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Super-resolution in linear systems with noise***

The two-point super-resolution for partially coherent light is discussed by using the Hermite polynomials corresponding to the use of analytic continuation with the Taylor power series cut by a Gaussian function even if considerable level of noise is present (up to 20%). For obtaining derivatives of the Fourier transform of the object function measured with error a recently proposed sampling formula by Ferwerda and Hoenders is adopted.

1. Introduction

Two-point resolution criteria have been examined recently for imaging with partially coherent light ([1–3] and references quoted therein). Although no special attention was payed to the possibility of reaching the super-resolution with partially coherent light, nevertheless this question has been discussed by a number of authors [4–7] for imaging with coherent or incoherent light. Some limitations for analytic continuation have been obtained at the presence of a level of noise [8] in particular, for the objects with greater number of degrees of freedom.

While the methods proposed in [5, 6] are based on the use of prolate spheroidal functions (also with additional noise), the paper [7] makes use of the orthogonal (Laguerre, Hermite, or Legendre) polynominals, in a way similar to that employed in reformulation of quantum optical eqivalence theorem [9]. The last method is based on the use of analytic continuation for the Fourier transform $\tilde{f}(\mu)$ of the object function $f(x)$, obtained only in a finite interval determined by the system. All further details are explained in [7]. An interesting result has been obtained in [10], i. e. that the reconstruction series in terms of the Hermite polynominals H_n corresponds to the Gaussian cutting of powers in the Taylor series for $\tilde{f}(\mu)$ used for the analytic continuation. The author [10] suggests also a complex basis leading to a decomposition theorem in terms of the Laguerre polynominals, if a Gaussian cutting is carried out in the complex region. This paper contains, also expressions with modifications in the scale base.

The purpose of the present paper is to demonstrate this method applying it to the resolving of two Gaussian peaks (instead of considering a sum of δ-functions which are not square integrable) whose tails decrease rapidly so that their sum can be considered as non-zero in a finite interval only (it can be considered as a function with a finite support). No wonder that analytic continuation can provide again the linear system with lost information, and that the super-resolution is obtained, if there are no errors and noise in the system. We show however that super-resolution is possible also in case of a rather high level of noise and errors (up to 20%). In order to obtain the derivatives of such an inaccurate function we use a sampling theorem, proved recently in [11], which is the key tool of our procedure.

2. Theory

We assume that the object function $f(x)$ and its image function $g(x)$ are connected by the usual convolution integral

$$g(x') = \int_{-\infty}^{+\infty} h(x'-x)f(x)\,dx; \tag{1}$$

g and f are amplitudes for imaging with coherent light and h is the diffraction function; if the

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imaging with incoherent light is performed, then g and f represent the intensities and h is the squared modulus of the diffraction function.

Assuming for simplicity the slit function

$$h(x) = \frac{\sin x}{\pi x},$$

and coherent light, we have for the Fourier transform

$$\tilde{h}(\mu) = \int_{-\infty}^{+\infty} h(x) \exp(i\mu x)\, dx = \begin{cases} 1, & |\mu| < 1, \\ 0, & |\mu| > 1. \end{cases} \quad (2)$$

Consequently, all spatial frequencies $|\mu| > 1$ are filtered out in the image ($\tilde{g}(\mu) = \tilde{h}(\mu)\tilde{f}(\mu)$, where $\tilde{f}$ and $\tilde{g}$ are the Fourier transforms of f and g, respectively). The object function is given by the formula

$$f(x) = \frac{1}{2(2\pi)^{1/2} a} \left\{ \exp\left[-\frac{(x-b)^2}{2a^2} \right] + \exp\left[-\frac{(x+b)^2}{2a^2} \right] \right\} \quad (3)$$

with the Fourier transform

$$\tilde{f}(\mu) = \cos \mu b \exp\left(-\frac{\mu^2 a^2}{2} \right), \quad (4)$$

