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Application of double complex numbers to the description of the polarization state

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The paper presents a new way of describing the resultant polarization vector in which two mutually perpendicular physical components of electric field – expressed in the form of complex numbers – are put in a complex sum. The number obtained in this way is double complex and is governed by simple rules of arithmetic, due to which the calculations concerning any changes of polarization state are clear and simple. A number of examples of calculations of this kind are shown concerning, among others, the changes in the state of polarization due to the passage of light through such elements as: polarizer, birefringent plate, and beam-splitting mirror. Also a relation between this way of description of polarization state and that based on Stokes and Jones matrices is presented.

1. Introduction

The elliptic polarization is the most general state of polarization of a monochromatic light wave. Two different but equivalent ways may be used to describe this state [1], [2].

The first one employs the parameters describing straightforwardly a polarization ellipse in its plane. These are: φ – azimuth, i.e., the angle between the positive direction of the x axis and the major axis of the ellipse (cf. Fig. 1), $\tan\vartheta = b/a$ – ellipticity describing the shape of the ellipse and the polarization helicity direction, $m = \sqrt{a^2+b^2} = \sqrt{m_x^2+m_y^2}$ – amplitude, i.e., the size of the ellipse, ψ – relative time phase. The space orientation of the polarization plane is determined here by unit vector, normal to this plane, which defines simultaneously the direction of wave propagation (it is enough to know two of its directional cosines since the third one may be calculated from the sum of their squares that equals to unity).

The second way consists in vector superposition of three mutually perpendicular linear harmonic vibrations of independent amplitudes and phases. For a complete description, the knowledge of these amplitudes and phases is necessary, thus of six quantities in all (which is the same number as in the first case).

On the assumption that the positive direction of the axis z of the orthogonal right-handed Cartesian coordinate system determines the direction of the plane wave propagation, the number of parameters necessary to describe completely the state of polarization is reduced to four. The space orientation of the unit vector, normal to the polarization plane, is explicitly defined under these circumstances; namely, it is colinear with the z axis and thus, perpendicular to the xy plane.

To describe the state of polarization in the first way, it suffices to know the parameters φ, ϑ, m and ψ.

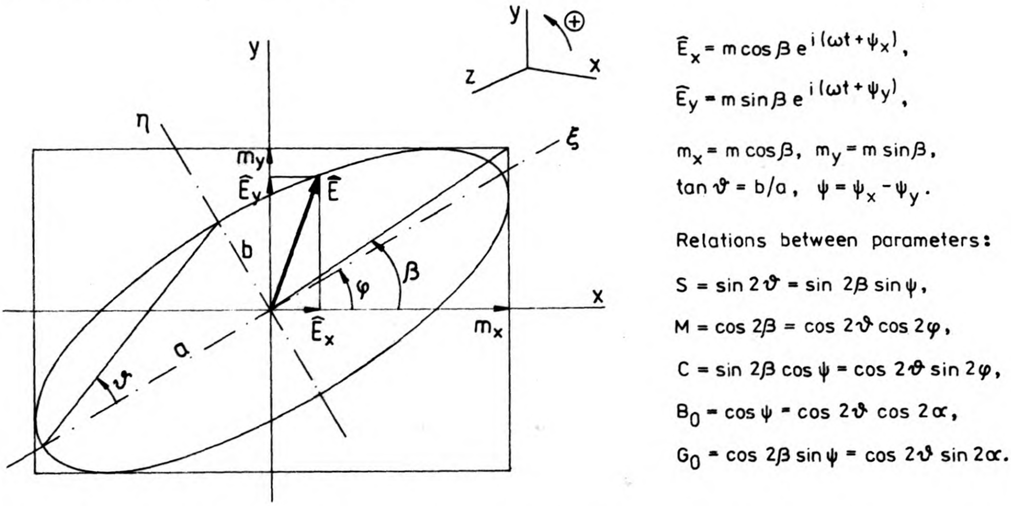

Fig. 1. Ellipse of polarization. Parameters of polarization ellipse: ϑ – ellipticity, φ – azimuth, β – diagonal angle, ψ – phase difference, α – general equiphase, a, b – semi-major and semi-minor axes, respectively

The second way involves the superposition of two mutually perpendicular harmonic oscillations of directions consistent with the axes x and y:

$$E_x = E_{0x}\cos(\omega t+\psi_x) = m\cos\beta\cos(\omega t+\psi_x),$$

$$E_y = E_{0y}\cos(\omega t+\psi_y) = m\sin\beta\cos(\omega t+\psi_y).$$

2. Basic notations

For the plane monochromatic wave travelling along the z axis, two mutually perpendicular harmonic oscillations located in the xy plane may be written down with the help of usual complex numbers as follows:

$$\hat{E}_x = \hat{E}_{0x}e^{i\omega t} = m\cos\beta e^{i(\omega t+\psi_x)},$$

$$\hat{E}_y = \hat{E}_{0y}e^{i\omega t} = m\sin\beta e^{i(\omega t+\psi_y)}.$$

The resultant polarization vector written down with the help of double complex number is

$$\hat{\hat{E}} = \hat{E}_x + j\hat{E}_y = \bar{E} + i\bar{H} = m(\cos\beta e^{i\psi_x} + j\sin\beta e^{i\psi_x})\, e^{i\omega t}$$

$$= m\,\{\cos\beta\cos(\omega t+\psi_x) + j\sin\beta\cos(\omega t+\psi_y)$$

$$+\, i\,[\cos\beta\sin(\omega t+\psi_x) + j\sin\beta\sin(\omega t+\psi_y)]\} \quad (1)$$

where: $\hat{E}_x, \psi_x$ – first linear component and its phase,

$\hat{E}_y, \psi_y$ – second linear component and its phase,

$\bar{E} = m\{\cos\beta\cos(\omega t+\psi_x)+j\sin\beta\cos(\omega t+\psi_y)\}$ – real elliptic component (vector) of polarization.

$\bar{H} = m\{\cos\beta\sin(\omega t+\psi_x)+j\sin\beta\sin(\omega t+\psi_y)\}$ – imaginary elliptic component (vector) of polarization (the so-called "shadow"),

i – first imaginary (ordinary) unit number,

j – second imaginary unit number,

m, ω, t – amplitude, frequency, time.

