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Holographic lens recorded on conical surfaces. Aberration analysis

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The paper contains an analysis of aberration of holograms recorded on conical surfaces. The case of holographic lens is studied in more details. The possibilities of compensating third order coma are examined in particular.

1. Introduction

In modern optics unconventional imaging elements are more and more widely used in practice. The holographic lenses are a very important example of above. The problem of the quality of the images obtained by the holographic lens (holo-lens), particularly those recorded on a spherical surface, has been investigated by many workers [1]–[7]. The first, who considered a spherical holo-lens was Welford [1], but a full analysis of its third order aberrations was given by Mustafin [2], [3]. Kijek in [5] tried to extend the formulas obtained by Mustafin into the more general case of holo-lens recorded on conical surface. His results, however, were not satisfactory. Verboven and Langesse have found the formulas for aberration terms of holograms made on surfaces of any shape [8]. It seems that they have not known the Mustafin's work, but which is important for the particular case of spherical surface, their results are in agreement with those of Mustafin's. It is important that, for conical surface, the Verboven's and Lagasse's formulas take a simple form. In present work, we investigated some characteristic features of the holo-lens recorded on conical surfaces.

2. Analytical formulas

Let P_o, P_r, P_c and P_i (Fig. 1) be the coordinates of the object point source, the reference point source, the reconstruction point source and the trial image location, respectively. The total wavefront aberration can be expressed [8] as

$$W = \sum_{n=0}^{\infty} \sum_{k=0}^{n} W_{n,k}. \tag{1}$$

The aberrations' terms $W_{n,k}$ are:

$$W_{n,k} = b_{n,k}[V_c^{2n-1}\omega_c^{n-k}\xi_c^k - V_i^{2n-1}\omega_i^{n-k}\xi_i^k \pm \mu(V_o^{2n-1}\omega_o^{n-k}\xi_o^k - V_r^{2n-1}\omega_r^{n-k}\xi_r^k)]. \quad (2)$$

This sum may bé written for short as

$$W_{n,k} = b_{n,k}\sum_q V_q^{2n-1}\omega_q^{n-k}\xi_q^k \quad (3)$$

where

$$b_{n,k} = (-1)^{k+1}\frac{(2n-3)!!}{(n-k)!k!2!}, \qquad q \in (\mathrm{o, r, c, i}).$$

V_q is the inverse of the distance from the point P_q to the origin of the coordinates

$$\omega_q = xx_q + yy_q,$$

x, y, z are the coordinates of a point on the hologram surface

$$\xi_q = x^2 + y^2 + z^2 - 2zz_q. \quad (4)$$

μ represents the reconstructing-to-recording wavelength ratio. The sign $(+)$ by μ in the sum (2) corresponds to the primary image and the sign $(-)$ – to the secondary one. We will deal with the primary image only. The index n in the Eqs. (2) and (3) corresponds to the order of aberration. The terms corresponding to n are the $(2n-1)$th order. The index k corresponds to a fixed sort of aberration. For example, in the case of the third order aberration n is equal to 2 and the increasing values of $k = 0$, 1, 2 correspond to the third order astigmatism, the third order coma, and the third order spherical aberration, respectively.

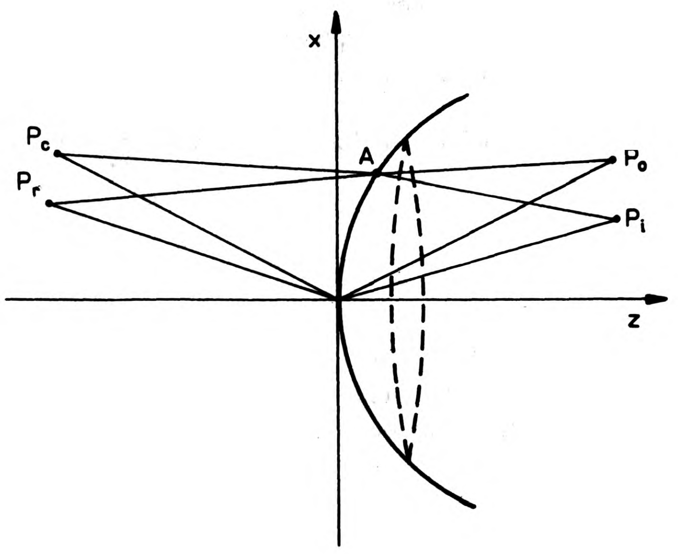

Fig. 1. Considered setup

Let us consider the equation below

$$2z = 1/\rho(x^2+y^2+z^2), \tag{5}$$

which describes a family of conical surfaces. The family depends on parameters δ, ε, which take the following values:

$\sigma = 1$,	$\varepsilon = 1$	for the sphere,
$\sigma = 1$,	$\varepsilon > 1$ and $0<\varepsilon<1$	for the ellipsoid,
$\sigma = 1$,	$\varepsilon = 0$	for the paraboloid,
$\sigma = 1$,	$\varepsilon < 0$	for the hyperboloid of one sheet,
$\sigma = -1$,	$\varepsilon < 0$	for the hyperboloid of two sheets,
$\sigma = -1$,	$\varepsilon = 0$	for the hyperbolic paraboloid.

We consider here the proper conical surfaces only.

Inserting the fixed number of the family (5) into Eq. (4), we can eliminate the variable z. Thus, for the paraboloidal and hyperbolic paraboloid, we get

$$\xi_q = r^2+C[0.25\,C(x^2+y^2)-1](x^2+\sigma y^2), \tag{6}$$

for the hyperboloid of two sheets we get

$$\xi_q = r^2+(x^2-y^2)\varepsilon^{-1}+2(C\varepsilon)^{-2}\{1-[1-C^2(x^2-y^2)\varepsilon]^{1/2}\}(1-C\varepsilon z_q), \tag{7}$$

and for the others surfaces we get

$$\xi_q = r^2(1-1/\varepsilon)+2(C\varepsilon)^{-2}[1-(1-C^2r^2\varepsilon)^{1/2}](1-C\varepsilon z_q) \tag{8}$$

where: $C = 1/\rho$, and $r^2 = x^2+y^2$.

It is easy to see that for $\varepsilon = 1$ the first factor in Eq. (8) vanishes ($\varepsilon = 1$ corresponds to a spherical surface with a radius ρ). In this case, each aberration term can be represented by

$$W_{n,k} = \sum_i f_i(\mathrm{A},\alpha)\sum_q g_{qi}(P_q,\alpha) \tag{9}$$

where: $f_i(\mathrm{A},\alpha)$ is a function of the coordinates of the point A on the hologram surface (Fig. 1), and the parameters $\alpha = (\rho,\ \sigma,\ \varepsilon)$ of the surfaces; $g_{qi}(P_q,\alpha)$ is a function of the coordinates of the point P_q and the parameters α. For example, for the $W_{2,\,1}$ the functions f_i and g_{qi} are:

$$f_i = 2\eta_i C^{-2}[1-(1-C^2r^2)^{1/2}], \tag{10}$$

$$g_{qi} = \eta_{qi}V_q^3(1-Cz_q) \tag{11}$$

where: $i = 1,\ 2$

$\eta_i = x$ for $i = 1$ and $\eta_i = y$ for $i = 2$,

$\eta_{qi} = x_q$ for $i = 1$ and $\eta_{qi} = y_q$ for $i = 2$.

Functions $g_q = \sum_i g_{qi}$ are called aberration coefficients.

When we are able to represent the aberration terms in the form (9), it is reasonable to find such values of the parameters α and the coordinates of the point sources P_q that the aberration coefficients vanish. It is not always possible, of course, but when it is, then for each point $A(x, y, z)$ on the hologram surface the aberration term $W_{n,k}$ corresponding to the considered g_{qi} is equal to zero. This fact allows us to compensate some aberrations. For other conical surfaces and additional factor for $W_{n,k}$ emerges in the equation. The $W_{n,k}$ (except for $W_{n,0}$) take the form

$$W_{n,k} = \sum_{qi} h_{qi}\ (A,\ P_q,\ \alpha) + \sum_i f_i(A,\ \alpha) \sum_q g_{qi}(P_q,\ \alpha) \tag{12}$$

where: $h_{qi}(A,\ P_q,\ \alpha)$ is a function of the entire set of parameters and variables described above. For example, in the case of ellipsoid we get for $W_{2,\,1}$:

$$f_i = 2\eta_i (C\varepsilon)^{-2}[1-(1-C^2 r^2 \varepsilon)^{1/2}], \tag{13}$$

$$g_{qi} = \eta_{qi} V_q^3 (1 - C\varepsilon z_q), \tag{14}$$

$$h_{qi} = \eta_i \eta_{qi} V_q^3\, r^2 (1 - 1/\varepsilon). \tag{15}$$

We cannot exclude from both factors of the sum (12) its common part which is independent of the coordinates of points on the hologram surface. Hence, it is impossible to get any $W_{n,k}$ equal to zero for each point $A(x, y, z)$ on the hologram surface simultaneously. Thus we have shown that the only surface, among the conical surfaces, that allows us to eliminate some aberrations, is sphere. For the $W_{n,0}$ of any order the situation is different. It is easy to obtain from Eq. (2) that the $W_{n,0}$ can be represented in the form (9) for any shape of a hologram. In this case, however, the $W_{n,0}$ does not depend on parameters (α) of hologram surface, so we cannot improve the quality of image by changing the hologram surface geometry. To make it clearer, we want to give some numerical examples.