where a is the standard deviation. Thus 2 is the distance between the peaks, and for $a \to 0$

$$f(x) = \frac{\delta(x-b) + \delta(x+b)}{2}$$

and $\tilde{f}(\mu) = \cos \mu b$. The derivatives at $\mu = 0$ are

$$\tilde{f}^{(2j)}(0) = \left(\frac{a}{\sqrt{2}} \right)^{2j} H_{2j} \left(\frac{ib}{\sqrt{2}a} \right), \quad (5)$$

$$\tilde{f}^{(2j+1)}(0) = 0, \; j = 0, 1, \ldots, .$$

Fort the reconstruction of the object function $f(x)$ we have used the formula [7, 10]

$$f(x) = \exp\left(-i\mu_0 x - \frac{x^2}{2\sigma^2} \right) \sum_{n=0}^{\infty} c_n H_n \left(\frac{x}{\sqrt{2}\sigma} \right), \quad (6a)$$

where

$$c_n = \frac{(-i)^n}{\sqrt{2\pi}\sigma} \sum_{j=0}^{[n/2]} \frac{\tilde{f}^{(n-2j)}(\mu_0)}{j!(n-2j)!2^{2j}(\sqrt{2}\sigma)^{n-2j}}, \quad (6b)$$

$$\left[\frac{n}{2} \right] = \frac{n}{2}$$

if n is even, and $(n-1)2$ if n is odd, μ_0 is a point from which the analytic continuation starts, and σ is a scale parameter used to regulation of convergence.

In general, (6a) is a generalized function; it is however, a square integrable function if

$$\int_{-\infty}^{+\infty} |f(x)|^2 d\mu(x) = \sigma\sqrt{2\pi} \sum_{n=0}^{\infty} 2^n n!\, |c_n|^2 < \infty, \quad (7)$$

where

$$d\mu(x) = \exp\left(\frac{x^2}{2\sigma^2} \right) dx,$$

and the orthogonality condition for the Hermite polynomials,

$$\int_{-\infty}^{+\infty} H_m \left(\frac{x}{\sqrt{2}\sigma} \right) H_n \left(\frac{x}{\sqrt{2}\sigma} \right) \exp\left(-\frac{x^2}{2\sigma^2} \right) \frac{dx}{\sqrt{2}\sigma} = \delta_{mn} \sqrt{\pi} 2^n n! \quad (8)$$

is used.

Separating the real and imaginary parts in (6a, b) we get

$$\begin{aligned} f(x) = {} & \frac{\exp\left(-\frac{x^2}{2\sigma^2} \right)}{\sqrt{2\pi}\sigma} \times \\ & \times \left\{ \cos \mu_0 x \sum_{n=0}^{\infty} (-1)^n a_n H_{2n} \left(\frac{x}{\sqrt{2}\sigma} \right) - \right. \\ & - \sin \mu_0 x \sum_{n=0}^{\infty} (-1)^n b_n H_{2n+1} \left(\frac{x}{\sqrt{2}\sigma} \right) - \\ & - i \left[\sin \mu_0 x \sum_{n=0}^{\infty} (-1)^n a_n H_{2n} \left(\frac{x}{\sqrt{2}\sigma} \right) + \right. \\ & \left. \left. + \cos \mu_0 x \sum_{n=0}^{\infty} (-1)^n b_n H_{2n+1} \left(\frac{x}{\sqrt{2}\sigma} \right) \right] \right\}, \end{aligned} \quad (9a)$$

where

$$a_n = \sum_{j=0}^{n} \frac{\tilde{f}^{(2n-2j)}(\mu_0)}{j!(2n-2j)!\, 2^{2j} (\sqrt{2}\sigma)^{2n-2j}}, \quad (9b)$$

and b_n are obtained from a_n by substituting $2n \to 2n+1$. These formulae are suitable for numerical calculations.

