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Fresnel field of a multiple diffraction grating system illuminated by a point source

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General relations in the Fresnel area describing the field propagation by a system consisting of an arbitrary number of diffraction gratings have been derived. The considerations are based on the analyses of changes of the Fourier spectrum during the field propagation through the gratings. The application to the study of the moiré fringes formed in a double-diffraction system has been demonstrated.

1. Introduction

Systems of separated diffraction gratings are applied in various metrological problems [1]. The utility of such systems follows from their simplicity and flexibility to construct different measuring configurations. Additionally, the filtering of the grating spectra enhances these attributes.

The analyses of the Fresnel field of double-diffraction systems are known [2], [3]. In order to determine the field in the image space of the last grating, the field distributions in the plane of every grating are found successively. This way of conducting of the analysis focuses our attention on the fields in the grating planes. The diffraction field equations are derived using the distances between the gratings as the functional parameters. Though such an approach seems very natural, it does not take into account fundamental property of the phenomenon. The centre of the propagating field is situated at the point source and this fact occurs for every grating in a system, independently of the grating position.

In the paper a new approach for studying the multiple diffraction grating system has been proposed. It is based on the analysis in the Fourier plane. For the first time such an approach has been applied to the Talbot effect interpretation in the case of one diffraction grating [4]. If an infinitely large diffraction grating is illuminated by a point source, then the Fourier plane coincides with this source and the Fourier distribution is discrete. In this plane we have a set of points appearing at equal distances along the direction perpendicular to the diffraction grating lines. Every light spot in the Fourier plane of the first diffraction grating may be taken as the point source for the second one. This spot also generates discrete spectrum in the Fourier plane of the second diffraction grating. If the lines of both gratings are inclined, then two-dimensional set of points in the Fourier plane will be seen [5]. The influence of the next gratings can be analysed in the same way. It means that the changes of the field for all gratings will be seen with the aid of the Fourier spectra only. The advantages of such approach follow from the constant position of the Fourier plane, the discrete form of the Fourier distributions and the simple mathematical operations defining the transitions from the spectrum to the spectrum of adjacent gratings. At the end of the analysis we can admit that the field distribution in a plane of the image space of the last diffraction grating is a result of the interference of light emitted by all the points of the Fourier plane. In the mathematical sense we shall determine the Fourier transform of the respective Fourier distribution. In addition, the proposed approach facilitates the harmonic analysis and allows the direct derivation of general conclusions.

2. Field propagation through one diffraction grating

Let $V_{\Sigma}(\bar{a})$ be a field distribution on a sphere Σ, which coincides with a diffraction grating G (Fig. 1), $\bar{a}$ is the radial position vector on Σ, S – centre of the sphere Σ. Directly behind the diffraction grating on the sphere Σ' the field distribution arises

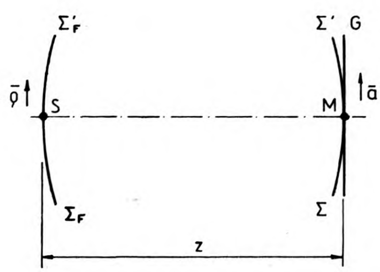


Fig. 1. Field propagation through a diffraction grating G

in the form $V' = V_{\Sigma} T$, where T is the diffraction grating transmittance. Although the spheres Σ and Σ' are coincident in the figure, they are in the different spaces of G. The sphere Σ is in the object space, and Σ' in the image space of G. The field distribution $V'(\bar{a})$ from the sphere Σ' generates its own Fourier transform distribution $V'_{\mathrm{F}}(\bar{\varrho})$ on the sphere Σ'_{F} with the centre M. For the case shown in the figure the Fourier spectrum is virtual with respect to the grating. $\bar{\varrho}$ is the radial position vector on the sphere Σ'_{F}.

Denoting

$$\bar{A} = k\bar{a}/z, \tag{1}$$

we can write

$$V'_{\mathrm{F}}(\bar{\varrho}) = \lambda z\, \mathrm{FT}^{+}\, [V_{\Sigma}(\bar{A})\, T(\bar{A})] \tag{2}$$

where z is the distance between the centres M and S, $k = 2\pi/\lambda$, λ – wavelength, FT^{+} – the operator of two-dimensional inverse Fourier transform. The modulus

A is a parametrized measure of the angular position of points on the spheres Σ and Σ', as seen from their centres. The introduction of the quantity $\bar{A}$ will be very useful, because the field changes during the propagation in space represent some characteristics of the grating geometrical shadow projected from the point S. This means that the field distributions ad different distances from S are in some degree invariant with regard to the angular variable $\bar{A}$, but it is not a case for the linear variable $\bar{a}$.

If the direction of the periodical changes of the diffraction grating G is given by the versor $\bar{\omega}^0$, then

$$T(\bar{A}) = \sum_m c_m \exp(im\bar{\omega}\bar{A}) \tag{3}$$

where c_m are the coefficients of one-dimensional Fourier series. The modulus $\omega = 2\pi/W_A$ is the fundamental angular frequency of the diffraction grating, W_A – the period of the diffraction grating related to the parametrized quantity $\bar{A}$ and, according to (1), $W_A = kW_a/z$ (W_a is the period of the diffraction grating in the linear measure). The sum in the relation (3) is taken in the infinite limits.

