponents $3p$ and $(p + 2)$ leads to $p = 1$. Then, setting the coefficients of the linearly independent functions $\operatorname{sech}^{p+j}$ (for $j = 0, 2$) to zero leads to:

$$k = \alpha(B^2 v^2 - \omega^2) \tag{13}$$

$$B = \sqrt{\frac{\beta + \omega(\gamma - \delta)}{2\alpha v^2}}\, A \tag{14}$$

which are the relations for the wave number and the amplitude-width. The second relation given by Eq. (14) introduces the constraint condition $\alpha(\beta + \omega\gamma - \delta\omega) > 0$. Thus the bright 1-soliton solution to Eq. (1) as obtained by the ansatz method is given by Eq. (11), where the amplitude-width relation is given by Eq. (14) and the wave number is given by Eq. (13). The velocity of the soliton and the constraint condition are still given by Eq. (5). This method introduces a new constraint condition: $\alpha(\beta + \omega\gamma - \delta\omega) > 0$ as a consequence of Eq. (14).

Now let us look for the dark-soliton solution to Eq. (1). In order to obtain the dark 1-soliton solution to Eq. (1), we assume

$$P(z, \tau) = A \tanh^p(\xi) \tag{15}$$

In this case, the parameters A and B are referred to as the free parameters, while remaining parameters have the same interpretation. Adopting the similar procedure as above, we obtain:

$$k = -\alpha(\omega^2 + 2B^2 v^2) \tag{16}$$

$$B = \sqrt{-\frac{\beta + \omega\gamma - \omega\delta}{2\alpha v^2}}\, A \tag{17}$$

Equation (17) shows that the constraint condition in this case is given by $\alpha(\beta + \omega\gamma - \delta\omega) < 0$. It is interesting to note that this condition is opposite to that of the bright soliton case. Hence the dark 1-soliton solution is given by

$$u(z, \tau) = A \tanh\left[B(z - v\tau)\right] \exp(kz - \omega\tau + \theta)$$

The velocity is again given by Eq. (5). It is worthwhile to note that the constrains seem to suggest that one could reverse the standard conditions for bright and dark solitons, $\alpha\beta < 0$ and $\alpha\beta > 0$, for the standard NLSE. In passing, it should be noted that the travelling wave solution is employed to find the bright soliton solution while the ansatz method is employed to find both the bright and dark soliton solution.

2. Results

In order to check the stability of the both bright and dark soliton solution, we solve Eq. (1) by the so-called split-step Fourier method [26]. The parameters are chosen such

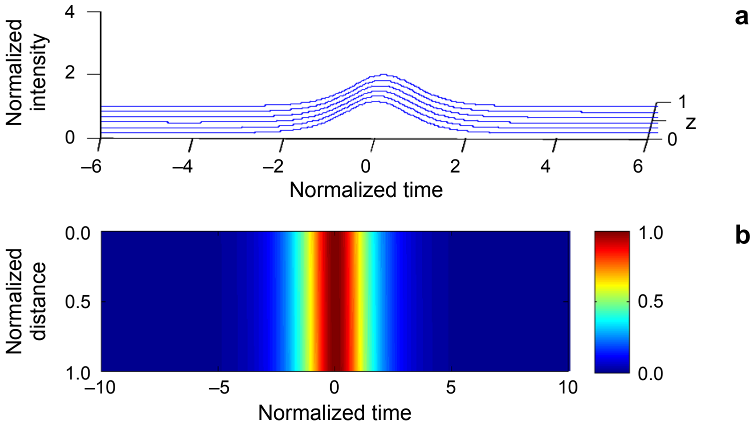

Fig. 1. Spatio-temporal evolution of bright-soliton propagation (**a**), and the corresponding contour plot (**b**). The parameters taken are: $\alpha = 1$, $\beta = 1$, $\gamma = -0.02$, $\delta = 0.02$ with $A = 1$ and $k = 0.5$.

