Optica Applicata, Vol. XVII, No. 4, 1987

Rapid evaluation of the zero-order Hankel transform for optical diffraction problems

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A sampling theorem based on the Dini expansion is employed for the numerical evaluation of the zero-order Hankel transform. In order to compute the optical field in sampling points both fast and accurately, basic Hankel transforms for polynomial representations of pupil functions are considered, and the concept of Elementary Sampling Coefficients (ESC) is proposed. For moderate values of a space-bandwidth product of the transform, our technique seems to be superior to the commonly used algorithms.

1. Introduction

In many optical diffraction problems involving cylindrical geometry, the aim is to evaluate Hankel transforms (HT) numerically with high computational efficiency. In the case of rotational symmetry, commonly encountered in optics [1], [2], this leads to the fast computation of the following transform:

$$G(z) = 2\int_0^1 T(r) J_0(zr) r dr \tag{1}$$

where J_0 is the Bessel function of the first kind and of zero-order. There are several general techniques to handle the transform (1), but all of them have substantial disadvantages [2]–[9]. For instance, the Abel-transform-based algorithm [3], [4] as well as Candel's dual algorithms [5] are very involved and slow. The QFHT algorithm according to SIEGMAN [6] requires exponential sampling grids both in the original and transform domain; in addition, owing to the artificial obscuration, appropriate analytic end correction techniques of the QFHT result are to be used [7]–[9]. Among various methods of computing the HT, the classical Baracat's approach that utilizes Fourier–Bessel (FB) sampling theorem as a computational tool (1) is still recommended because of rather small number of sampling points that are necessary in typical optical applications. The problems, however, is that the corresponding samples play the role of interpolation nodes, and, therefore, must be calculated very exactly with the aid of a separate numerical quadrature. Since the integrand involves the highly oscillatory function J_0 the quadrature is neither typical, nor rapid [1].

In our previous paper [10], we have introduced the Dini sampling theorem instead of the standard Fourier–Bessel one, and proposed it as an improved computational tool. Dini-series-based approach was originally introduced into optical imaging by ROISEN–DOSSIER [11], who discussed the usefullness of that approach to solve analytically certain apodisation problems. The great usefullness of the Dini sampling formula in numerical work results from the fact that it converges much more rapidly than the FB one [10]. The number of sampling points can be therefore reduced, but the question how to calculate the samples efficiently and with high accuracy is again crucial in practical computational work. In what follows here, we shall propose a useful method of both fast and accurate calculation of the samples that are necessary for employing the Dini sampling formula as the computational tool.

2. Basic considerations

In order to evaluate the transform (1) it is possible to use the band-limited properties of the function $G(z)$ to develop sampling formulae that can be employed for computational purposes. Our approach is based on a Dini expansion of the original function $T(r)$, which leads to the following sampling formula [10]:

$$G(z) = [2J_1(z)/(z)]\{G(0) + \sum_{L=1}^{L=M} [G(z_L)/J_0(z_L)]/[1-(z_L/z)^2]\} \tag{2}$$

where $z = 0$, $z = z_L$, and the discrete sampling points, $z = z_L$, are nothing but successive zeros of the familiar Airy pattern, $2J_1(z)/z$. The above relationship permits one to compute $G(z)$ anywhere in between the sampled values $G(z_L)$, basing on these values as the sole data. In practice, $G(z)$ may be evaluated with arbitrary precision over any prespecified interval of z, by using sufficiently large finite number M of the samples: the greater both the interval of z and the precision required, the higher the number M of the discrete samples that would be demanded. In fact, the relation (2) can be interpreted as a sort of interpolation formula, in which the samples $G(z_L)$ play the role of interpolation nodes. It is important that in typical optical applications the small and moderate ($z < 20$) values in the transform domain are of greatest interest, and, in this case, the number of samples M is surprisingly small: in certain applications concerning small intervals ($z < z_1$) even first-sampling-point approach ($M = 1$) has proved to be fully sufficient [9], [12].