After substituting Eq. (3) into Eq. (2) we obtain

$$V'_{\mathrm{F}}(\bar{\varrho}) = \sum_m c_m V_{\mathrm{F}}(\bar{\varrho}+m\bar{\omega}) \tag{4}$$

where

$$V_{\mathrm{F}}(\bar{\varrho}) = \lambda z\, \mathrm{FT}^+ [V_\Sigma(\bar{A})]. \tag{5}$$

The distribution $V_{\mathrm{F}}(\bar{\varrho})$, as the Fourier transform of the field distribution from the sphere Σ, arises on the sphere Σ_{F} in the object space of the diffraction grating. According to Eq. (4) we can conclude that the transmission of a field distribution through a diffraction grating leads to the multiplication of the Fourier spectrum of this distribution. The effect is known as the image multiplication by the sampling of the Fourier spectrum [6]. In this case, roles of the image and its spectrum are interchanged.

In particular case, the diffraction grating may be illuminated by a point source S. We have: $V_\Sigma(\bar{A}) = V_0$, and from (5): $V_{\mathrm{F}}(\bar{\varrho}) = \lambda z V_0 \delta(\bar{\varrho})$. V_0 is the constant amplitude of the wavefront Σ. On account of (4) the spectrum in the image space of the diffraction grating is given by

$$V'_{\mathrm{F}}(\bar{\varrho}) = \lambda z V_0 \sum_m c_m \delta(\bar{\varrho}+m\bar{\omega}). \tag{6}$$

The last relation represents a discrete distribution on the sphere Σ'_{F}. The equidistant points are situated on the line, which coincides with the direction of periodical changes of the grating (coincident with $\bar{\omega}^0$). The distance between the adjacent points of the Fourier spectrum equals ω (see Fig. 2).

The relation (4) and, in particular case, the relation (6) allow the determination of the change of the Fourier spectrum related to the transmission of a field

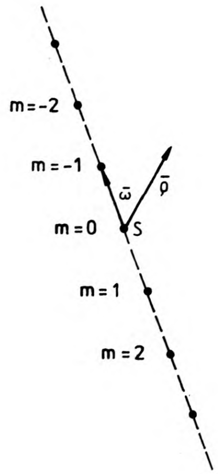


Fig. 2. Fourier spectrum of one diffraction grating illuminated by a point source S

distribution through a linear diffraction grating. To follow the field propagation through a system of diffraction gratings it is necessary to find the relation defining the changes of the Fourier spectrum related to the field propagation in free space between the gratings.

With reference to Fig. 1 let spheres Σ' and Σ'_F be the spheres with the known field distributions in the image space of a diffraction grating (Fig. 3). It is important to know that on the sphere Σ' with the centre S we have the field distribution directly behind the grating, and on the sphere Σ'_F with the centre M – the Fourier spectrum of the field distribution from the sphere Σ'. To find the field distribution on a sphere Σ'_d with the centre located at the point S, it is sufficient to determine, in agreement with our approach, the Fourier spectrum of this distribution. The Fourier distribution occurs on the sphere Σ'_{Fd} with the centre M_d, and the relation between the spectra from the spheres Σ'_{Fd} and Σ'_F, according to Fig. 3, is given by

$$V'_{Fd}(\bar{\varrho}) = V'_F(\bar{\varrho})\exp(ikp\varrho^2) \tag{7}$$

where

$$p = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z_d}\right). \tag{8}$$

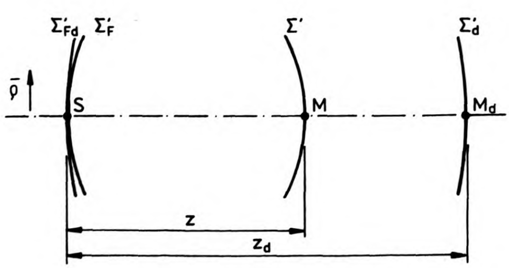


Fig. 3. Propagating field distributions on the spheres Σ' and Σ'_d and their Fourier transforms on the spheres Σ'_F and Σ'_{Fd}, respectively

From the relation (7) it follows that the propagation of the field with the discrete spectrum does not change its discrete character. Indeed, substituting Eq. (6) into Eq. (7) we have

$$V'_{\mathrm{Fd}}(\bar{\varrho}) = \lambda z V_0 \sum_m c_m \exp(ikpm^2\omega^2)\,\delta(\bar{\varrho}+m\bar{\omega}). \tag{9}$$

Then, the propagation of the field with the discrete spectrum introduces the respective phase shifts in the spectrum only.

In general case, the relations (4) and (7) solve our problem of the field propagation through the arbitrary diffraction grating with the aid of the Fourier spectra. The field transmission through a diffraction grating can be described by the spectrum multiplication (see Eq. (4)). On the other hand, the propagation in free spece can be expressed by the product of the spectrum distribution and the phase factor (see Eq. (7)). The simplicity of mathematical operations and the constant position of Fourier transforms are the essential advantages of the proposed method of analysis. The additional advantage of our approach is related to the system illuminated by a point source, because in this case the spectra are in the form of two-dimensional sets of points.