Each double complex number possesses three corresponding conjugate forms:

– conjugation with respect to the i unit

$$\overset{\triangle}{E}{}^* = \hat{E}_x^* + j\hat{E}_y^* = \bar{E} - i\bar{H}, \tag{1a}$$

– conjugation with respect to the j unit

$$\overset{\triangle}{E}{}' = \hat{E}_x - j\hat{E}_y = \bar{E}' + i\bar{H}', \tag{1b}$$

– conjugation with respect to both the imaginary units (double conjugated number)

$$\overset{\triangle}{E}{}^{*\prime} = \hat{E}_x^* - j\hat{E}_y^* = \bar{E}' - i\bar{H}'. \tag{1c}$$

On the basis of these conjugate forms the Stokes parameters can be calculated in the following way:

$$\boldsymbol{J} = \frac{1}{2}(\overset{\triangle}{E}\overset{\triangle}{E}{}^{*\prime} + \overset{\triangle}{E}{}^*\overset{\triangle}{E}{}') = \hat{E}_x\hat{E}_x^* + \hat{E}_y\hat{E}_y^* = \bar{E}\bar{E}' + \bar{H}\bar{H}' = m^2, \tag{2a}$$

$$\boldsymbol{S} = -\frac{1}{2ij}(\overset{\triangle}{E}\overset{\triangle}{E}{}^{*\prime} - \overset{\triangle}{E}{}^*\overset{\triangle}{E}{}') = i(\hat{E}_x\hat{E}_y^* - \hat{E}_y\hat{E}_x^*)$$

$$= j(\bar{E}\bar{H}' - \bar{H}\bar{E}') = m^2\sin 2\beta\sin\psi = m^2\sin 2\vartheta, \tag{2b}$$

$$\boldsymbol{M} = \frac{1}{2}(\overset{\triangle}{E}\overset{\triangle}{E}{}^* + \overset{\triangle}{E}{}'\overset{\triangle}{E}{}^{*\prime}) = \hat{E}_x\hat{E}_x^* - \hat{E}_y\hat{E}_y^* = \frac{1}{2}(\bar{E}^2 + \bar{E}'^2 + \bar{H}^2 + \bar{H}'^2)$$

$$= m^2\cos 2\beta = m^2\cos 2\vartheta\cos 2\varphi, \tag{2c}$$

$$\boldsymbol{C} = \frac{1}{2j}(\overset{\triangle}{E}\overset{\triangle}{E}{}^* - \overset{\triangle}{E}{}'\overset{\triangle}{E}{}^{*\prime}) = \hat{E}_x\hat{E}_y^* + \hat{E}_y\hat{E}_x^*$$

$$= \frac{1}{2j}(\bar{E}^2 - \bar{E}'^2 + \bar{H}^2 - \bar{H}'^2) = m^2\sin 2\beta\cos\psi = m^2\cos 2\vartheta\sin\varphi. \tag{2d}$$

A vector for which $J = m^2 = 1$ is defined as unit vector

$$\overset{\triangle}{E} = (\cos\beta e^{i\psi_x} + j\sin\beta e^{i\psi_y})e^{i\omega t}. \tag{3}$$

Substituting the following expressions into (1):

$\psi = \psi_x - \psi_y$ – phase difference,

$\psi_p = \frac{1}{2}(\psi_x - \psi_y)$ – general initial phase it may be reduced to the following form:

$$\hat{\hat{E}} = m\hat{\hat{E}} = m(\cos\beta e^{i\psi/2} + j\sin\beta e^{-i\psi/2})\, e^{i(\omega t + \psi_p)} = \hat{m}/\hat{\hat{E}}$$

in which $\hat{\hat{E}}$ is a unit vector (cf. (3)), and

$$/\hat{\hat{E}} = /\hat{\hat{E}}_0\, e^{i\omega t} = (\cos\beta e^{i\psi/2} + j\sin\beta e^{-i\psi/2})\, e^{i\omega t} = \left(\cos\frac{\psi}{2} e^{j\beta} + i\sin\frac{\psi}{2} e^{-j\beta}\right) e^{i\omega t} \tag{4}$$

is the so-called standard form of a double complex vector, for which the phase-amplitude coefficient is

$$(\hat{m} = me^{i\psi_p} = 1) \rightarrow \{(m = 1) \wedge (\psi_p = 0)\}.$$

The Stokes parameters calculated according to Eqs. (2a)–(2d) for a unit or standard vector should be reduced by m^2.

Taking advantage of the relations (1)–(1c), the parameters defining the ellipticity funtion of the vector $\hat{\hat{E}}$

$$\Theta = \hat{\hat{E}}\hat{\hat{E}}^{*}\,\hat{\hat{E}}'\,\hat{\hat{E}}^{*\prime} = \cos^2 2\vartheta \tag{2e}$$

may also be determined.

The real component $\bar{E}$ (similarly to the imaginary one $\bar{H}$) performs a rotational motion as a function of time parameter ωt in the xy plane, being elliptic in the general case. This motion is right-handed (i.e., positive in the assumed right-handed coordinate system xyz, Fig. 1) if it is performed counterclockwise, and *vice versa*. The sense may be determined on the basis of the following relations between the signs of the basic ellipse parameters

– right-hand sense

$$\{[(\pi > \psi > 0) \wedge (\tan\beta > 0)] \vee [(-\pi < \psi < 0) \wedge (\tan\beta < 0)]\} \Rightarrow (\vartheta > 0),$$

– left-hand sense

$$\{[(-\pi < \psi < 0) \wedge (\tan\beta > 0)] \vee [(\pi > \psi > 0) \wedge (\tan\beta < 0)]\} \Rightarrow (\vartheta < 0)$$

while $\hat{E}_x/\hat{E}_y = e^{i\psi}/\tan\beta$.

3. Rotational transformation of the coordinate system

Let the polarization vector $\hat{\hat{E}} = \hat{E}_x + j\hat{E}_y$ be written down in primary coordinate system xy, while the same vector $\hat{\hat{E}}_{(\xi\eta)} = e^{-j\varphi_0}\,\hat{\hat{E}} = \hat{E}_\xi + j\hat{E}_\eta$ (the same polarization ellipse) is written in the secondary system $\xi\eta$, rotated through an angle φ_0 (Fig. 2). In the case where in the primary system the vector is written in the standard form, and in the secondary one it is to be written in such form, too, the shift of general time phase ψ_0 between both systems must be considered in the formula

$$/\hat{\hat{E}}_{(\xi\eta)} = e^{-j\varphi_0}/\hat{\hat{E}}e^{i\psi_0}. \tag{5a}$$

Assuming, for example, that the system $\xi\eta$ and the principal axes of the ellipse overlap, i.e., that $\varphi_0 = \varphi$ (angle between both the coordinate systems is equal to ellipse azimuth in the xy coordinate system), the equation of the standard

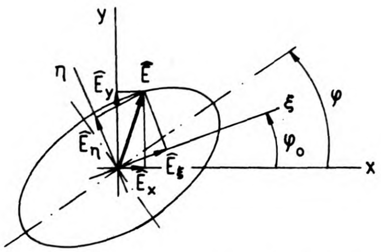

Fig. 2. Transformation of the coordinate system rotation

polarization vector (4) takes the canonical form in the $\xi\eta$ system (for which $\beta = \vartheta$ and $\psi = \pi/2$)

$$/\hat{\bar{E}}_{K(\zeta\eta)} = (\cos\vartheta e^{i\pi/4} + j\sin\vartheta e^{-i\pi/4})\, e^{i\omega t} = (\cos\vartheta - ij\sin\vartheta)\, e^{i(\omega t + \alpha_K)}$$

where $\alpha_K = \pi/4$ – general equiphase of the standard canonical form.