Derivatives $\tilde{f}^{(j)}(\mu_0)$ have been computed with the help of the sampling formula derived by Ferwerda and Hoenders [11]. This formula is useful as far as derivatives of an inaccurate function are to be obtained. After a slight modification it has the following form

$$\tilde{f}^{(j)}_{\text{approx}}(\mu_0) \approx j! \sum_{l=1}^{p} \tilde{f}(\mu_l) \sum_{k=j+1}^{p} \times \\ \times \frac{B_{p-k}(\mu_l-\mu_0)^{k-j-1}}{\prod_{\substack{m=1\\m\neq l}}^{p}(\mu_l-\mu_m)}, \qquad (10a)$$

where

$$B_0 = 1,$$

$$B_1 = -\sum_{l=1}^{p}(\mu_l-\mu_0),$$

$$B_k = -\frac{1}{k}(s_k+s_{k-1}B_1+s_{k-2}B_2+\ldots+s_1B_{k-1}), \\ k = 1,2,\ldots,p, \qquad (10b)$$

$$s_k = \sum_{l=1}^{p}(\mu_l-\mu_0)^k$$

and $\mu_l \neq \mu_0$ are arbitrary points in the support of $\tilde{f}(\mu)$, i.e. in our case $\langle -1, +1 \rangle$, and p is the number of these sampling points. The accuracy of this formula is given by the estimate

$$\frac{1}{j!}\left|\tilde{f}^{(j)}(\mu_0)-\tilde{f}^{(j)}_{\text{approx}}(\mu_0)\right| \leqslant \frac{\xi}{(\xi-1)^{j+1}} \frac{\underset{C}{\mathrm{Max}}\,|f(\mu)|}{|\mu_0|^j} \times \\ \times\left(\frac{|\mu_0|+a}{\xi|\mu_0|-a}\right)^p \sum_{l=0}^{j}\frac{p!}{l!(p-l)!}\left[\frac{|\mu_0|(\xi+1)}{|\mu_0|+a}\right]^l \qquad (10c)$$

where a is the cut-off frequency, C represents the circle $|\mu| \leqslant \xi|\mu_0|$, and ξ is chosen in such a way that C contains all μ_l; in our case $a = 1$. In particular we have chosen μ_l equidistantly and put $\mu_0 = 0$. In this case only the first term in (9a) is appropriate (cf. also (5) from which $b_n = 0$). In (10c) $\xi \to \infty$ to have $\xi\,|\mu_0|$ finite.

Further we have simulated the noise by inserting

$$\tilde{f}(\mu)\left(1 \pm \frac{\varepsilon}{100}\right)$$

into (10a), where ε denotes the percentage error and the $+$ and $-$ signs have been chosen at random by a noise generator at various points μ_l.

The image function is given by

$$g(x') = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{h}(\mu)\tilde{f}(\mu)\exp(-i\mu x)\,d\mu \\ = \frac{1}{\pi}\int_{0}^{1}\exp\left(-\frac{\mu^2a^2}{2}\right)\cos\mu b\cos\mu x'd\mu, \qquad (11a)$$

as follows from (4) and (2).

By substituting (3) into (1) and taking account of

$$h(x) = \frac{\sin x}{\pi x}$$

we get an equivalent expression

$$g(x') = \frac{1}{2\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\sum_{j=0}^{k}\frac{\left(\frac{a^2}{2}\right)^j}{j!(2k-2j)!}\times \\ \times[(x'-b)^{2(k-j)}+(x'+b)^{2(k-j)}]. \qquad (11b)$$

The half-distance b_0 which can still be resolved classicaly is given by the equation

$$\left.\frac{d^2g(x')}{dx'^2}\right|_{x'=0} = \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{2m+3}\sum_{j=0}^{m}\times \\ \times\frac{\left(\frac{a^2}{2}\right)^j b^{2(m-j)}}{j!(2m-2j)!} = 0, \qquad (12a)$$

which provides for $a = 0$ [12]

$$\tan b = \frac{2b}{2-b^2}. \qquad (12b)$$

Cases for $b > b_0$ are resolved classically, while those with $b \leqslant b_0$ correspond to the super-resolutions.