3. Field propagation through a system of diffraction gratings

First, we shall combine the Eqs. (7) and (4), and in this manner we shall obtain one relation describing the transmission and the field propagation simultaneously for one grating. For this purpose let us consider K-th diffraction grating of a multiple grating system (see Fig. 4), $K = 1, 2, \ldots, N$ (N – number of grating in the system). Moreover, let $V_{\mathrm{F},K}(\bar{\varrho})$ be the object spectrum distribution of diffraction grating G_K. It means that this distribution occurs in the object space of G_K on the sphere $\Sigma_{\mathrm{F},K}$ with the centre at M_K. In order to determine the object spectrum distribution $V_{\mathrm{F},K+1}(\bar{\varrho})$ of the diffraction grating G_{K+1} according to (7) and (4) we can write

$$V_{\mathrm{F},K+1}(\bar{\varrho}) = \sum_{m_K} c_{m_K} \exp(ikp_{K,K+1}\varrho^2)\, V_{\mathrm{F},K}(\bar{\varrho}+m_K\bar{\omega}_K) \tag{10}$$

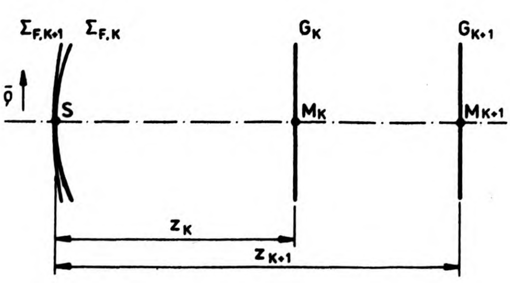


Fig. 4. Field propagation through one element of the multiple diffraction grating system as the change of the Fourier spectrum distributions on the spheres $\Sigma_{\mathrm{F},K}$ and $\Sigma_{\mathrm{F},K+1}$, respectively

where

$$p_{K,K+1} = \frac{1}{2}\left(\frac{1}{z_K} - \frac{1}{z_{K+1}}\right). \tag{11}$$

The spectrum $V_{F,K+1}(\bar{\varrho})$ occurs in the object space of the grating G_{K+1} on the sphere $\Sigma_{F,K+1}$ with the centre at M_{K+1}. $\bar{\omega}_K$ is the fundamental angular frequency in the vectorial form of K-th grating, c_{m_K} – coefficients of the Fourier series of the same grating.

Now, let G be a system of N diffraction gratings, illuminated by a point source S (Fig. 5). For simplicity, the first and last gratings are marked in the figure only.

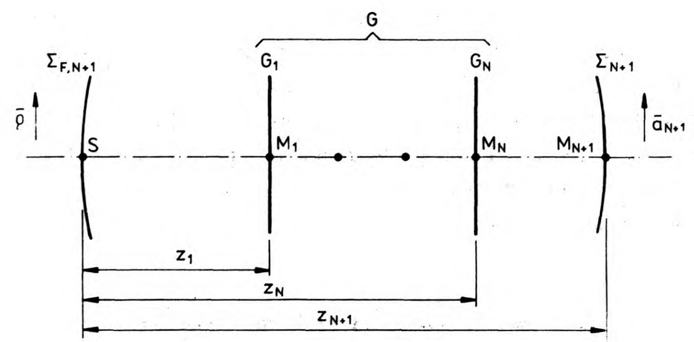


Fig. 5. Propagating field distribution on the sphere Σ_{N+1} and its Fourier transform on the sphere $\Sigma_{F,N+1}$ as a result of the propagation of a spherical wave through a multiple diffraction grating system G (S – light point source)

Taking into account the relation (10), for the first two diffraction gratings we can write

$$V_{F,2}(\bar{\varrho}) = \lambda z_1 V_0 \sum_{m_1} c_{m_1} \exp[ikp_{1,2}(m_1\bar{\omega}_1)^2]\,\delta(\bar{\varrho}+m_1\bar{\omega}_1), \tag{12a}$$

$$\begin{aligned} V_{F,3}(\bar{\varrho}) = \lambda z_1 V_0 \sum_{m_1}\sum_{m_2} c_{m_1} \exp[ikp_{1,2}(m_1\bar{\omega}_1)^2] \\ \times c_{m_2}\exp[ikp_{2,3}(m_1\bar{\omega}_1+m_2\bar{\omega}_2)^2]\,\delta[\bar{\varrho}+(m_1\bar{\omega}_1+m_2\bar{\omega}_2)], \end{aligned} \tag{12b}$$

because $V_{F,1}(\bar{\varrho}) = \lambda z_1 V_0\,\delta(\bar{\varrho})$.