In the xy coordinate system (rotated with respect to the coordinate system $\xi\eta$ through $-\varphi$), the equation of this vector is

$$/\hat{\bar{E}} = e^{i\varphi}\, /\hat{\bar{E}}_{K(\zeta\eta)}\, e^{i\varphi_0} = e^{j\varphi}(\cos\vartheta - ij\sin\vartheta)\, e^{i(\omega t + \alpha)}$$
$$= (\cos\beta e^{i\psi/2} + j\sin\beta e^{-i\psi/2})\, e^{i\omega t} \tag{6}$$

where: $\alpha = \alpha_K - \psi_0$, $\psi_0 = \alpha_K - \alpha$, or generally

$$\psi_0 = \alpha_{(\xi\eta)} - \alpha_{(xy)}. \tag{7}$$

In Eqs. (6) and (7), there has appeared a new ellipse parameter – general equiphase α. It occurs in the relations constituting a completion of the Stokes parameters, which, however, unlike the Stokes parameters, are the functions of the time variable ωt and the general initial phase ψ_p, namely

$$B = \frac{1}{2}(\hat{\bar{E}}\hat{\bar{E}}' + \hat{E}^*\hat{\bar{E}}^{*\prime}) = \frac{1}{2}(\hat{E}_x^2 + \hat{E}_x^{*2} + \hat{E}_y^2 + \hat{E}_y^{*2}) = \bar{E}\bar{E}' - \bar{H}\bar{H}'$$
$$= \cos 2\vartheta \cos 2(\omega t + \psi_p + \alpha), \tag{2f}$$

$$G = \frac{1}{2i}(\hat{\bar{E}}\hat{\bar{E}}' - \hat{E}^*\hat{\bar{E}}^{*\prime}) = \frac{1}{2i}(\hat{E}_x^2 - \hat{E}_x^{*2} + \hat{E}_y^2 - \hat{E}_y^{*2}) = \bar{E}\bar{H}' + \bar{H}\bar{E}'$$
$$= \cos 2\vartheta \sin 2(\omega t + \psi_p + \alpha). \tag{2g}$$

Using the parameter B the instantaneous values of the amplitudes of the vectors $\bar{E}$ and $\bar{H}$ may be calculated

$$E = \pm\sqrt{\bar{E}\bar{E}'} = \pm\sqrt{\frac{1}{2}(1+B)} = \pm\sqrt{\frac{1}{2}\{1 + \cos 2\vartheta \cos 2(\omega t + \psi_p + \alpha)\}},$$

$$H = \pm\sqrt{\bar{H}\bar{H}'} = \pm\sqrt{\frac{1}{2}(1-B)} = \pm\sqrt{\frac{1}{2}\{1 - \cos 2\vartheta \cos 2(\omega t + \psi_p + \alpha)\}}.$$

For the initial moment, when $\omega t = \psi_p = 0$, i.e., $\hat{E} \equiv /\hat{E}_0$ (cf. (4))

$$\left.\begin{aligned} B \equiv B_0 &= \cos 2\vartheta \cos 2\alpha = \cos\psi \\ G \equiv G_0 &= \cos 2\vartheta \sin 2\alpha = \sin\psi \cos 2\beta \end{aligned}\right\} \tag{8}$$

and

$$E \equiv E_0 = \sqrt{\frac{1}{2}(1+\cos 2\vartheta \cos 2\alpha)} = \sqrt{\frac{1}{2}(1+\cos\psi)} = \cos\frac{\psi}{2},$$

$$H \equiv H_0 = \sqrt{\frac{1}{2}(1-\cos 2\vartheta \cos 2\alpha)} = \sqrt{\frac{1}{2}(1-\cos\psi)} = \sin\frac{\psi}{2}.$$

The parameters described by formulae (2a)–(2g) may be taken in one common relation. By creating for the unit vector

$$\hat{E} = (\cos\beta e^{i\psi/2} + j\sin\beta e^{-i\psi/2})\, e^{i(\omega t+\psi_p)} = e^{j\varphi}(\cos\vartheta - ij\sin\vartheta)\, e^{i(\omega t+\psi_p+\alpha)}$$

the following products:

$$\hat{E}\hat{E}^* = M + jC = \cos 2\vartheta e^{j2\varphi},$$

$$\hat{E}\hat{E}^{*\prime} = J - ijS = 1 - ij\sin 2\vartheta,$$

$$\hat{E}\hat{E}' = B + iG = \cos 2\vartheta e^{i2(\omega t+\psi_p+\alpha)},$$

we may then calculate the total product to obtain the expression

$$\hat{E}\hat{E}^*\, \hat{E}\hat{E}^{*\prime}\, \hat{E}\hat{E}' = (M+jC)(J-ijS)(B+iG) = \Theta\hat{E}^2 = (\cos 2\vartheta \hat{E})^2. \tag{2}$$

From the transformation formula (6) the following equality follows:

$$e^{j\varphi}(\cos\vartheta - ij\sin\vartheta)\, e^{i\alpha} = \cos\beta e^{i\psi/2} + j\sin\beta e^{-i\psi/2},$$

from which – by comparing the respective parts of double complex numbers on both sides – we obtain the formulae joining the trigonometric functions of single angles being parameters of ellipse, for instance,

$$\cos\varphi\cos\vartheta = \cos\beta\cos\left(\frac{\psi}{2}-\alpha\right), \quad \cos\alpha\cos\varphi\cos\vartheta - \sin\alpha\sin\varphi\sin\vartheta = \cos\beta\cos\frac{\psi}{2},$$

$$\sin\varphi\cos\vartheta = \sin\beta\cos\left(\frac{\psi}{2}+\alpha\right), \quad \cos\alpha\sin\varphi\cos\vartheta + \sin\alpha\cos\varphi\sin\vartheta = \sin\beta\cos\frac{\psi}{2},$$

$$\cos\varphi\sin\vartheta = \sin\beta\sin\left(\frac{\psi}{2}+\alpha\right), \quad \cos\alpha\sin\varphi\sin\vartheta + \sin\alpha\cos\varphi\cos\vartheta = \cos\beta\sin\frac{\psi}{2},$$

$$\sin\varphi\sin\vartheta = \cos\beta\sin\left(\frac{\psi}{2}-\alpha\right), \quad \cos\alpha\cos\varphi\sin\vartheta - \sin\alpha\sin\varphi\cos\vartheta = \sin\beta\sin\frac{\psi}{2},$$

and so on.

In the transformation formulae, the parameters of ellipse β, φ, ψ, and α refer to concrete coordinate system in which the ellipse is described. This remark does not concern the parameter ϑ which is an invariant.