While computing the derivatives $\tilde{f}^{(j)}(\mu_0)$, the difference between results obtained from (5) and (10a), with $p = 40$, is less than $2\,\%$ for $j < 10$, but it quickly increases for $j \geqslant 10$. To reconstruct $f(x)$ about 15 terms of the reconstructing series (9a) have been used. The scale parameter σ suitably chosen made it possible to neglect the terms with $(2n-2j) \geqslant 10$ in (9b).

3. Discussion of numerical results

When solving (12a) we have found the limit b_0 for the classical resolution given in table 1. It can be seen that its value is about 2.1 and

Table 1

a	b_0
0.0	2.0815760
0.1	2.0822621
0.3	2.0877852
0.4	2.0926683
0.5	2.0990151
$1/2^{1/2}$	2.1170563

slightly depends on a for the values shown. Small increase of b_0 with a is comprehensible.

In figs 1–3 the object function $f(x)$ is denoted by broken lines for $a = 0.3$ in (3), the image function $g(x)$ given in (11a, b) (in which the spatial frequencies $|\mu| > 1$ are filtered) is given by dotted lines. The object function $f(x)$ reconstructed with the help of (9a, b) and (10a, b) with $\sigma = 5$, are denoted by full lines for $b = 2$ (fig. 1), $b = 1.8$ (fig. 2) and $b = 1.4$ (fig. 3) (in all these cases $b < b_0 = 2.088$) if randomly generated errors 0, 1, 5, 10 and 20 $^0/_0$ are introduced. Only one half of lines is shown (they are symmetrical with respect to the line $x = 0$). We see that the reconstruction procedure operates up to the errors amounting to 20 $^0/_0$ for $b = 2$ and to 5 $^0/_0$ for $b = 1.8$ and 1.4. Some reversion in the sequence of the lines with respect to ε occurring in figs. 1 and 2 is due to the fact that in various cases the signs were randomly generated in the noise generator. It should be noticed that except for the values of ε an additional error, arising from (10a), and estimated in (10c) is introduced. In the lines shown for $\varepsilon = 0$ only this error is involved. In a determined value of b this error is about 10 $^0/_0$, in some cases smaller.

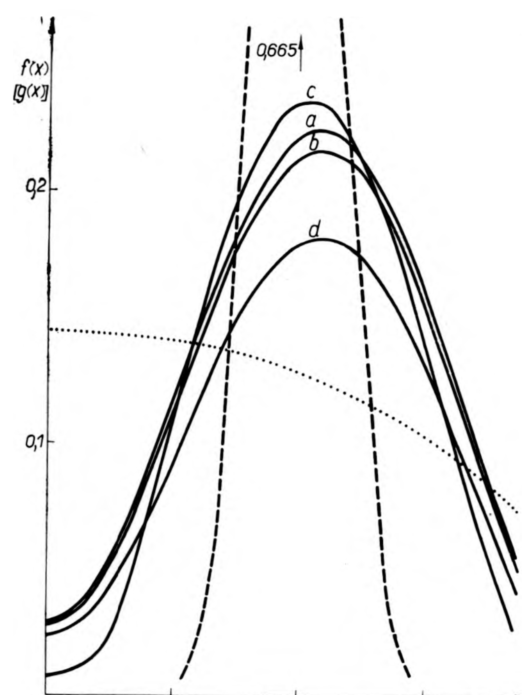

Fig. 1. The object function $f(x)$ for $b = 2$ and $a = 0.3$ (broken curve), the image function $g(x)$ (dotted curve), and the reconstructed object function $f(x)$ (full curves) for the errors $\varepsilon = 1^0/_0$ (curve a), $5^0/_0$ (curve b), $10^0/_0$ (curve c) and $20^0/_0$ (curve d)