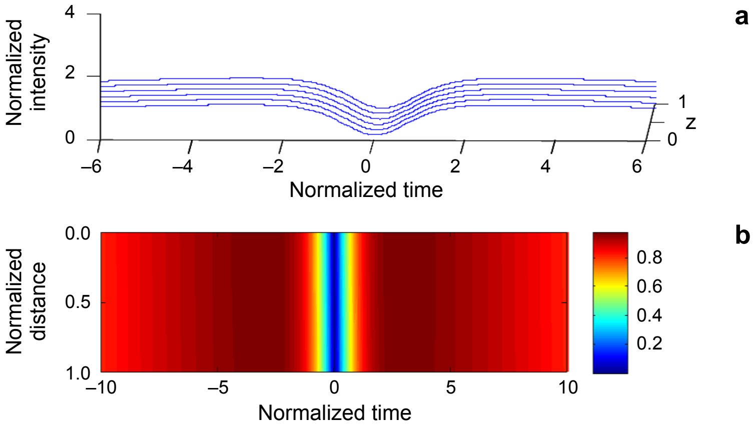

Fig. 2 Spatio-temporal evolution of dark-soliton propagation (**a**), and the corresponding contour plot (**b**). The parameters taken are: $\alpha = 1$, $\beta = -1$, $\gamma = -0.02$, $\delta = 0.02$ with $A = 1$ and $k = 0.5$.

that the respective constraints conditions are satisfied. In Fig. 1 we depict the spatio -temporal evolution and the corresponding contour plots of the bright 1-soliton propagation. Figure 2 depicts the contour plot of the dark 1-soliton propagation through the cascaded quadratic medium.

It could be observed that for the chosen parameters, the propagation of both bright and dark soliton is relatively stable. In fact, our numerical exploration shows that, both the bright and dark solitons are stable against their slight variation in amplitude during their propagation through the media. It should be noted that in the context of the usual NLSE, without any higher order terms, a soliton is stable as long as $0.5 \le A \le 1.5$ [26], derived from the inverse scattering transform method. In the context of cascaded nonlinearity, the nonlinear evolution equation gets quite complicated. So we cannot claim that the same condition applies rigorously here also. However, if the perturbations are weak, there may be a slight change only in the above condition. One can easily study the impact of various parameters on soliton propagation and draw useful conclusions, however in this report our objective is to report the soliton solution and the constraints conditions under which the bright or dark soliton may exist in a cascaded quadratic medium. This study may enhance the interest in exploring soliton phenomena in cascaded quadratic nonlinear media both theoretically and experimentally.

3. Conclusion

To conclude, we have reported an exact bright and dark one-soliton solution to the nonlinear evolution equation derived by Moses and Wise for cascaded quadratic media beyond the slowly varying envelope approximations. The corresponding constraint conditions are obtained in order for the soliton solutions to exist. Numerical simulation indicates stability of the solitons subject to the judicious choice of parameters.

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References

[1] Goulielmakis E., Uiberacker M., Kienberger R., Baltuska A., Yakovlev V., Scrinzi A., Westerwalbesloh Th., Kleineberg U., Heinzmann U., Drescher M., Krausz F., *Direct measurement of light waves*, Science **305**(5688), 2004, pp. 1267–1269.
[2] Brabec T., Krausz F., *Intense few-cycle laser fields: Frontiers of nonlinear optics*, Reviews of Modern Physics **72**(2), 2000, pp. 545–592.
[3] Wegener M., *Extreme Nonlinear Optics*, Springer-Verlag, Berlin, 2005.
[4] Brabec T., *Strong Field Laser Physics*, Springer, Berlin, 2008.
[5] Kumar P., Sarma A.K., *Optical force on two-level atoms by few-cycle-pulse Gaussian laser fields beyond the rotating-wave approximation*, Physical Review A **84**(4), 2011, article 043402.
[6] Kumar P., Sarma A.K., *Gaussian and sinc-shaped few-cycle-pulse-driven ultrafast coherent population transfer in Λ-like atomic systems*, Physical Review A **85**(4), 2012, article 043417.
[7] Kumar P., Sarma A.K., *Optical dipole force on ladderlike three-level atomic systems induced by few-cycle-pulse laser fields*, Physical Review A **86**(5), 2012, article 053414.
[8] Paul P.M., Toma E.S., Breger P., Mullot G., Augé F., Balcou Ph., Muller H.G., Agostini P., *Observation of a train of attosecond pulses from high harmonic generation*, Science **292**(5522), 2001, pp. 1689–1692.
[9] Chang J., *Fundamentals of Attosecond Optics*, CRC, Boca Raton, 2011.
[10] Krausz F., Ivanov M., *Attosecond physics*, Reviews of Modern Physics **81**(1), 2009, pp. 163–234.
[11] Brabec T., Krausz F., *Nonlinear optical pulse propagation in the single-cycle regime*, Physical Review Letters **78**(17), 1997, pp. 3282–3285.
[12] Gaeta A.L., *Catastrophic collapse of ultrashort pulses*, Physical Review Letters **84**(16), 2000, pp. 3582–3585.