4. Rotational transformation of the vector with respect to the coordinate system

Rotational transformation of the vector with respect to the coordinate system is an operation reciprocal to the rotational transformation of the coordinate system. Therefore, if $\overset{\triangle}{E}_p = \hat{E}_{px}+j\hat{E}_{py}$ is a vector in the primary position with respect to the xy system, then

$$\overset{\triangle}{E}_w = e^{j\varphi_0}\overset{\triangle}{E}_p = \hat{E}_{wx}+j\hat{E}_{wy} \tag{9}$$

describes the vector in the same system in the secondary position, i.e., rotated through an angle φ_0 with respect to the primary position (Fig. 3). The angle $\varphi_0 = \varphi_w - \varphi_p$ is the difference of the azimuths of both the ellipses measured in the

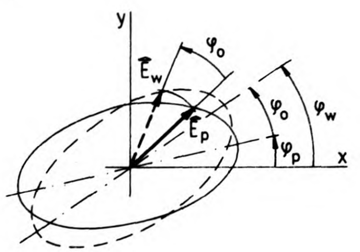

Fig. 3. Transformation of the vector rotation

same coordinate system. When the vector in the primary position is written in the standard form $|\overset{\triangle}{E}_p$, and the notation of the vector in the secondary position is to be of the standard form $|\overset{\triangle}{E}_w$ as well, then the shift of the general time phase ψ_0 should be taken into account in (9)

$$/\overset{\triangle}{E}_w = e^{j\varphi_0}/\overset{\triangle}{E}_p e^{-i\psi_0}. \tag{9a}$$

The formulae (9) and (9a) differ from the formulae (5) and (5a) only in the signs for φ_0 and ψ_0 for positive direction of rotation of the system in the first case or that of the vector in the second case.

4.1. Linear polarization

In the case where the ellipticity angle $\vartheta = 0$ and thus, the diagonal angle is equal to azimuth, $\beta = \varphi$, and by the same means the phase difference $\psi = 0$, Eq. (3) describes the linear polarization vector that forms the φ angle with respect to x axis

$$\overset{\triangle}{E} = (\cos\varphi + j\sin\varphi)\, e^{i(\omega t+\psi_p)} = e^{j\varphi}\, e^{i(\omega t+\psi_p)}. \tag{10}$$

4.2. Circular polarization

In this case $\vartheta = \beta = \pi/4$, while the phase difference $\psi = \pi/2$

$$\overset{\triangle}{E} = \left(\cos\frac{\pi}{4} e^{i\pi/4} + j\sin\frac{\pi}{4} e^{-i\pi/4}\right) e^{i(\omega t+\psi_p)}$$

$$= \frac{1}{\sqrt{2}}(1-ij)\, e^{i(\omega t+\psi_p+\pi/4)} = \frac{1}{\sqrt{2}}(i+j)\, e^{i(\omega t+\psi_p-\pi/4)}$$

5. Transformation of polarization state

As it is well known, each state of polarization may be expressed as a sum of two component polarizations. The components are in the general case of elliptic polarization types being in mutually orthogonal states. This means that the respective axes of ellipses are mutually perpendicular, while their senses of helicity are opposite.

The following notions are to be introduced:

– eigenvector, determining one of the two orthogonal directions of distribution and being a unit vector

$$\overset{\triangle}{E} = \hat{E}_x + j\hat{E}_y,$$

– initial polarization vector

$$\overset{\triangle}{E}_p = \hat{E}_{px} + j\hat{E}_{py},$$

– two orthogonal vector being components of $\overset{\triangle}{E}_p$ vector

$$\overset{\triangle}{E}_1 = \hat{E}_{1x} + j\hat{E}_{1y},$$

$$\overset{\triangle}{E}_2 = \hat{E}_{2x} + j\hat{E}_{2y}.$$

From the above definitions it follows that $\overset{\triangle}{E}_1 + \overset{\triangle}{E}_2 = \overset{\triangle}{E}_p$.

The operation of decomposition of the vector $\overset{\triangle}{E}_p$ is carried out taking account of the identity

$$\overset{\triangle}{E}_p \equiv \frac{1}{2}(\overset{\triangle}{E}\overset{\triangle}{E}{}^{*\prime} + \overset{\triangle}{E}{}^{*}\overset{\triangle}{E}{}') \overset{\triangle}{E}_p \tag{12}$$

following from the formula (2a).

For the unit eigenvector $\overset{\triangle}{E}$, the factor appearing in front of $\overset{\triangle}{E}_p$ on the right-hand side of the identitiy is equal to unity. The identity (12) may be transformed to the following form

$$\overset{\triangle}{E}_p = \frac{1}{2}\{\overset{\triangle}{E}(\overset{\triangle}{E}{}^{*\prime}\overset{\triangle}{E}_p) + \overset{\triangle}{E}{}^{*}(\overset{\triangle}{E}{}'\overset{\triangle}{E}_p)\} = \frac{1}{2}\{(\hat{E}_x^* - j\hat{E}_y^*)(\hat{E}_{px} + j\hat{E}_{py})\overset{\triangle}{E}$$

$$+ (\hat{E}_x - j\hat{E}_y)(\hat{E}_{px} + j\hat{E}_{py})\overset{\triangle}{E}{}^{*}\} = \frac{1}{2}(\hat{E}_x^*\hat{E}_{px} + \hat{E}_y^*\hat{E}_{py})\overset{\triangle}{E}$$

$$+j(\hat{E}_x\hat{E}_{py}-\hat{E}_y\hat{E}_{px})\overset{\triangle}{E}^*+(\hat{E}_x\hat{E}_{px}+\hat{E}_y\hat{E}_{py})\overset{\triangle}{E}^*+j(\hat{E}_x^*\hat{E}_{py}-\hat{E}_y^*\hat{E}_{px})\overset{\triangle}{E}\}$$

$$=\frac{1}{2}\{(\overset{\triangle}{E}_{11}+j\overset{\triangle}{E}_{21})+(\overset{\triangle}{E}_{12}+j\overset{\triangle}{E}_{22})\}.$$

The above result shows that the operation of decomposing the vector $\overset{\triangle}{E}_p$ into two component elliptical polarizations may be performed in the two following ways:

$$\overset{\triangle}{E}_p\equiv\overset{\triangle}{E}_{11}+j\overset{\triangle}{E}_{21}=(\hat{E}_x^*\hat{E}_{px}^*+\hat{E}_y^*\hat{E}_{py})\overset{\triangle}{E}+j(\hat{E}_x\hat{E}_{py}-\hat{E}_y\hat{E}_{px})\overset{\triangle}{E}^*, \tag{13a}$$

$$\overset{\triangle}{E}_p\equiv\overset{\triangle}{E}_{12}+j\overset{\triangle}{E}_{22}=(\hat{E}_x\hat{E}_{px}+\hat{E}_y\hat{E}_{py})\overset{\triangle}{E}^*+j(\hat{E}_x^*\hat{E}_{py}-\hat{E}_y^*\hat{E}_{px})\overset{\triangle}{E}. \tag{13b}$$

5.1. Linear polarizer

The eigenvector of the polarizer $\overset{\triangle}{E}=e^{j\varphi_H}e^{i\omega t}$ has the direction of the ζ axis, consistent with that of polarizer transmittance (Fig. 4). If $\overset{\triangle}{E}_p=\hat{E}_{px}+j\hat{E}_{py}$ is a