3. Numerical examples

An analysis of the third order aberrations of a holo-lens recorded on a spherical surface is given in [7]. The holo-lens is a hologram under conditions:

$$x_o = y_o = 0, \qquad x_r = y_r = 0. \tag{16}$$

The coordinates of the Gaussian image are given as:

$$V_i = V_c \pm \mu (V_o - V_r), \tag{17}$$

$$V_i x_i = V_c x_c. \tag{18}$$

For the holo-lens the location of the Gaussian image does not depend on surface's parameters. In the general case this fact is not true.

In paper [7], it is showed that for a fixed value of ρ and coordinates of the source points P_q the aberrations coefficients corresponding to the third order coma and the third order spherical aberration vanish simultaneously. Those authors based on the

MUSTAFIN'S works [2], [3]. We present the value of the $W_{2,1}$ aberration term, which corresponds to the third order coma for the holo-lens recorded on sphere, ellipsoid, paraboloid. Our examples will be confined to the one-dimensional case, i.e., we put $y = y_q = 0$. We put next $\mu = 1$ and $V_r^{-1} = V_c^{-1} = \infty$, $V_i = V_o = 100^{-1}$. These values satisfy Eq. (17). All examples will be calculated for $x_c V_c = 0.02$. It is worth noting that the product $x_c V_c$ corresponds to the value of sine of an angle between the z axis and beam of the reconstruction wave. (The distances are given in mm).

3.1. Sphere ($\varepsilon = 1$, $\sigma = 1$)

To examine the spherical holo-lens, it is sufficient to deal with the aberration coefficients. For coma the aberration coefficient takes the form

$$g_i = -V_c x_c (C V_o + V_o^2). \tag{19}$$

It is easy to show that for $\rho = 100$ the g_i coefficient is equal to zero. Hence, $W_{2,1} = 0$ for each point on the holo-lens surface.

3.2. Ellipsoid ($\varepsilon > 0$, $\varepsilon \neq 1$, $\sigma = 1$)

Now, because of the additional factor in Equation (12), we have to examine the whole expression for the $W_{2,1}$. From Eqs. (2) and (8) we get

$$W_{2,1} = 0.5\, V_c x_c \{-x^3(1-1/\varepsilon)\, V_o^2 + 2x(C\varepsilon)^{-2}\, [1-(1-C^2 r^2 \varepsilon)^{1/2}](V_o \varepsilon C + V_o^2)\}.$$

Let a be the radius of ellipsoid in the direction of x axis, and b the second radius of ellipsoid in the direction of z axis. Let $\varepsilon' = a/b$, then from Eq. (5) we get $\varepsilon = \varepsilon'^2$. We will consider the family of ellipsoids which intersect the sphere with radius $\rho = 100$ at the points: $x = z = 0$, $x = 5$, $z = 0.125078223$ (Fig. 2). The values of $W_{2,1}$ for different

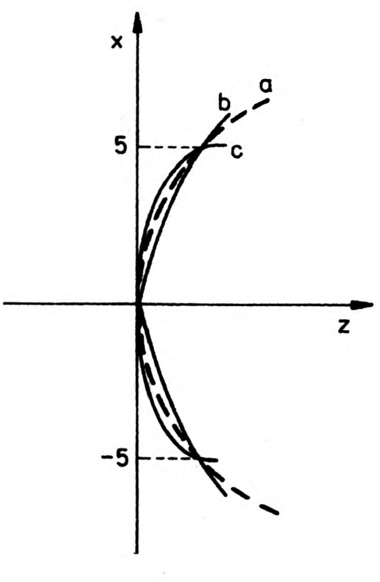

Fig. 2. Ellipsoid and paraboloid intersecting the sphere. a – sphere, b – paraboloid, c – ellipsoid