As it was mentioned earlier, the truncated series (2) is useful for our purposes provided that the following sampled values:

$$G(z_L)/J_0(z_L) = [2/J_0(z_L)] \int_0^1 T(r) J_0(z_L r) r dr \tag{3}$$

may be exactly computed without considerable effort. Now, we propose the simple method of evaluation of the normalised samples (3). The technique is based on the following assumption:

$$T(r) = T(r^2), \tag{4}$$

that is commonly fulfilled in optical diffraction problems involving pure rotational symmetry [2]. In consequence, the original function $T(r)$ can be expanded in terms of powers, r^{2p} or represented by an appropriate polynomial. Namely,

$$T(r) = \sum_{p=0}^{p=N} a_p r^{2p}. \tag{5}$$

That is why the following set of basic functions

$$(p+1)\, r^{2p}, \quad (p = 0, 1, \ldots) \tag{6}$$

was introduced by Boivin to deal with rotationally-symmetric optical diffraction problems, and corresponding Hankel transforms of these functions are extensively studied and tabulated [2]. The transform

$$2\,(p+1) \int_0^1 r^{2p} J_0(zr)\, r dr \tag{7}$$

is defined as the Boivin function of $(p+1)$-th order, and designated as $A_{p+1}(z)$. Selected mathematical properties of the Boivin's function $A_p(z)$ create the basis for our further considerations. In fact, after taking into account Eqs. (3) and (5), it is easily seen that the whole problem reduces to the computation of the following numbers:

$$S_{p+1,L} = [2/J_0(z_L)] \int_0^1 r^{2p} J_0(z_L r)\, r dr \tag{8}$$

that are nothing but properly normalised values of the Boivin function of $p+1$ order. If the numbers (8) are calculated, the whole procedure of evaluation of the samples (3) will be reduced to a single summation

$$G(z_L)/J_0(z_L) = \sum_{p=0}^{p=N} a_p S_{p+1,L}, \tag{9}$$

being analogous to the original function representation (5) except that the powers r^{2p} are replaced with corresponding values $S_{p+1,L}$ determined for successive sampling points. For this reason, the numbers $S_{p+1,L}$ are interpreted as ESC. The set of ESC for a given L-th sampling point can easily be generated with the help of the following recurrence formula, derived in the Appendix,

$$S_{p+1,L} = (2p/z_L)^2\, [(i/t) - S_{p,L}] \tag{10}$$

where $p = 0, 1, 2, \ldots$; and $S_{i,L} = 0$. Hence, the values of the ESC can be successively computed and stored in a look-up table, if necessary. The sets of ESC for the first few sampling points are listed in Table 1. It should be noted that the ESC tabulated there enable one to make computations for the polynomials $T(r)$ up to the order 20. The first few values for $L = 1$ were listed in our previous works [9], [12]. However, neither the further ESC values nor the method of calculation were published there.

Summing up, the whole procedure of computation of the transform (1) for the

Table 1. Elementary sampling coefficients $S_{p,L}$ tabulated for increasing orders of p and first few sampling points L

p	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
2	.272443	.081270	.038647	.022532	.014745
3	.247985	.136121	.071320	.043034	.028620
4	.209272	.144247	.091135	.058870	.040436
5	.177535	.137513	.098235	.068906	.049440
6	.153009	.126959	.098323	.073847	.055500
7	.133952	.116173	.095086	.075293	.059009
8	.118879	.106260	.090464	.074597	.060580
9	.106727	.097473	.085421	.072684	.060791
10	.096753	.089780	.080420	.070134	.060099
11	.088439	.083060	.075671	.067296	.058834

original functions represented by Eq. (3) reduces to the evaluation of the sampling series (2), in which the necessary samples can be obtained by the summation (9). The algorithm is both very simple and efficient. Excluding the computation of the familiar Airy pattern $2J_1(z)/z$ the algorithm does not require any Bessel nor special functions to be precomputed. In order to keep good accuracy, the required zeros z_L should be stored very exactly. The known Olver's tables are recommended as a source of the exact values of the zeros of Bessel functions.

3. Examples

At first, we consider a couple of analytic examples closely related to the optical diffraction theory. We assume that the original $T(r)$ stands for the pupil function of a rotationally-symmetric optical system; in consequence, the transform $G(z)$ describes the associated impulse response of the system. For convenience, we slightly redefine the set of basic functions (5), namely, we take

$$T_p(r) = 2(p+1)r^{2p}. \tag{11}$$

The reason for the change is that

$$2(p+1)\int_0^1 r^{2p}\,r\,dr = 1. \tag{12}$$

Now, we consider the limiting cases when $p \to \infty$, and $p = 0$, respectively. It is easy to see that in the first case