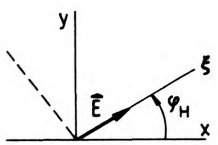

Fig. 4. Linear polarizer

vector of initial polarization written in the xy system, and which in the $\xi\eta$ system has the form

$$\overset{\triangle}{E}_{p(\xi\eta)}=e^{-j\varphi_H}\overset{\triangle}{E}_p=\cos\varphi_H\hat{E}_{px}+\sin\varphi_H\hat{E}_{py}$$
$$+j(-\sin\varphi_H\hat{E}_{px}+\cos\varphi_H\hat{E}_{py})=\hat{E}_{p\xi}+j\hat{E}_{p\eta},$$

then, only the $E_{p\xi}$ component is transmitted through the polarizer. The resultant polarization vector is thus a linear vector. In the $\xi\eta$ system it has the form

$$\overset{\triangle}{E}_{w(\xi\eta)}=\hat{E}_{p\xi}=\cos\varphi_H\hat{E}_{px}+\sin\varphi_H\hat{E}_{py}, \tag{14a}$$

while after having been transformed to the xy coordinate system

$$\overset{\triangle}{E}_w=e^{j\varphi_H}\overset{\triangle}{E}_{w(\xi\eta)}=e^{j\varphi_H}(\cos\varphi_H\hat{E}_{px}+\sin\varphi_H\hat{E}_{py})=\frac{1}{2}(\overset{\triangle}{E}_p+e^{j2\varphi_H}\overset{\triangle}{E}'_p) \tag{14b}$$

where $\overset{\triangle}{E}'_p=\hat{E}_{px}-j\hat{E}_{py}$ is the conjugate version of the number $\overset{\triangle}{E}_p$ with respect to j.

If the directions of polarizer transmittance were consistent with the η axis, the equation for the resultant polarization linear vector in the $\xi\eta$ system would be

$$\overset{\triangle}{E}_{w(\xi\eta)}=j\hat{E}_{p\eta}=j(-\sin\varphi_H\hat{E}_{px}+\cos\varphi_H\hat{E}_{py}),$$

and after transformation to the xy system

$$\overset{\triangle}{E}_w=e^{j\varphi_H}\overset{\triangle}{E}_{w(\xi\eta)}=je^{j\varphi_H}(-\sin\varphi_H\hat{E}_{px}+\cos\varphi_H\hat{E}_{py})=\frac{1}{2}(\overset{\triangle}{E}_p-e^{j2\varphi_H}\overset{\triangle}{E}'_p).$$

5.2. Linear double-refracting plate

The double-refracting plate performs a decomposition of the initial polarization into two linear component polarizations being in mutually orthogonal states shifted in phase by $\psi_F = \psi_\xi - \psi_\eta$. The first direction of the decomposition is represented by the eigenvector $\overset{\triangle}{E}$ of the plate, determining the axis ξ of the $\xi\eta$ system together with its azimuth φ_F (Fig. 5). The respective phase shifts introduced

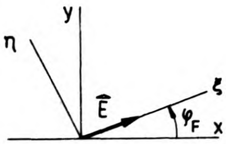

Fig. 5. Linear double-refracting plate

by the plate for both optical axes of the $\xi\eta$ system are ψ_ξ and ψ_η. If $\overset{\triangle}{E}_p = \hat{E}_{px} + j\hat{E}_{py}$ is an initial polarization vector written in the xy system, and

$$\overset{\triangle}{E}_{p(\xi\eta)} = e^{-j\varphi_F}\overset{\triangle}{E}_p = \hat{E}_{p\xi} + j\hat{E}_{p\eta} = \cos\varphi_F\,\hat{E}_{px} + \sin\varphi_F\,\hat{E}_{py} + j(-\sin\varphi_F\,\hat{E}_{px} + \cos\varphi_F\,\hat{E}_{py})$$

is the same vector written in the $\zeta\eta$ system, then

$$\overset{\triangle}{E}_{w(\xi\eta)} = e^{i\psi_\xi}\hat{E}_{p\xi} + je^{i\psi_\eta}\hat{E}_{p\eta} = (\cos\varphi_F\,\hat{E}_{px} + \sin\varphi_F\,\hat{E}_{py})e^{i\psi_\xi} + j(-\sin\varphi_F\,\hat{E}_{px} + \cos\varphi_F\,\hat{E}_{py})e^{i\psi_\eta} \tag{15a}$$

is the equation of the resultant polarization vector in the $\xi\eta$ system, which after having been transformed to the xy system takes the form

$$\begin{aligned}\overset{\triangle}{E}_w &= e^{j\varphi_F}\overset{\triangle}{E}_{w(\xi\eta)} = e^{j\varphi_F}\{(\cos\varphi_F\,\hat{E}_{px} + \sin\varphi_F\,\hat{E}_{py})e^{i\psi_\xi} + j(-\sin\varphi_F\,\hat{E}_{px} + \cos\varphi_F\,\hat{E}_{py})e^{i\psi_\eta}\} \\ &= \frac{1}{2}\{(\overset{\triangle}{E}_p + e^{j2\varphi_F}\overset{\triangle}{E}'_p)e^{i\psi_\xi} + (\overset{\triangle}{E}_p - e^{j2\varphi_F}\overset{\triangle}{E}'_p)e^{i\psi_\eta}\} \\ &= \frac{1}{2}\{(\overset{\triangle}{E}_p + e^{j2\varphi_F}\overset{\triangle}{E}'_p)e^{i\psi_F/2} + (\overset{\triangle}{E}_p - e^{j2\varphi_F}\overset{\triangle}{E}'_p)e^{i\psi_F/2}\}e^{i\psi_S} \\ &= \left(\cos\frac{\psi_F}{2}\overset{\triangle}{E}_p + i\sin\frac{\psi_F}{2}\overset{\triangle}{E}'_p e^{j2\varphi_F}\right)e^{i\psi_S}\end{aligned} \tag{15b}$$

where $\psi_S = \frac{1}{2}(\psi_\xi + \psi_\eta)$.

The formula (15b) results from the decomposition of the initial vector $\overset{\triangle}{E}_p$ into two orthogonal polarizations (linear in this case) and next ascribing to them the respective phase shifts ψ_ξ and ψ_η in accordance with the properties of the plate. This formula may also be derived starting from the relation (13a). Since the first eigenvector is

$$\overset{\triangle}{E} = e^{j\varphi_F}e^{i\omega t} = (\cos\varphi_F + j\sin\varphi_F)e^{i\omega t},$$

then

$$\hat{\hat{E}}_w = (\hat{E}_x^* \hat{E}_{px} + \hat{E}_y^* \hat{E}_{py}) \hat{\hat{E}} e^{i\psi_\xi} + j(\hat{E}_x \hat{E}_{py} - \hat{E}_y \hat{E}_{px}) \hat{\hat{E}}^* e^{i\psi_\eta}$$
$$= \{(\cos\varphi_F \hat{E}_{px} + \sin\varphi_F \hat{E}_{py}) e^{i\psi_\xi} + j(-\sin\varphi_F \hat{E}_{px} + \cos\varphi_F \hat{E}_{py}) e^{i\psi_\eta}\} e^{i\varphi_F}.$$

This is a result consistent with the relation (15b). An identical result is obtained for decomposition into two linear polarizations (i.e., for linear eigenvector) based on relation (13b).