ε' are given in Tab. 1. This table shows that the $W_{2,1}$ value depends strongly on the coordinate x of the points at the pupil. Equation (20) for $W_{2,1}$ is continuous in relation to variable x, hence between $\varepsilon' = 0.9$ and $\varepsilon' = 1.1$ there exists such ε' that $W_{2,1} = 0$. According to our theory, in the case of ellipsoid each x can correspond to different ellipsoids if $W_{2,1} = 0$. This is well shown in Tab. 2, where we have fixed the ε' ($\varepsilon' = 0.8$) and treated the $W_{2,1}$ as a function of ρ. The results show that the $W_{2,1}$ equal to zero corresponds to different ρ for different x. That difference is due to the fact that for the ellipsoid it is impossible to represent the aberration term $W_{2,1}$ by its aberration coefficients.

Table 1. Aberration term $W_{2,1}$ for the family of ellipsoids

ε'	$W_{2,1}$ ($x = 1$)	$W_{2,1}$ ($x = 5$)
0.7	3.06×10^{-10}	1.53×10^{-13}
0.8	2.16×10^{-10}	2.44×10^{-13}
0.9	1.14×10^{-10}	4.00×10^{-13}
1.1	-1.26×10^{-10}	-1.97×10^{-13}
1.2	-2.64×10^{-10}	-1.99×10^{-13}
1.3	-4.14×10^{-10}	-1.85×10^{-13}

Table 2. Aberration term $W_{2,1}$ as a function of ρ (for ellipsoids)

ρ	$W_{2,1}$ ($x = 1$)	$W_{2,1}$ ($x = 5$)
99.90	9.92×10^{-10}	9.70×10^{-8}
99.95	4.91×10^{-10}	3.44×10^{-8}
99.97	2.91×10^{-10}	9.36×10^{-9}
99.98	1.91×10^{-10}	-3.19×10^{-9}
99.99	9.10×10^{-11}	-1.56×10^{-8}
100	-9.02×10^{-12}	-2.81×10^{-8}

3.3. Paraboloid ($\varrho = 0$, $\sigma = 1$)

We have to examine, as previously, the whole expression for the $W_{2,1}$. From Equations (2) and (7) we get

$$W_{2,1} = -0.5x^3 V_o V_c x_c (V_o + C + 0.25 V_o C^2). \tag{21}$$

Table 3. Aberration term $W_{2,1}$ for the family of paraboloid

ρ	$W_{2,1}$ ($x = 1$)	$W_{2,1}$ ($x = 5$)
90	1.11×10^{-7}	1.38×10^{-5}
95	5.25×10^{-8}	6.50×10^{-6}
99.94	5.75×10^{-10}	-3.17×10^{-9}
105	-4.76×10^{-8}	-4.82×10^{-8}
110	-9.10×10^{-8}	-1.14×10^{-5}

We will consider five paraboloids. The paraboloid number 3 intersects the sphere with radius $\rho = 100$ in the same way as ellipsoids in the previous example. The results given in the Tab. 3 are similar to those obtained for ellipsoids.

4. Conclusions

We have shown that for the holograms recorded on the conical surface (except the sphere), it is impossible to represent the aberration terms $W_{n,k}$ by the aberration coefficients. Hence, it is impossible to find such values of surface parameters and coordinates of source points P_q that the $W_{n,k}$ is equal to zero for each beam going through any point of a pupil. It is not sure, however, that the most appropriate way to obtain the best image is to compensate completely some aberrations. For this reason, it is necessary to make numerical investigations in order to get the full characteristic of an image.

References

[1] WELFORD W. T., Opt. Commun. **9** (1973), 268.
[2] MUSTAFIN K. S., Opt. Spektrosk. **37** (1974), 1198 (in Russian).
[3] SATAROV F. A., MUSTAFIN K. S., Opt. Spektrosk. **47** (1979), 1204 (in Russian).
[4] SMITH R. W., Opt. Commun. **21** (1977), 102.
[5] GAJ M., KIJEK A., Opt. Appl. **10** (1980), 341.
[6] JAGOSZEWSKI E., Optik **69** (1985), 85.
[7] ZAJĄC M., NOWAK J., GADOMSKI A., Opt. Appl. **19** (1989), 229.
[8] VERBOVEN P. E., LAGASSE P. E., Appl. Opt. **25** (1986), 4150.

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Голографические линзы, полученные на поверхностях второго порядка. Анализ аберрации

Проведен анализ аберрации голограмм, нолученных на поверхностях второго порядка. Пример голографических линз рассматривали более точно. Исследовали корректировочные возможности голографических линз для комы третьего порядка.