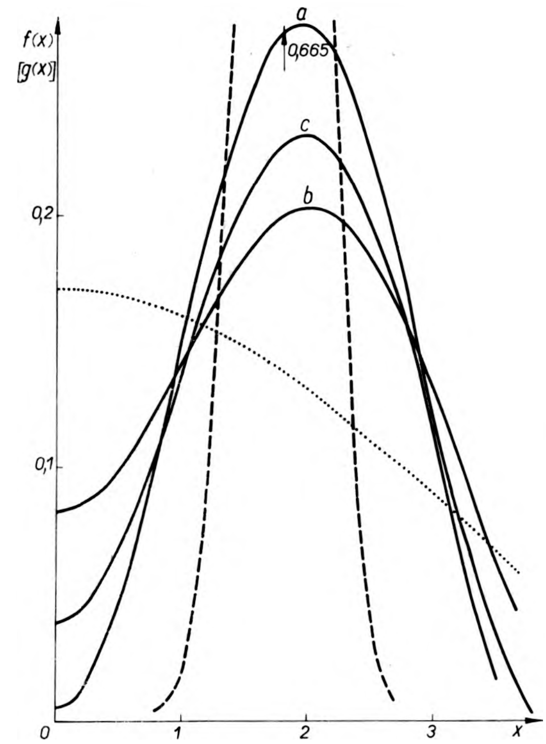

Fig. 2. The same as in fig. 1 for $b = 1.8$ (full curves); a, b and c are for $\varepsilon = 0.1$ and $5^0/_0$ respectively

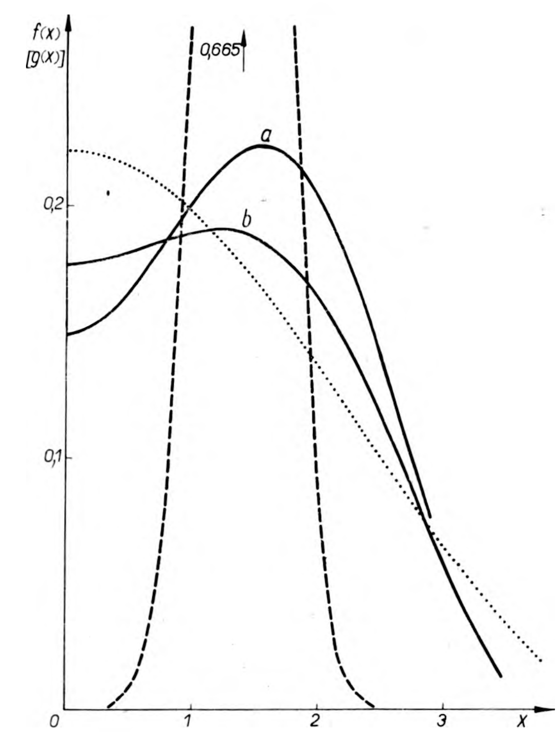

Fig. 3. The same as in fig. 1 for $b = 1.4$ (full curves); a and b are for $\varepsilon = 0$ and $5^0/_0$ respectively

Thus we have demonstrated that for finite objects having rather small number of degrees of freedom the object function can be well reconstructed by the method proposed in [7], and with the sampling formula for obtaining derivatives of an inaccurate function derived in [11] even if rather high level of noise is present.

Сверхразрешение в линейных системах с шумом

Анализировалось двухточечнсе Сверхразрешение для света частично когерентного при помощи полиномов Гермита спссобом, ссответствующим применению степенных рядов Тайлора, с сбрезксй при помощи функции Гауса. В анализе допускается присутствие значительных шумов, доходящих до 20%. Для получения производных трансформации Фурье предметной функции, измеряемой с погрешностью, применили фсрмулу, предложенную в последнее время Фервердсм и Гендерссм.

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