5.3. Elliptic double-refracting plate

The initial polarization is decomposed in this case into two orthogonal elliptic polarizations shifted with respect to each other in phase by $\psi_F = \psi_\xi - \psi_\eta$. Assuming that the first eigenvector of the plate is

$$\hat{\hat{E}} = \hat{E}_x + j\hat{E}_y = (\cos\beta e^{i\psi/2} + j\sin\beta e^{-i\psi/2}) e^{i\omega t} = e^{j\varphi_F}(\cos\vartheta - ij\sin\vartheta) e^{i(\omega t + \alpha)},$$

and taking account of (13a) and (13b), the following results may be obtained

$$\hat{\hat{E}}_{w1} = e^{i\psi_\xi} \hat{\hat{E}}_{11} + je^{i\psi_\eta} \hat{\hat{E}}_{21} = e^{i\psi_\xi}(\hat{E}_x^* \hat{E}_{px} + \hat{E}_y^* \hat{E}_{py}) \hat{\hat{E}} + je^{i\psi_\eta}(\hat{E}_x \hat{E}_{py} - \hat{E}_y \hat{E}_{px}) \hat{\hat{E}}^*$$
$$= \frac{1}{2}\{e^{i\psi_\xi}(\hat{\hat{E}}\hat{\hat{E}}^{*\prime} \hat{\hat{E}}_p + \hat{\hat{E}}\hat{\hat{E}}^* \hat{\hat{E}}_p') + e^{i\psi_\eta}(\hat{\hat{E}}^* \hat{\hat{E}}' \hat{\hat{E}}_p - \hat{\hat{E}}\hat{\hat{E}}^* \hat{\hat{E}}_p')\}$$
$$= \frac{1}{2}\{(e^{i\psi_F/2} \hat{\hat{E}}\hat{\hat{E}}^{*\prime} + e^{-i\psi_F/2} \hat{\hat{E}}^* \hat{\hat{E}}') \hat{\hat{E}}_p + (e^{i\psi_F/2} - e^{-i\psi_F/2}) \hat{\hat{E}}\hat{\hat{E}}^* \hat{\hat{E}}_p'\} e^{i\psi_S}$$
$$= \left\{\left(\cos\frac{\psi_F}{2} + j\sin 2\vartheta \sin\frac{\psi_F}{2}\right)\hat{\hat{E}}_p + i\cos 2\vartheta \sin\frac{\psi_F}{2} e^{j2\varphi_F} \hat{\hat{E}}_p'\right\} e^{i\psi_S}, \quad (16a)$$

and

$$\hat{\hat{E}}_{w2} = e^{i\psi_\xi} \hat{\hat{E}}_{12} + je^{i\psi_\eta} \hat{\hat{E}}_{22} = e^{i\psi_\xi}(\hat{E}_x \hat{E}_{px} + \hat{E}_y \hat{E}_{py}) \hat{\hat{E}}^* + je^{i\psi_\eta}(\hat{E}_x^* \hat{E}_{py} - \hat{E}_y^* \hat{E}_{px}) \hat{\hat{E}}$$
$$= \frac{1}{2}\{e^{i\psi_\xi}(\hat{\hat{E}}^* \hat{\hat{E}}' \hat{\hat{E}}_p + \hat{\hat{E}}\hat{\hat{E}}^* \hat{\hat{E}}_p') + e^{i\psi_\eta}(\hat{\hat{E}}\hat{\hat{E}}^{*\prime} \hat{\hat{E}}_p - \hat{\hat{E}}\hat{\hat{E}}^* \hat{\hat{E}}_p')\}$$
$$= \frac{1}{2}(e^{i\psi_F/2} \hat{\hat{E}}^* \hat{\hat{E}}' + e^{-i\psi_F/2} \hat{\hat{E}}\hat{\hat{E}}^{*\prime}) \hat{\hat{E}}_p + (e^{i\psi_F/2} - e^{-i\psi_F/2}) \hat{\hat{E}}\hat{\hat{E}}^* \hat{\hat{E}}_p'\} e^{i\psi_S}$$
$$= \left\{\left(\cos\frac{\psi_F}{2} - j\sin 2\vartheta \sin\frac{\psi_F}{2}\right)\hat{\hat{E}}_p + i\cos 2\vartheta \sin\frac{\psi_F}{2} e^{j2\varphi_F} \hat{\hat{E}}_p'\right\} e^{i\psi_S} \quad (16b)$$

where $\psi_F = \psi_\xi - \psi_\eta$, $\psi_S = \frac{1}{2}(\psi_\xi + \psi_\eta)$.

In deriving the formulae (16a) and (16b) the following substitutions were made use of:

$$\hat{E}^{*\prime}\hat{E}_p = \hat{E}_x^*\hat{E}_{px}+\hat{E}_y^*\hat{E}_{py}+j(\hat{E}_x^*\hat{E}_{py}-\hat{E}_y^*\hat{E}_{px}),$$

$$\hat{E}^*\hat{E}_p' = \hat{E}_x^*\hat{E}_{px}+\hat{E}_y^*\hat{E}_{py}+j(\hat{E}_x^*\hat{E}_{py}-\hat{E}_y^*\hat{E}_{px}),$$

$$\hat{E}'\hat{E}_p = \hat{E}_x\hat{E}_{px}+\hat{E}_y\hat{E}_{py}+j(\hat{E}_x\hat{E}_{py}-\hat{E}_y\hat{E}_{px}),$$

$$\hat{E}\hat{E}_p' = \hat{E}_x\hat{E}_{px}+\hat{E}_y\hat{E}_{py}-j(\hat{E}_x\hat{E}_{py}-\hat{E}_y\hat{E}_{px}),$$

$$\hat{E}\hat{E}^{*\prime} = J-ijS = 1-ij\sin 2\vartheta,$$

$$\hat{E}^*\hat{E}' = J+ijS = 1+ij\sin 2\vartheta,$$

$$\hat{E}\hat{E}^* = M+jC = \cos 2\vartheta e^{j2\varphi}.$$

5.4. Linear quarter-wave plate

A quarter-wave plate is a double refracting plate for which $\psi_F = \psi_\xi - \psi_\eta = \pm\pi/2$ (the plus sign denoting the dextrorotatory quarter-wave, while the minus sign – the laevorotation quarter-wave). In accordance with (15b) the equation for the resultant polarization vector in the xy system is

$$\hat{E}_{\lambda/4} = e^{j\varphi_F}\{(\cos\varphi_F\hat{E}_{px}+\sin\varphi_F\hat{E}_{py})e^{\pm i\pi/4}+j(-\sin\varphi_F\hat{E}_{px}$$
$$+\cos\varphi_F\hat{E}_{py})e^{\pm i\pi/4}\}e^{i\psi_S} = \frac{1}{\sqrt{2}}(\hat{E}_p \pm ie^{j2\varphi_F}\hat{E}_p')e^{i\psi_S}. \quad (17)$$

