Optica Applicata, Vol. XXII, No. 3–4, 1992

Calculation of the form of stationarity region for speckle refractometry of the eye

P. Kašpar, K. Tomančák

Department of Optics, Natural Science Faculty, Palacký University, 771 46 Olomouc, Czech and Slovak Federal Republic.

In this paper, the influence of the position of a coherent source with respect to an illuminated rough surface in the measurement of the refractive state of the eye with the aid of speckle pattern on the form and the position of the so-called region of stationarity is considered. This problem is connected with the design of a compact measuring apparatus for refractometry.

1. Introduction

If a diffused reflective rough surface is illuminated by a coherent beam, the observer of the surface gets an impression of a granular structure, since the light intensity changes from point to point. This phenomenon is known as the speckle pattern [1], which is due to the interference produced by coherent light waves which are backscattered from the illuminated area of the rough surface. This interference pattern may be observed either as real images at many different distances in front of the rough surface or as virtual images in many planes behind the surface. Thus, the images are created in all possible planes and may be conjugated with the retinas of all possible refractive abilities. If the scattered radiation passes through an optical system, including that of the eye, and this image is recorded, we call it the image speckle pattern. The appearance of speckle is not limited to imagery formed with reflected light. If a photographic transparency is illuminated through a diffuser, then in the image we again find large fluctuations of irradiance caused by the overlapping of a multitude of dephased amplitude spread functions.

The essence of the refraction measurement by using the image speckle pattern is that the patient observes the illuminated rough surface from a distance of about 6 m. If the subject eye does not move and the surface rotates at a constant velocity, then the direction of the speckle movement depends on the kind of refraction error of the eye [2]. If the patient's eye is hypermetropic, then he reports the direction of the speckle movement opposite to that of the surface movement. If the patient is myopic, he reports the speckle movement direction consistent with that of surface movement. For the astigmatic eye, none of the principal meridians is identical to the surface movement, and consequently the speckle pattern movement is oblique to that of the surface. The speckle appears stationary for the emmetropic eye (the subject reports a whirling movement of the speckle pattern). In this case the retina is conjugate to a plane, which is called the plane of stationarity. However, in general we feel that this is not a plane but a region. For our purposes it can be called the region of stationarity and we will now determine its position and form. The answer to this question is very important for the creation of a compact laser refractor. Earlier, this problem was studied only as a plane problem and the position of the "plane" of stationarity was determined in paper [3].

2. Region of stationarity

The cylindrical coordinates are appropriate to perform the due calculations. The source of coherent radiation placed at the point Q $[r, \beta, Z_q]$ illuminates the moving rough surface (a cylinder rotating around its own axis). Let P $[R, \alpha, Z_p]$ be an

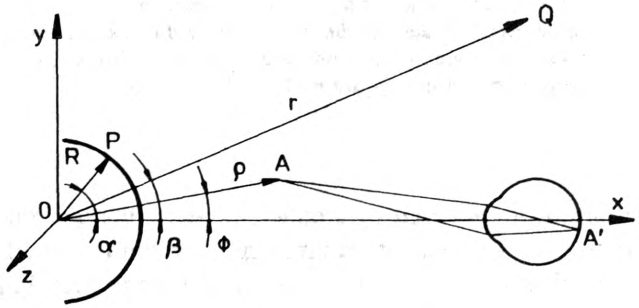


Fig. 1. Scheme of our arrangement

arbitrary point on the surface. A $[\rho, \varphi, Z]$ is a point optically conjugated to the point A' placed on the retina of the investigated eye (Fig. 1). Then the total optical length between points Q and A' is

$$S = S_1 + kS_2 + S_3 + \delta_s \tag{1}$$

where: S_3 is the optical length between points A and A', being constant because these points are optically conjugated, δ_s is a parameter which represents the profile differences caused by the roughness of the surface, $k = 1$ if the point A is placed in front of the surface, and $k = -1$ if it is placed behind the surface. The optical lengths S_1 (between the points Q and P) and S_2 (between the points A and P) may be expressed, respectively, as