$$T_p(r) = \delta(r-1), \qquad (p \to \infty) \tag{13}$$

where δ denotes the Dirac delta function. In fact, T_p vanishes everywhere except for the point $r = 1$, in which $T_p \to \infty$. Nevertheless the condition (12) is still satisfied. Therefore, the function (13) describes an infinitely narrow, uniformly illuminated circular rim, that can be interpreted as a limiting case of a uniformly illuminated ring pupil of the unity radius. As it is well known [2], the associated impulse response is simply the Bessel function of zero-order, i.e., $G(z) = J_0(z)$. As a result, all the normalized samples in the formula (3) are now equal to unity

$$G(z_L)/J_0(z_L) = 1, \tag{14}$$

regardless the sampling point number L. Further, if we analogously rescale our ESC as defined by Eq. (8) in such a manner that

$$G_{2(p+1),L} = 2(p+1)S_{p+1,L}, \tag{15}$$

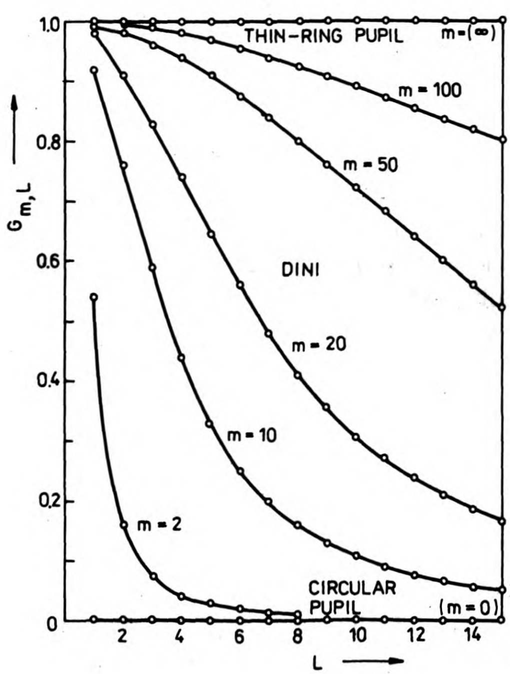
we obtain, for the thin-ring pupil,

$$G_{2(p+1),L} = 1, \quad \text{if} \quad p \to \infty. \tag{16}$$

Turning back to the second case ($p = 0$) it is easy to see that it reduces to the trivial case of a uniformly illuminated circular pupil $T_0(r) = 1$. In such a case the sampling formula (2) immediately gives the familiar result $G(z) = 2J_1(z)/z$ and both the normalized samples and corresponding ESC must be equal to zero, regardless the sampling order L. Thus,

$$G_{2(p+1),L} = S_{p+1,L} = 0 \quad \text{if} \quad p = 0. \tag{17}$$

The rescaled sets of ESC are shown graphically in the Figure which seems to be a useful aid for better understanding of our computational technique, and gives a



Graphical interpretation of the rescaled elementary sampling coefficients $G_{m,L}$ (note that $m = 2p$)

clear physical interpretation of it. Excellent numerical features of the ECS are also evident from the Figure.

Finally, to demonstrate numerical advantages of the proposed algorithm, let us consider the following original function;

$$T(r) = \exp(-12.5\,r^2). \tag{18}$$

Since $T(1) = 0.000\ 004$, it may be treated as a nontruncated Gaussian; in consequence the HT is also Gaussian. Namely,

$$G(z) = \exp(-z^2/50). \tag{19}$$

By using the sampling formula (2), the HT of the Gaussian function (18) is calculated an compared with the exact values obtained from Eq. (19). Firstly, the necessary samples of $G(z)$ are computed: although the nodes may be evaluated by the use of Eq. (19), they are determined with the aid of the ESC after fitting the Gaussian with a polynomial of 20th order. Selected numerical results are presented in Table 2.

Table 2. Values of the Gaussian response function calculated with the help of Dini sampling technique using 7 sampling points, compared with those obtained exactly

z	Dini	Exact
0.0	1.000000	1.000000
2.5	0.882502	0.882499
5.0	0.606535	0.606531
7.5	0.324653	0.324652
10.0	0.135337	0.135335
12.5	0.043936	0.043937
15.0	0.011109	0.011109
17.5	0.002188	0.002187
20.0	0.000497	0.000335
22.5	0.000041	0.000040
25.0	0.000003	0.000004

It is found that to hold the maximum absolute error at the level of 10^{-5} over the whole interval $z\,[0, 25]$, seven sampling points ($M = 7$) are sufficient. It should be stressed that the same example was originally investigated by AGRAWAL and LAX [7], who applied the QFHT algorithm with and without the additional end correction. They showed that the same level of accuracy for the space-bandwidth product $z = 25$ can be obtained with 512 radial samples without the end correction, or with 64 radial samples with the end correction.