Going on with the recurrent Eq. (10) for the subsequent gratings, we shall obtain for $N-1$ gratings the following relation:

$$V_{F,N}(\bar{\varrho}) = \lambda z_1 V_0 \sum_{m_1}\sum_{m_2}\dots\sum_{m_{N-1}} d_{N-1}\,\delta(\bar{\varrho}+\bar{\Omega}_{N-1}) \tag{13}$$

where the following denotations have been introduced:

$$d_M = \prod_{s=1}^{M} C_s, \tag{14a}$$

$$\bar{\Omega}_M = \sum_{t=1}^{M} m_t\bar{\omega}_t, \qquad M = 1, 2, \dots, N, \tag{14b}$$

and

$$C_s = c_{m_s} \exp(ikp_{s,s+1}\,\Omega_s^2). \tag{15}$$

The field distribution $V_{F,N}(\bar{\varrho})$ is the object spectrum of the N-th diffraction grating and it arises in the sphere $\Sigma_{F,N}$ not marked in the figure. The surface $\Sigma_{F,N}$, as the Fourier sphere, coincides with the point S, and its centre is located at the point M_N. Now, the question is what the field distribution is on a sphere Σ_{N+1} located in the image space of the whole system at the distance z_{N+1} from the source S (see Fig. 5). Using once more the relation (10), we shall obtain the spectrum of the demanded distribution in the form

$$V_{F,N+1}(\bar{\varrho}) = \lambda z_1 V_0 \sum_{m_1}\sum_{m_2}\ldots\sum_{m_{N-1}}\sum_{m_N} d_N\,\delta(\bar{\varrho}+\bar{\Omega}_N). \tag{16}$$

The last distribution arises on the sphere $\Sigma_{F,N+1}$ with the centre M_{N+1}. The field distribution on the sphere Σ_{N+1} can be found from the relation

$$V_{N+1}(\bar{A}) = \frac{1}{\lambda z_{N+1}} \mathrm{FT}^{-}\,[V_{F,N+1}(\bar{\varrho})] \tag{17}$$

where FT^{-} is the Fourier transform operator. Now, the quantity $\bar{A}$ equals $k\bar{a}_{N+1}/z_{N+1}$, where $\bar{a}_{N+1}$ is the radial vector position on the sphere Σ_{N+1}. Substituting Eq. (16) into (17) it will be

$$V_{N+1}(\bar{A}) = \frac{z_1}{z_{N+1}} V_0 \sum_{m_1}\sum_{m_2}\ldots\sum_{m_{N-1}}\sum_{m_N} d_N \exp(i\bar{A}\bar{\Omega}_N). \tag{18}$$

Summing up, the relations (18) and (16) give the field distributions $V_{N+1}(\bar{A})$ and $V_{F,N+1}(\bar{\varrho})$ on two characteristic spheres Σ_{N+1} and $\Sigma_{F,N+1}$, respectively, in the image space of a system of N diffraction gratings illuminated by a point source S (see Fig. 5). N is the arbitrary number of gratings, and in the extreme case it can be $N = 1$ (for the proof compare the relations (16) and (12a)). According to the Eq. (16) the Fourier spectrum is a set of points. The field distribution on the sphere Σ_{N+1} may be treated as a result of the interference of the light emitted by the points sources of the spectrum, which is presented in the mathematical form by the relation (18). The distribution of the field amplitude of the original source among the spectrum points is expressed by the moduli $|d_N|$, which result from (14a) and (15) as follows:

$$|d_N| = \prod_{s=1}^{N} |c_{m_s}|. \tag{19}$$

It means that the amplitude distribution depends on the ones in all grating spectra only. The amplitude distribution for s-th grating is defined by the amplitudes of the Fourier series harmonics $|c_{m_s}|$, $m = 0, \pm 1, \pm 2, \ldots$, and $s = 1, 2, \ldots, N$.

If we assume $d_N = |d_N|\exp(i\varphi_N)$, then according to (16), (14a) and (15) the secondary point sources of the spectrum are phase shifted with respect to the sphere $\Sigma_{F,N+1}$. The phase shifts are given by the relation

$$\varphi_N = \sum_{s=1}^{N} [\operatorname{Arg}(c_{m_s}) + kp_{s,s+1}\Omega_s^2]. \tag{20}$$

Thus, the phase shifts of points consist of the phase corrections related to the construction parameters of gratings $[\operatorname{Arg}(c_{m_s})]$ and the phase corrections depending on the grating positions. If all gratings are amplitude type and symmetrical, then $\operatorname{Arg}(c_{m_s})$ may equal 0 or π only for every s and m. In this case it is better to neglect $\operatorname{Arg}(c_{m_s})$ in (20) assuming the negative modulus of coefficients c_{m_s}. It means that Eq. (19) should have the form $|d_N| = \prod_{s=1}^{N} c_{m_s}$.