5.5. Linear half-wave plate

In this case the phase difference $\psi_F = \pi$, and thus

$$\hat{E}_{\lambda/2} = e^{j\varphi_F}\{(\cos\varphi_F\hat{E}_{px}+\sin\varphi_F\hat{E}_{py})e^{i\pi/2}+j(-\sin\varphi_F\hat{E}_{px}$$
$$+\cos\varphi_F\hat{E}_{py})e^{-i\pi/2}\}e^{i\psi_S} = ie^{j2\varphi_F}\hat{E}_p'e^{i\psi_S} = e^{j2\varphi_F}\hat{E}_p'e^{i(\psi_S+\frac{\pi}{2})}. \quad (18)$$

5.6. Beam-splitter (beam-splitting mirror)

The light beam incident on the mirror surface at the angle r (Fig. 6), with the polarization determined by the vector $\hat{E} = \hat{E}_x + j\hat{E}_y$ composed of:

$\hat{E}_x$ – lying in the plane of incidence,

$\hat{E}_y$ – lying in the plane perpendicular to the plane of incidence, is decomposed into the ray of the polarization vector $\hat{E}_r = \hat{R}_x\hat{E}_x + j\hat{R}_y\hat{E}_y$ reflected at the angle r and the ray of the polarization vector $\hat{E}_t = \hat{T}_x\hat{E}_x + j\hat{T}_y\hat{R}_y$ refracted at the angle t.

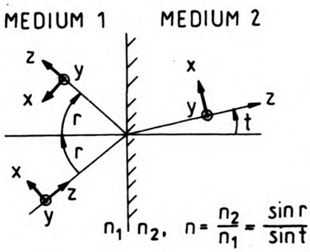

Fig. 6. Beam-splitting mirror xyz – the dextrorotatory reference system related to the direction of light propagation in the following way: the x axis lies in the plane of incidence, the y axis is perpendicular to the plane of incidence (in the figure it is directed upwards, which is denoted by the symbol ⊙), the z axis determines the directions of incident, reflected and transmitted light beams, respectively

The complex amplitude-phase coefficients appearing in the formulae given above are:

$\hat{R}_x = R_x e^{-i\psi_{R_x}}$ – reflectance of the light reflected for the component of the polarization vector lying in the plane of incidence (which is determined by the z and x axes).

$\hat{R}_y = R_y e^{-i\psi_{R_y}}$ – reflectance of the light reflected for the polarization vector lying in the plane perpendicular to the plane of incidence (which is determined by the z and y axes).

$\hat{T}_x = T_x e^{-i\psi_{T_x}}$ – trnasmittance of the light transmitted for the polarization vector component lying in the plane of incidence.

$\hat{T}_y = T_y e^{-j\psi_{T_y}}$ – transmittance of the light transmitted for the polarization vector component lying in the plane perpendicular to that of incidence.

The real coefficients R_x, R_y, T_x, and T_y define the degree of the attenuation of amplitudes for the respective components of vectors $\overset{\triangle}{E}_r$ and $\overset{\triangle}{E}_t$; arguments of the complex coefficients ψ_{R_x}, ψ_{R_y}, ψ_{T_x}, and ψ_{T_y} represent the phase jumps of the respective components of vectors $\overset{\triangle}{E}_r$ and $\overset{\triangle}{E}_t$. Since, for the light transmitted, the phase jumps $\psi_{T_x} = \psi_{T_y} = 0$, then finally:

$$\overset{\triangle}{E}_r = \hat{R}_x \hat{E}_x + j\hat{R}_y \hat{E}_y = R_x \hat{E}_x e^{-i\psi_{R_x}} + jR_y \hat{E}_y e^{-i\psi_{R_y}}, \tag{19}$$

$$\overset{\triangle}{E}_t = T_x \hat{E}_x + jT_y \hat{E}_y,$$

The reflectance and transmittance coefficients are functions of absolute indices of refraction n_1 for the medium 1, and n_2 for the medium 2 (Fig. 6).

6. Conclusions

The description of the polarization state by means of the double complex numbers seems to be more perspicuous in comparison with the respective matrix methods. A transition from the double complex to the matrix notation renders no difficulties.

In order to determine the matrix components of the Stokes vector, the relations (2a)–(2d) may be used. The Stokes vector expressed by the double complex numbers is

$$W_S = \begin{bmatrix} J \\ M \\ C \\ S \end{bmatrix} = \overset{\triangle}{M} + \overset{\triangle}{M}', \quad \text{where } \overset{\triangle}{M} = \frac{1}{2}\overset{\triangle}{E}\begin{bmatrix} \overset{\triangle}{E}^{*\prime} \\ \overset{\triangle}{E}^{*} \\ -j\overset{\triangle}{E}^{*} \\ -ji\overset{\triangle}{E}^{*\prime} \end{bmatrix},$$

$\overset{\triangle}{M}'$ – the form conjugate with respect to j of the matrix $\overset{\triangle}{M}$.

The Jones vector may be written immediately on the basis of the double complex notation, since

$$\hat{E} = (\cos\beta e^{i\psi_x} + j\sin\beta e^{i\psi_y})\, e^{i\omega t} = \begin{bmatrix} \cos\beta e^{i\psi_x} \\ \sin\beta e^{i\psi_y} \end{bmatrix} e^{i\omega t}.$$

Hence,

$$W_J = \begin{bmatrix} \cos\beta e^{i\psi_x} \\ \sin\beta e^{i\psi_y} \end{bmatrix}$$

is the matrix of the Jones vector.

The way to write the Jones transformation matrices will be shown by two examples. The relation (5) makes it possible to determine the matrix of the coordinate system rotation

$$\hat{E}_{(\xi\eta)} = e^{-j\varphi_0}\hat{E} = (\cos\varphi_0 - j\sin\varphi_0)(\hat{E}_x + j\hat{E}_y)$$
$$= \cos\varphi_0\,\hat{E}_x + \sin\varphi_0\,\hat{E}_y + j(-\sin\varphi_0\,\hat{E}_x + \cos\varphi_0\,\hat{E}_y)$$
$$= \begin{bmatrix} \cos\varphi_0\,\hat{E}_x + \sin\varphi_0\,\hat{E}_y \\ -\sin\varphi_0\,\hat{E}_x + \cos\varphi_0\,\hat{E}_y \end{bmatrix} = \begin{bmatrix} \cos\varphi_0 & \sin\varphi_0 \\ -\sin\varphi_0 & \cos\varphi_0 \end{bmatrix} \begin{bmatrix} \hat{E}_x \\ \hat{E}_y \end{bmatrix}.$$