4. Conclusion

It is demonstrated throughout the paper that for the class of originals $T(r)$ that can be fitted with a polynomial or represented by a truncated power series, there exists a profitable technique of evaluation of the corresponding zero-order HT. The method is based on the single Dini sampling series (2) and all the necessary samples can be calculated by the single-summation formula (9). Excluding the familiar Airy pattern function, no special functions are required to perform the HT calculation. Owing to the previously reported numerical advantages of the Dini approach [10], as well as due to the introduction of the elementary sampling coefficients, the algorithm appears to be very useful for optical diffraction calculations, as well as in related areas of imaging technology.

Appendix

Here, the recurrence formula (10) will be derived. Our starting point is the following recurrence formula developed by Boivin [2]

$$A_p+\{z^2/[4t(t+1)]\}A_{p+1} = A_\infty+[z^2/(4t)]A_1 \tag{A1}$$

where $A_p = A_p(z)$ denotes Boivin functions of appropriate orders, and

$$A_\infty = J_0(z), \tag{A2}$$

$$A_1 = 2J_1(z)/z. \tag{A3}$$

According to the definitions (7) and (8) the ESC can be reexpressed as

$$S_{p+1,L} = [1/(p+1)]A_{p+1}(z_L)/A_\infty(z_L). \tag{A4}$$

Further, since $A_1(z_L) = 0$, the recurrence formula (A1) reduces to

$$A_p(z_L)+\{z_L^2/[4p(p+1)]\}A_{p+1}(z_L) = A_\infty(z_L).$$

After multiplying both sides of the above equation by

$$i/[pA_\infty(z_L)],$$

and taking into account Eq. (A4), we immediately obtain

$$S_{p,L}+[z_L/(2p)]^2 S_{p+1,L} = 1/p, \quad S_{1,L} = 0,$$

which is in perfect accordance with Eq. (10). Finally, we note that, owing to Eq. (A4), further interesting properties of the ESC may be directly developed by using the Boivin function formalism. For instance, having in mind Eq. (A2) and (A3) one can easily see that $S_{p,L} \sim 1/p$, when $p \to \infty$.

Acknowledgements — The author wishes to thank Professor Alberic Boivin from the Laval University, Canada, and Professor Anthony E. Siegman of Stanford University, USA, for their comments and exchange of information.

References

[1] Barakat R., [In] *The Computer in Optical Research*, Frieden B. R., [Ed.], Springer–Verlag, New York 1980, Chap. 2.

[2] Boivin A., *Theorie et Calcul des Figures de Diffraction de Revolution*, Laval U. Press, Quebec 1964.

[3] Oppenheim A. V., Frisk G. V., Martinez D. R., Proc. IEEE **66** (1978), 264.
[4] Vest C. M., J. Opt. Soc. Am. **64** (1974), 1215.
[5] Candel S. M., IEEE Trans. Acoust. Speech Signal Process. ASSP–**29** (1981), 963.
[6] Siegman A. E., Opt. Lett. **1** (1977), 13.
[7] Agrawal G. P., Lax M., Opt. Lett. **6** (1981), 171
[8] Sheng S.–Ch., Doctor's Thesis, Stanford University, Stanford 1980.
[9] Szapiel S., J. Opt. Soc. Am. **A4** (1987), 625.
[10] Szapiel S., Opt. Lett. **6** (1981), 604.
[11] Roisen–Dossier B., Doctor's Thesis, Paris 1954.
[12] Szapiel S., Appl. Opt. **25** (1987), 244.

Received June 17, 1987

Быстрый метод численного определения изображения функции Ганкеля нулевого порядка в вопросах оптической дифракции

В работе показано применение теоремы о выборке для поворотно-симметрических систем, использующей ряды Дини для численного определения изображений функции Ганкеля нулевого порядка. Рассуждены свойства изображений функции Ганкеля из полиномов, а также введены т.наз. Элементарные Коэффициенты Выборки, которые способствуют быстрому и точному определению образцов изображения функции, необходимых для вычислений. Было численно показано, что для умеренных значений аргуманта изображения функции представленный метод является конкурентным по отношению к известным алгоритмам, применяемым в оптических вычислениях.