The form of the phase corrections in Eq. (20) related to the positions of gratings is adapted for the discription of the transitions between the adjacent gratings, because every component $\operatorname{Arg}(C_s) = kp_{s,s+1}\Omega_s^2$ concerns the phase changes introduced by the field propagation between the gratings s and $s+1$ ($s = 1, 2, \ldots, N-1$), as well as between N-th grating and the sphere Σ_{N+1}. The form of the phase corrections results from the applied process of analysis. In this case the influence of the position of a grating may be considered with comparison to the self-image of the preceding grating. For example, the fulfilment of the condition

$$kp_{K,K+1}\omega_K^2 = 2\pi M, \qquad M = \pm 1, \pm 2, \ldots, \tag{21}$$

signifies that the M-th self-image of the grating K is located in the plane of the grating $K+1$ [4]. Therefore, according to the definition (14b), the component $kp_{K,K+1}m_K^2\omega_K^2$ in (20) may be neglected. However, from practical point of view, the coincidences of the self-images of different gratings with the image sphere Σ_{N+1} (Fig. 5) are more essential. In order to adapt the relation (20) to the analysis of this coincidence, the order of the elements in it have to be transposed. Squaring the sum in parentheses and rearranging the terms, using the relation $p_{s,t} + p_{t,u} = p_{s,u}$, we can write

$$\varphi_N = \sum_{s=1}^{N} [\operatorname{Arg}(c_{m_s}) + kp_{s,N+1}m_s^2\omega_s^2] + 2\sum_{s=2}^{N} kp_{s,N+1}m_s\bar{\omega}_s\bar{\Omega}_{s-1} \tag{22}$$

instead of (20). The above form of the phase shift φ_N is similar to the results of the paper [3], obtained for two gratings. Although the relations (20) and (22) are fully equivalent, the second form is less clear. Moreover, the position of the sphere Σ_{N+1} (the change of z_{N+1}) changes, in the sum (20), the last component with $p_{N,N+1}$ only. This may facilitate analyses of the field distributions at different distances from the grating system. On the other hand, the relation (22) allows, in a more easy way, to take into account the condition

$$kp_{K,N+1}\omega_s^2 = 2\pi M, \qquad M = \pm 1, \pm 2, \ldots, \tag{23}$$

concerning the coincidence of M-th self-image of the grating K with the sphere Σ_{N+1}. In this case, the components $kp_{s,N+1}m_s^2\omega_s^2$ with $s = K$ in (22) may be neglected.

According to Eq. (18), the phase factor $\exp(i\bar{A}\bar{\Omega}_N)$ describing the field distribution of the field harmonics on the sphere Σ_{N+1} does not change its form during the changes of the sphere position (the changes of the distance z_{N+1}, see Fig. 5). This convenient property has been obtained by the introduction of the parametrized vector position $\bar{A}$. In order to determine parameters registered in the image plane, it is necessary to find the relation between the angular frequency $\bar{\Omega}_N$ related to the parametrized vectorial coordinate $\bar{A}$ and the distribution period $W_a^{(N+1)}$ in the linear measure of the same harmonic on the sphere Σ_{N+1}. The vectorial form of $\bar{\Omega}_N$ may be neglected, because the period W_a is defined in the direction of the harmonic changes. The quantity A equals ka_{N+1}/z_{N+1}, where a_{N+1} is the radial distance on Σ_{N+1}. Moreover, from (18) it follows: $\Omega_N = 2\pi/W_A$, where W_A is the harmonic period related to the quantity A. This means that $W_A = kW_a^{(N+1)}/z_{N+1}$, and

$$W_a^{(N+1)} = \frac{2\pi z_{N+1}}{k\Omega_N} = \frac{\lambda z_{N+1}}{\Omega_N}. \tag{24}$$

The same relation is between the periods $W_a^{(s)}$ of s-th diffraction grating and their fundamental angular frequency ω_s, i.e.,

$$W_a^{(s)} = \frac{\lambda z_s}{\omega_s}. \tag{25}$$

From the last equation it results, in particular, that in our approach the position change of a grating with given period $W_a^{(s)}$ changes the fundamental angular frequency ω_s.

4. System of diffraction gratings illuminated by a plane wave

The above analysis and the form of the obtained relations have been well adapted to configurations with the light source located at a finite distance from the gratings. In this case, a system of diffraction gratings is illuminated by a spherical wave, and the spectra for all gratings are located in the same plane at a finite distance. On the other hand, the system illuminated by a plane wave is the case of a great practical importance as well. Consequently, the problem of necessary changes in our equations arises, including this particular case.

For the source located at the finite distance the utility of the parametrized coordinate $\bar{A}$ was based on the shadow property of the propagating field projected from the source. Now, the quantity $\bar{A}$ is useless and the analogical shadow property of the field is fulfilled for the linear coordinate $\bar{a}$. Moreover, because the Fourier plane shifts to infinity, it is necessary to introduce the notion of the angular field distribution for it. For that purpose let $V_{\mathrm{F}}(\bar{\varrho})$ be a field distribution on a sphere Σ_F with the centre M (see Fig. 6). Physically, the intensity distribution $I_F(\bar{\varrho})$, $[I_F(\bar{\varrho}) = V_F(\bar{\varrho}) V_F^*(\bar{\varrho})]$, is proportional to the surface power density of the distribution related to the elementary area of the sphere Σ_F, ($I_F \sim dW/dS_F$). For the sphere Σ_F located at infinity ($z \to \infty$) the intensity distributions I_F, as well as the field distribution V_F become useless too, because both quantites are infinitesimal.