$$S_1 = [r^2 + R^2 - 2rR\cos(\alpha - \beta) + (Z_p - Z_q)^2]^{1/2}, \tag{2a}$$
$$S_2 = [\rho^2 + R^2 - 2\rho R\cos(\alpha - \varphi) + (Z_p - Z)^2]^{1/2} \tag{2b}$$

where: r, R, ρ, α, β, φ are defined in Fig. 1, Z_p denotes the Z-coordinate of the point P, Z_q denotes the Z-coordinate of the point Q, Z is the Z-coordinate of the point A. The parameter δ_s is very small because the amplitude of the surface roughness is comparable with the light wavelength and therefore it can be neglected in our calculation. The dependence of the optical length S on the angle α does not change in time and therefore

$$\frac{d}{dt}\left[\frac{\partial S}{\partial \alpha}\right] = 0. \tag{3}$$

If an immobile eye is focused on an arbitrary plane, then

$$S = S(\alpha, Y, Z), \tag{4}$$

and we can write Eq. (3) in the form

$$\frac{d}{dt}\left[\frac{\partial S}{\partial \alpha}\right] = \frac{\partial^2 S}{\partial \alpha^2}\omega + \frac{\partial^2 S}{\partial \alpha \partial Y} v_y + \frac{\partial^2 S}{\partial \alpha \partial Z} v_z \tag{5}$$

where:

$\omega = \dfrac{d\alpha}{dt}$ – the angular velocity of cylinder rotation,

$v_y = \dfrac{dY}{dt}$ – the speckle pattern velocity in the direction of y axis,

$v_z = \dfrac{dZ}{dt}$ – the speckle pattern velocity in the direction of z axis.

If we wish to stop the speckle pattern movement ($v_y = 0$, $v_z = 0$), it is necessary either to stop the cylinder movement or fulfil the following condition:

$$\frac{\partial^2 S}{\partial \alpha^2} = 0. \tag{6}$$

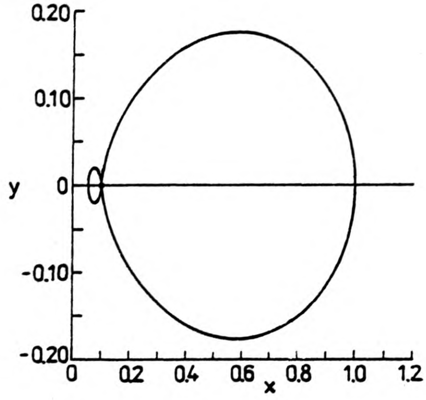
This condition is identical with that given by FERCHER and SPRONGL in [4] and it is the sufficient condition of the stationary speckle pattern in the case of a cylinder rotating around its own axis or a sphere rotating around its centre. After having calculated the second-order partial derivative of the optical length S with respect to α, we can write

$$\frac{rR\cos(\alpha-\beta)}{S_1} - \frac{r^2R^2\sin^2(\alpha-\beta)}{S_1^3} + k\left[\frac{\rho R\cos(\alpha-\varphi)}{S_2} - \frac{\rho^2R^2\sin^2(\alpha-\varphi)}{S_2^3}\right] = 0. \tag{7}$$

This equation has no solution for $k = 1$, while for $k = -1$ it is the equation for a region, its plane sections being shown in Figs. 2 and 3. They were calculated for $r = 1$, $\beta = 0$, $Z_q = 0$, $R = 0.1$, $\alpha = 0$, $Z_p = 0$. It is evident that we cannot regard them as an expression for the volume of stationarity. An envelope of the region determined by Eq. (7) will create the region of stationarity, because all points of the illuminated area contribute to the appearance of the speckle pattern. The points of the surface are determined by coordinates α and Z_p. We need to calculate:

$$\frac{\partial^3 S}{\partial \alpha^3} = -\frac{rR\sin(\alpha-\beta)}{S_1} - \frac{3r^2R^2\sin(\alpha-\beta)\cos(\alpha-\beta)}{S_1^3} + \frac{3r^3R^3\sin^3(\alpha-\beta)}{S_1^5}$$

$$+\frac{\rho R\sin(\alpha-\varphi)}{S_2} + \frac{3\rho^2R^2\sin(\alpha-\varphi)\cos(\alpha-\varphi)}{S_2^3} - \frac{3\rho^3R^3\sin^3(\alpha-\varphi)}{S_2^5}, \tag{8a}$$