[13] LEBLOND H., MIHALACHE D., *Few-optical-cycle solitons: modified Korteweg–de Vries sine-Gordon equation versus other non–slowly-varying-envelope-approximation models*, Physical Review A **79**(6), 2009, article 063835.
[14] MARANGONI M., SANNA G., BRIDA D., CONFORTI M., CIRMI G., MANZONI C., BARONIO F., BASSI P., DE ANGELIS C., CERULLO G., *Observation of spectral drift in engineered quadratic nonlinear media*, Applied Physics Letters **93**(2), 2008, article 021107.
[15] BACHE M., BANG O., ZHOU B.B., MOSES J., WISE F.W., *Optical Cherenkov radiation in ultrafast cascaded second-harmonic generation*, Physical Review A **82**(6), 2010, article 063806.
[16] CONFORTI M., BARONIO F., DE ANGELIS C., *Nonlinear envelope equation for broadband optical pulses in quadratic media*, Physical Review A **81**(5), 2010, article 053841.
[17] BACHE M., BANG O., MOSES J., WISE F.W., *Nonlocal explanation of stationary and nonstationary regimes in cascaded soliton pulse compression*, Optics Letters **32**(17), 2007, pp. 2490–2492.
[18] MOSES J., WISE F.W., *Controllable self-steepening of ultrashort pulses in quadratic nonlinear media*, Physical Review Letters **97**(7), 2006, article 073903.
[19] SARMA A.K., KUMAR P., *Modulation instability of ultrashort pulses in quadratic nonlinear media beyond the slowly varying envelope approximation*, Applied Physics B **106**(2), 2012, pp. 289–293.
[20] WEI-PING ZHONG, BELIĆ M., *Traveling wave and soliton solutions of coupled nonlinear Schrödinger equations with harmonic potential and variable coefficients*, Physical Review E **82**(4), 2010, article 047601.
[21] RUIYU HAO, LU LI, ZHONGHAO LI, GUOSHENG ZHOU, *Exact multisoliton solutions of the higher-order nonlinear Schrödinger equation with variable coefficients*, Physical Review E **70**(6), 2004, article 066603.
[22] PETROVIĆ N.Z., BELIĆ M., WEI-PING ZHONG, *Spatiotemporal wave and soliton solutions to the generalized (3+1)-dimensional Gross–Pitaevskii equation*, Physical Review E **81**(1), 2010, article 016610.
[23] SAHA M., SARMA A.K., BISWAS A., *Dark optical solitons in power law media with time-dependent coefficients*, Physics Letters A **373**(48), 2009, pp. 4438–4441.
[24] BELIĆ M., PETROVIĆ N., WEI-PING ZHONG, RUI-HUA XIE, GOONG CHEN, *Analytical light bullet solutions to the generalized (3+1)-dimensional nonlinear Schrödinger equation*, Physical Review Letters **101**(12), 2008, article 123904.
[25] CISNEROS-AKE L.A., MINZONI A.A., *Effect of hydrogen bond anharmonicity on supersonic discrete Davydov soliton propagation*, Physical Review E **85**(2), 2012, article 021925.
[26] AGRAWAL G.P., *Nonlinear Fiber Optics*, 4th Ed., Academic Press, San Diego, 2007.

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