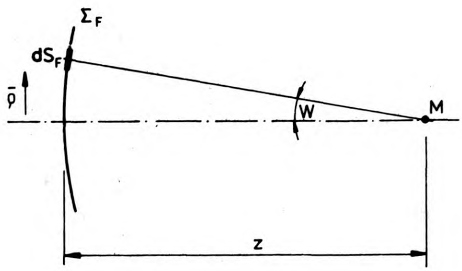


Fig. 6. Reference sphere Σ_F with the Fourier transform distribution and its elementary area dS_F

The angular intensity distribution $J_F(\bar{w})$ related to the intensity distribution $I_F(\bar{\varrho})$ is proportional to the optical power of the distribution per unit solid angle subtended by the element of the area dS_F at the point M. The vector $\bar{w}$ equals $\bar{\varrho}/z$, then the modulus w is the angular coordinate of the points of the sphere Σ_F as seen from the point M. The radial versor $\bar{w}^0$ defines the meridional plane. On account of the relation $J_F(\bar{w}) \sim z^2 dW/dS_F$ we can write $J_F(\bar{w}) = z^2 I_F(\bar{\varrho})$. It means that the angular field distribution $U_F(\bar{w})$ defined on the sphere Σ_F, $[J_F(\bar{w}) = U_F(\bar{w}) U_F^*(\bar{w})]$, and the field distribution $V_F(\bar{\varrho})$ are related by the expression

$$U_F(\bar{w}) = z V_F(\bar{\varrho}). \tag{26}$$

The quantity $U_F(\bar{w})$ is particularly useful for distributions located at infinity ($z \to \infty$).

Now, using Eqs. (25) and (26), as well as the general relation $\delta(c\bar{\varrho}) = \delta(\bar{\varrho})/c^2$, as in this case $z_s/z_t = 1$ ($s, t = 1, 2, \ldots, N+1$), Eqs. (16) and (18) with the defining expressions (14) and (15) can be easily transformed into the following forms of relations valid for the plane wave

$$U_{F,N+1}(\bar{R}) = \frac{4\pi^2 V_0}{\lambda} \sum_{m_1} \sum_{m_2} \cdots \sum_{m_{N-1}} \sum_{m_N} d'_N \delta(\bar{R} + \bar{\Omega}'_N), \tag{27}$$

$$V_{N+1}(\bar{a}) = V_0 \sum_{m_1} \sum_{m_2} \cdots \sum_{m_{N-1}} \sum_{m_N} d'_N \exp(i\bar{a}\bar{\Omega}'_N) \tag{28}$$

where:

$$\bar{R} = k\bar{w}, \tag{29a}$$

$$d'_N = \prod_{s=1}^{N} C'_s, \tag{29b}$$

$$\bar{\Omega}'_N = \sum_{s=1}^{N} m_s \bar{\omega}_s, \tag{29c}$$

$$C'_s = c_{m_s} \exp\left(\frac{i\Delta z_{s,s+1}\,\Omega_s'^2}{2k}\right), \tag{29d}$$

$$\omega'_s = \frac{2\pi}{W_a^{(s)}}, \tag{29e}$$

$V_{N+1}(\bar{a})$ is the field distribution on the plane Σ_{N+1} as the result of the propagation of the plane wave Σ_0 through the system G of N diffraction gratings (Fig. 7), $\bar{a}$ is the radial vector position, V_0 – amplitude of Σ_0. $U_{F,N+1}(\bar{R})$ designates the

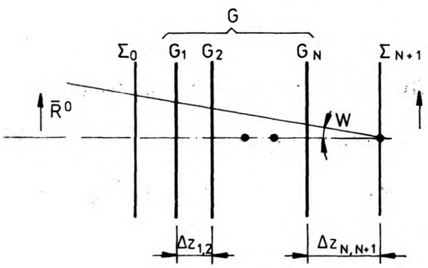


Fig. 7. Propagating field distribution on the plane Σ_{N+1} and its Fourier transform at infinity as a result of the propagation of the plane wave Σ_0 through a multiple diffraction grating system

angular field distribution in the angular spectrum of $V_{N+1}(\bar{a})$. The radial vector R is the parametrized angular coordinate of the spectrum, the angle w is marked in the figure. $W_a^{(s)}$ designates the fundamental period of the s-th diffraction grating in the linear measure. $\Delta z_{s,s+1}$ is equal to the distance between the gratings s and $s+1$ ($s = 1, 2, \ldots, N-1$). $\Delta z_{N,N+1}$ is the distance between the plane Σ_{N+1} and the last diffraction grating of the system. As before, c_{m_s} are the coefficients of the Fourier series of s-th diffraction grating. The direction of the vector $\bar{\omega}'_s$ coincides with the direction of periodical changes of the s-th diffraction grating.