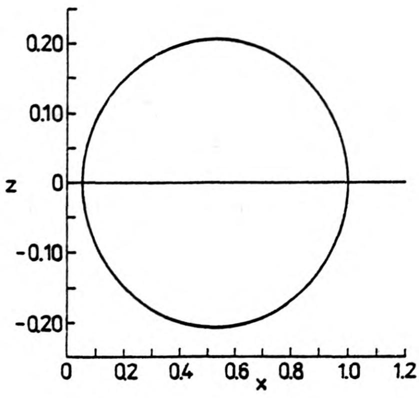


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Fig. 2. Plane section $z = 0$ of the region determined by Eq. (7)
Fig. 3. Plane section $y = 0$ of the region determined by Eq. (7)

$$\frac{\partial^3 S}{\partial\alpha^2 \partial Z_p} = -\frac{rR(Z_p - Z_q)\cos(\alpha-\beta)}{S_1^3} + \frac{3r^2R^2(Z_p - Z_q)\sin^2(\alpha-\beta)}{S_1^5} + \frac{\rho R(Z_p - Z)\cos(\alpha-\varphi)}{S_2^3} - \frac{3\rho^2R^2(Z_p - Z)\sin^2(\alpha-\beta)}{S_2^5}. \tag{8b}$$

The following conditions have to be valid for the envelope:

$$\frac{\partial^3 S}{\partial\alpha^3} = 0, \tag{9a}$$

$$\frac{\partial^3 S}{\partial\alpha^2 \partial Z_p} = 0. \tag{9b}$$

If we label

$$C_1 = \frac{rR\cos(\alpha-\beta)}{S_1} - \frac{r^2R^2\sin^2(\alpha-\beta)}{S_1^3}, \tag{10a}$$

$$C_2 = -\frac{rR\sin(\alpha-\beta)}{S_1} - \frac{3r^2R^2\sin(\alpha-\beta)\cos(\alpha-\beta)}{S_1^3} + \frac{3r^3R^3\sin^3(\alpha-\beta)}{S_1^5}, \tag{10b}$$

$$C_3 = -\frac{rR(Z_p - Z_q)\cos(\alpha-\beta)}{S_1^3} + \frac{3r^2R^2(Z_p - Z_q)\sin^2(\alpha-\beta)}{S_1^5}, \tag{10c}$$

$$v = \rho R\sin(\alpha-\varphi), \tag{10d}$$

$$u = \rho R\cos(\alpha-\varphi), \tag{10e}$$

then we get the set of equations for the region of stationarity in the form:

$$S_2^2 C_2 + S_2 v + 3vC_1 = 0, \tag{11a}$$

$$S_2^2 u + v^2 - S_2^3 C_1 = 0, \tag{11b}$$

$$S_2^5C_3+(Z_p-Z)(S_2^3C_1-2v^2)=0, \tag{11c}$$

$$\rho^2+R^2-2u+(Z_p-Z)^2-S_2^2=0, \tag{11d}$$

to which we add Eq. (2b). If we express v, u, Z_p-Z from Eqs. (11a)–(11c), and substitute them to (11d) and take into account that

$$\rho^2=\frac{1}{R^2}[v^2+u^2], \tag{12}$$

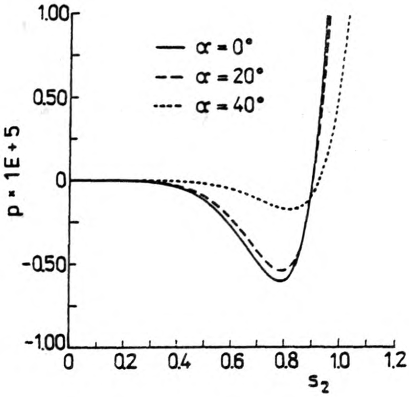
then we obtain the following equation for S_2:

$$\sum_{n=0}^{12} a_n S_2^n = 0, \tag{13}$$

where the coefficients a_n for $n = 1, \ldots, 12$ are expressed as follows:

$$a_0 = 6561R^4C_1^{10},$$
$$a_1 = 17496R^4C_1^9-13122R^2C_1^{11}-2916R^4C_1^7C_2^2,$$
$$a_2 = 20412R^4C_1^8-41553R^2C_1^{10}+6561C_1^{12}+324R^4C_1^4C_2^4+4374R^2C_1^8C_2^2 -5832R^4C_1^6C_2^2,$$
$$a_3 = 432R^4C_1^3C_2^4+11664R^2C_1^7C_2^2-1458C_1^9C_2^2-4860R^4C_1^5C_2^2+13608R^4C_1^7 -58320R^2C_1^9+17496C_1^{11},$$
$$a_4 = 13122R^2C_1^6C_2^2-2187C_1^8C_2^2-2160R^4C_1^4C_2^2-72R^2C_1^2C_2^6+6561R^2C_1^8C_3^2 -47628R^2C_1^8+5670R^4C_1^6+20412C_1^{10}-243C_1^6C_2^4-324R^2C_1^4C_2^4 +216R^4C_1^2C_2^4,$$
$$a_5 = 36C_1^3C_2^6-48R^2C_1C_2^6-648C_1^5C_2^4+48R^4C_1C_2^4-432R^2C_1^3C_2^4 +8100R^2C_1^5C_2^2-972C_1^7C_2^2-540R^4C_1^3C_2^2+17496R^2C_1^7C_3^2-24948R^2C_1^7 +1512R^4C_1^5+13608C_1^9,$$
$$a_6 = 20412R^2C_1^6C_3^2-8694R^2C_1^6+252R^4C_1^4+5670C_1^8+2970R^2C_1^4C_2^2 +135C_1^6C_2^2-72R^4C_1^2C_2^2-594C_1^4C_2^4+4R^4C_2^4+216R^2C_1^2C_2^4+60C_1^2C_2^6 -8R^2C_2^6+4C_2^8,$$
$$a_7 = 28C_1C_2^6-252C_1^3C_2^4-48R^2C_1C_2^4+648R^2C_1^3C_2^2+270C_1^5C_2^2-4R^4C_1C_2^2 +13608R^2C_1^5C_3^2+1512C_1^7-2016R^2C_1^5+24R^4C_1^3,$$
$$a_8 = 5670R^2C_1^4C_3^2-300R^2C_1^4+R^4C_1^2+252C_1^6+78R^2C_1^2C_2^2+99C_1^4C_2^2 -51C_1^2C_2^4-4R^2C_2^4+4C_2^6,$$
$$a_9 = 4R^2C_1C_2^2+16C_1^3C_2^2-4C_1C_2^4+1512R^2C_1^3C_3^2-26R^2C_1^3+24C_1^5,$$
$$a_{10} = 252R^2C_1^2C_3^2-R^2C_1^2+C_1^4+C_1^2C_2^2,$$
$$a_{11} = 24R^2C_1C_3^2,$$
$$a_{12} = R^2C_3^2,$$

depend only on α and Z_p. Equation (13) has only two real roots greater than zero and we have solved it by computer. The first one is $S_2 = S_1$. It is an evident solution of (7) and because, in this case, the point A coincides with the point Q in front of the surface, it is not the solution we seek. The second root is within the range of $(0, R)$.