Hence, the Jones matrix for the transformation of the coordinate system is

$$T_{(\varphi_0)} = \begin{bmatrix} \cos\varphi_0 & \sin\varphi_0 \\ -\sin\varphi_0 & \cos\varphi_0 \end{bmatrix}.$$

The other example is the determination of the polarizer matrix. Taking account of the relation (16a) we obtain

$$\hat{E}_w = (\hat{E}_x^*\hat{E}_{px} + \hat{E}_y^*\hat{E}_{py})(\hat{E}_x + j\hat{E}_y)e^{i\psi_\xi} + j(\hat{E}_x\hat{E}_{py} - \hat{E}_y\hat{E}_{px})(\hat{E}_x^* + j\hat{E}_y^*)e^{i\psi_\eta}$$
$$= \begin{bmatrix} \hat{E}_x^*\hat{E}_x\hat{E}_{px} + \hat{E}_x\hat{E}_y^*\hat{E}_{py} \\ \hat{E}_x^*\hat{E}_y\hat{E}_{px} + \hat{E}_y^*\hat{E}_y\hat{E}_{py} \end{bmatrix} e^{i\psi_\xi} + \begin{bmatrix} \hat{E}_y^*\hat{E}_y\hat{E}_{px} - \hat{E}_x\hat{E}_y^*\hat{E}_{py} \\ -\hat{E}_x^*\hat{E}_y\hat{E}_{px} + \hat{E}_x^*\hat{E}_x\hat{E}_{py} \end{bmatrix} e^{i\psi_\eta}$$
$$= \begin{bmatrix} \hat{E}_x^*\hat{E}_x & \hat{E}_x\hat{E}_y^* \\ \hat{E}_x^*\hat{E}_y & \hat{E}_y^*\hat{E}_y \end{bmatrix} \begin{bmatrix} \hat{E}_{px} \\ \hat{E}_{py} \end{bmatrix} e^{i\psi_\xi} + \begin{bmatrix} \hat{E}_y^*\hat{E}_y & -\hat{E}_x\hat{E}_y^* \\ -\hat{E}_x^*\hat{E}_y & \hat{E}_x^*\hat{E}_x \end{bmatrix} \begin{bmatrix} \hat{E}_{px} \\ \hat{E}_{py} \end{bmatrix} e^{i\psi_\eta}$$
$$= \left\{ \begin{bmatrix} \hat{E}_x^*\hat{E}_x & \hat{E}_x\hat{E}_y^* \\ \hat{E}_x^*\hat{E}_y & \hat{E}_y^*\hat{E}_y \end{bmatrix} e^{i\psi_F/2} + \begin{bmatrix} \hat{E}_y^*\hat{E}_y & -\hat{E}_x\hat{E}_y^* \\ -\hat{E}_x^*\hat{E}_y & \hat{E}_x^*\hat{E}_x \end{bmatrix} e^{-i\psi_F/2} \right\} \begin{bmatrix} \hat{E}_{px} \\ \hat{E}_{py} \end{bmatrix} e^{i\psi_S}$$
$$= (J_1 e^{i\psi_F/2} + J_2 e^{-i\psi_F/2}) \begin{bmatrix} \hat{E}_{px} \\ \hat{E}_{py} \end{bmatrix} e^{i\psi_S}$$

where

$$J_1 = \begin{bmatrix} \hat{E}_x^*\hat{E}_x & \hat{E}_x\hat{E}_y^* \\ \hat{E}_x^*\hat{E}_y & \hat{E}_y^*\hat{E}_y \end{bmatrix}$$

is the Jones matrix built up on the basis of the first eigenvector of the $\hat{E}$ polarizer

$$J_2 = \begin{bmatrix} \hat{E}_y^*\hat{E}_y & -\hat{E}_x\hat{E}_y^* \\ -\hat{E}_x^*\hat{E}_y & \hat{E}_x^*\hat{E}_x \end{bmatrix}$$

is the Jones matrix built up on the basis of the second eigenvector of the $-j\hat{E}^*$ vector, while

$$J = J_1 e^{i\psi_F/2} + J_2 e^{-i\psi_F/2}$$

$$= \begin{bmatrix} \hat{E}_x^* \hat{E}_x e^{i\psi_F/2} + \hat{E}_y^* \hat{E}_y e^{-i\psi_F/2} & 2i \sin\frac{\psi_F}{2} \hat{E}_x \hat{E}_y^* \\ 2i \sin\frac{\psi_F}{2} \hat{E}_x^* \hat{E}_y & \hat{E}_y^* \hat{E}_y e^{i\psi_F/2} + \hat{E}_x^* \hat{E}_x e^{-i\psi_F/2} \end{bmatrix}$$

is the Jones matrix of a double-refracting plate.

The whole reasoning given in this work has been carried out basing (as it has been pointed out at the beginning and as it is usually assumed in all the methods of describing the polarization state [1]–[4]) on the assumption that the direction of the wave polarization is consistent with the z axis, while the polarization vector $\hat{E}$ lies in the xy plane. In this formulation the description of the state of polarization is a two-dimensional problem. There obviously exists a possibility of leaving out this assumption and considering the problem in three-dimensional complex space, so that the direction of propagation be completely arbitrary. However, as far as the optical systems are concerned, such generalization is usually unnecessary. It may be useful in using the double complex numbers for simultaneous description of both fields of an electromagnetic wave (in which the real vector $\bar{E}$ may be identified with the polarization ellipse of an electric field, while the imaginary vector $\bar{H}$ may be identified with the ellipse of the magnetic field polarization).

In the complex three-dimensional space, the imaginary unit i remains assigned to the time variable, while separate imaginary unit numbers (imaginary versors) are assigned to each of the three axes of the xyz system. These three numbers j_x, j_y, j_z form the base of a complex space. They possess the following properties:

$$\begin{aligned} j_x j_x &= -1, \quad j_x j_y = j_z, \quad j_y j_x = -j_z, \\ j_y j_y &= -1, \quad j_y j_z = j_x, \quad j_z j_y = -j_x, \\ j_z j_z &= -1, \quad j_z j_x = j_y, \quad j_x j_z = -j_y. \end{aligned}$$

More detailed presentation of the problem in this formulation lies beyond the extent of this paper.

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Двухкомплексные числа используемые для описания состояния поляризации

В работе представлен новый способ записи результирующего вектора поляризации, в котором два взаимно перпендикулярных физических компонента электрического поля — представленные в виде комплексных чисел — взяты в комплексной сумме. Создано таким образом двухкомплексное число подчиняется простым законам арифметики, благодаря чему расчёты касающиеся любых изменений состояния поляризации несложны и наглядны. Представлен ряд примеров такого пода расчетов, касающихся между прочим изменений состояния поляризации в результате прохождения света через такие элементы как: поляризатор, двухпреломленная пластинка, зеркало разделяющее свет. Представлена также взаимосвязь между этим способом описания состояния поляризации и описанием при помощи матрицы Стоукса и Джоунса.