It should be remarked that in both cases of the light source located at finite and infinite positions, the different quantities have been parametrized, i.e., the quantity $\bar{A}$ is related to the image sphere (Eq. (1)), and $\bar{R}$ – to the spectrum (Eq. (29a)). We have assumed that it is convenient to parametrize the quantities describing the distributions which can be displaced to infinity. Certainly, for the finite distance of the source the determination of the field distribution at infinity may be interesting. Unfortunately, the assumed parametrization would complicate parallel analyses of the field propagation through a system with the source at finite and infinite positions. For such comparative considerations, the equations related to the plane wave are recommended, except for Eq. (29d), because the image sphere Σ_{N+1} (or plane) must be at the finite and constant position. In place of it, according to Eqs. (15), (14b) and (25), we can write

$$C'_s = c_{m_s} \exp\left[\frac{ip_{s,s+1}}{k}\Big(\sum_{t=1}^{s} m_t z_t \bar{\omega}'_t\Big)^2\right]. \tag{30}$$

5. Two-grating system

Two-grating system is chosen to demonstrate the utility of our approach (see Fig. 8). In the place of Eqs. (18) and (16) we can write

$$V_3(\bar{A}) = \frac{z_1 V_0}{z_3} \sum_{m_1} \sum_{m_2} d_2 \exp(i\bar{A}\bar{\Omega}_2), \tag{31}$$

$$V_{F,3}(\bar{\varrho}) = \lambda z_1 V_0 \sum_{m_1} \sum_{m_2} d_2 \delta(\bar{\varrho}+\bar{\Omega}_2) \tag{32}$$

where, according to (14) and (15),

$$d_2 = c_{m_1} c_{m_2} \exp(ikp_{1,2} m_1^2 \omega_1^2) \exp[ikp_{2,3}(m_1\bar{\omega}_1 + m_2\bar{\omega}_2)^2], \tag{33a}$$

$$\bar{\Omega}_2 = m_1\bar{\omega}_1 + m_2\bar{\omega}_2. \tag{33b}$$

As $p_{1,2}+p_{2,3} = p_{1,3}$, the Eq. (33a) may be easily rearranged into the form

$$d_2 = c_{m_1} c_{m_2} \exp[ik(p_{1,3} m_1^2 \omega_1^2 + p_{2,3} m_2^2 \omega_2^2 + 2p_{2,3} m_1 m_2 \bar{\omega}_1 \bar{\omega}_2)], \tag{34}$$

that corresponds to $\mathrm{Arg}(d_2)$ given by Eq. (22). It may be shown that the Eq. (34) is in full agreement with results of the paper [3].

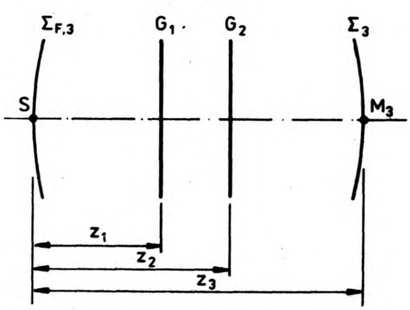
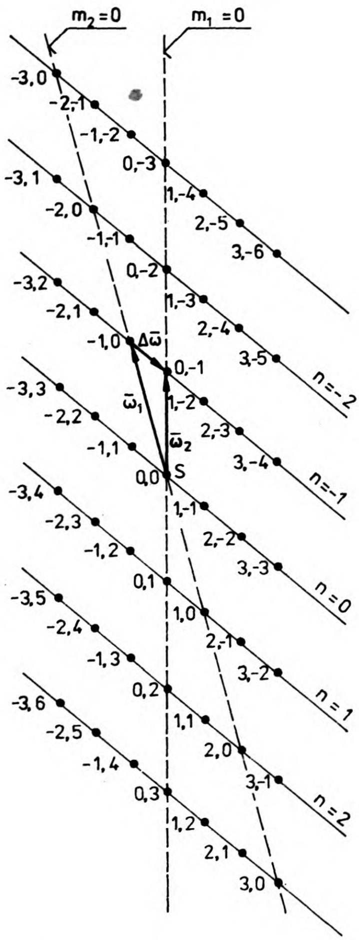


Fig. 8. Propagating field distribution on the sphere Σ_3 and its Fourier transform as a result of the propagation of the spherical wave with the centre S through a double-diffraction system (G_1, G_2 – diffraction gratings)

It is worth remembering that the field distribution $V_3(\bar{A})$ arises on the sphere Σ_3 with the centre S, and its spectrum $V_{F,3}(\bar{\varrho})$ – on the sphere $\Sigma_{F,3}$ with the centre M_3. The central part of two-grating spectrum is shown in Fig. 9. The grating system is illuminated by a point source S. Every point of the spectrum is marked in the figure by labels composed of two numbers m_1, m_2, which are the numbers of the harmonics of the Fourier series for first and second gratings, respectively. The directions, perpendicular to the lines of both gratings, are marked by the sections $m_1 = 0$ and $m_2 = 0$. V_0 in the Eq. (31) is the wave amplitude in the plane of the first grating. Therefore, the factor $z_1 V_0/z_3$ can be interpreted as the wave amplitude on the sphere Σ_3 with no gratings present.