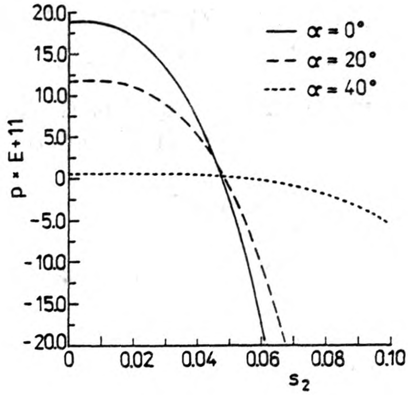


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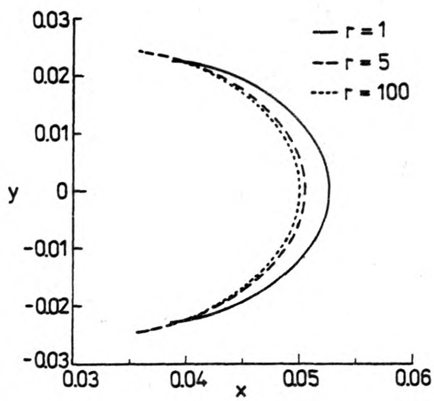
Fig. 4. Dependence of the polynomial from (13) (labelled p) on S_2

Fig. 5. Dependence of the polynomial from (13) (labelled p) on S_2



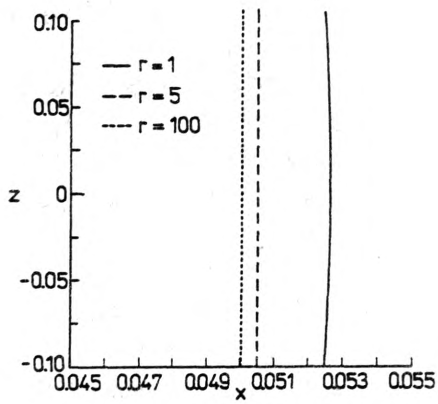


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Fig. 6. Plane sections $z = 0$ of the region of stationarity

Fig. 7. Plane sections $y = 0$ of the region of stationarity

The graphs of the dependence of the polynomial (13) on S_2 are in Fig. 4 and Fig. 5. They were calculated for $r = 1$, $\beta = 0$, $Z_q = 0$, $R = 0.1$, $Z_p = 0$. It is therefore possible to write $S_2 = S_2(\alpha, Z_p)$, and the parametric representation of the region of stationarity is

$$\rho = \frac{1}{R}\left\{\frac{S_2^4 C_2^2}{[S_2+3C_1]^2} + \left[S_2 C_1 - \frac{S_2^2 C_2^2}{[S_2+3C_1]^2}\right]^2\right\}^{1/2}, \tag{12a}$$

$$\varphi = \alpha + \arcsin \frac{\dfrac{S_2^2 C_2}{S_2+3C_1}}{\left\{\dfrac{S_2^4 C_2^2}{[S_2+3C_1]^2} + \left[S_2 C_1 - \dfrac{S_2^2 C_2^2}{[S_2+3C_1]^2}\right]^2\right\}^{1/2}}, \tag{12b}$$

$$Z = Z_p + \frac{S_2^2 C_3 [S_2 + 3C_1]^2}{C_1 [S_2 + 3C_1]^2 - 2S_2^2 C_2^2}, \tag{12c}$$

where S_2 is the solution of Eq. (13). The plane sections of the region of stationarity are in Fig. 6 and Fig. 7. They were calculated for the case when the source was placed on the optical axis ($\beta = 0°$, $Z_q = 0$), the radius of the cylinder $R = 0.1$ m and the illuminated area of the surface $\alpha \in (-50°, 50°)$, $Z_p \in (-0.1, 0.1)$.

3. Conclusions

From this parametric representation, we find out that the change of the source position has only a small influence on the form of the region of stationarity. The approach of the source towards the surface results only in the shift of the region towards this surface. In the case of the source located at infinity ($r \to \infty$) on the optical axis, the region of stationarity intersects the optical axis at the point $\rho = R/2$. Consequently, it is possible to illuminate the moving surface from a short distance by a diverging laser beam with its centre located at a point which is placed off axis. We then have to take account of the changes of the position of the region of stationarity depending on the position of the coherent source towards the moving surface in the astigmatism measurement. In this case, we rotate the cylinder axis around the optical axis.

Acknowledgements — This work was partially supported by a special grant from Palacký University, Olomouc.

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Received December 2, 1992