If the second grating is located in one of the self-images of the first grating, then $kp_{1,2}\omega_1^2 = 2\pi M$ $(M = \pm 1, \pm 2, \ldots)$, and in (33a) we can put $\exp(ikp_{1,2} m_1^2 \omega_1^2) = 1$. The simultaneous coincidences of the self-images of the first



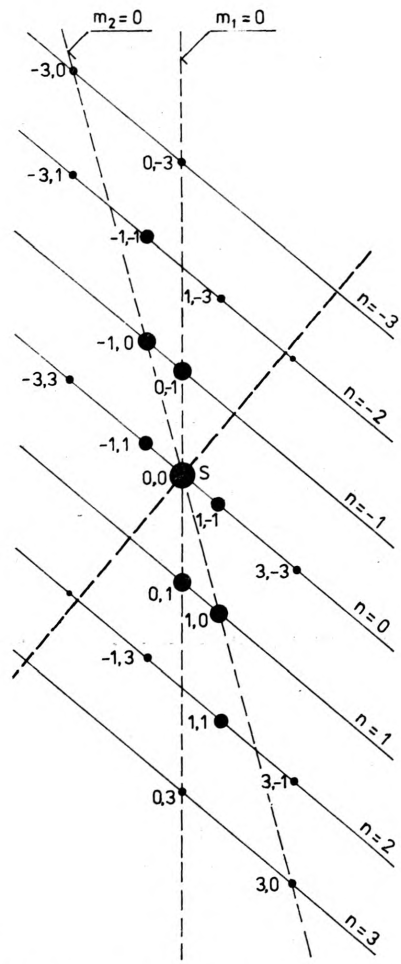


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Fig. 9. Fourier spectrum of two inclined diffraction gratings illuminated by the point source S

Fig. 10. Fourier spectrum of two inclined Ronchi gratings illuminated by the point source S

and second gratings with the sphere Σ_3 relate to the fulfilment of the two following conditions together

$$\left.\begin{aligned} kp_{1,3}\,\omega_1^2 &= 2\pi M_1 \\ kp_{2,3}\,\omega_2^2 &= 2\pi M_2 \end{aligned}\right\} \tag{35}$$

(M_1, M_2 – integers), in this case the relation (34) can be reduced to

$$d_2 = c_{m_1} c_{m_2} \exp(2ikp_{2,3}\, m_1 m_2 \bar{\omega}_1 \bar{\omega}_2). \tag{36}$$

5.1. Perpendicular lines of gratings

Let us denote the components of the vectors $\bar{\varrho}$ and $\bar{A}$ in the directions coincident with $\bar{\omega}_1$ and $\bar{\omega}_2$ by ϱ_1, ϱ_2 and A_1, A_2, respectively. As $\bar{\omega}_1 \bar{\omega}_2 = 0$, $\delta(\bar{\varrho})$

$= \delta(\varrho_1)\delta(\varrho_2)$, $\bar{\Omega}_2 \bar{A} = m_1 \omega_1 A_1 + m_2 \omega_2 A_2$, we can write (using (34) in the place of relations (31) and (32))

$$V_3(\bar{A}) = \frac{z_1 V_0}{z_3} \Big[\sum_{m_1} c'_{m_1} \exp(im_1 \omega_1 A_1)\Big] \Big[\sum_{m_2} c'_{m_2} \exp(im_2 \omega_2 A_2)\Big], \tag{37}$$

$$V_{F,3}(\bar{\varrho}) = \lambda z_1 V_0 \Big[\sum_{m_1} c'_{m_1} \delta(\varrho_1 + m_1 \omega_1)\Big] \Big[\sum_{m_2} c'_{m_2} \delta(\varrho_2 + m_2 \omega_2)\Big] \tag{38}$$

where:

$$c'_{m_1} = c_{m_1} \exp(ikp_{1,3} m_1^2 \omega_1^2), \tag{39a}$$

$$c'_{m_2} = c_{m_2} \exp(ikp_{2,3} m_2^2 \omega_2^2). \tag{39b}$$

Taking into account the above equations it can be concluded that the propagation of the spherical wave through a system of two diffraction gratings with the mutually perpendicular lines may be reduced to the independent considerations of two component fields for two principal sections. The resultant field equals the product of the component fields.

5.2. Moiré phenomenon in coherent light

It has been shown that the field in the image space of a system of diffraction gratings illuminated by a point source appears as a result of the interference of the spherical waves generated by all point sources of the Fourier spectrum. The interference of waves emitted by two points gives the linear fringe structure, for which the direction of the field changes is parallel to the line joining both sources, and the frequency of fringes is proportional to the distance between the sources. It is necessary to remark that the linear fringes arise on the sphere. On the plane the fringes are in the form of hyperbolae, which can be approximated in the paraxial region by the straights. Considering the Fig. 9, if the difference of the fundamental frequencies $\Delta\bar{\omega} = \bar{\omega}_2 - \bar{\omega}_1$ of both gratings is considerably smaller than the component frequencies ($\Delta\omega \ll \omega_1$ and $\Delta\omega \ll \omega_2$), the fringes may arise with small frequency $\Delta\bar{\omega}$. In this case, one says about moiré fringes.

According to Eq. (26), the intensity distribution generated by the double-diffraction system can be found from the expression

$$I_3(\bar{A}) = V_3(\bar{A}) V_3^*(A). \tag{40}$$

In the case of moiré phenomenon analyses the intensity changes with high frequencies are not detected. This may be obtained using a detector integrating the intensity in the direction perpendicular to $\Delta\bar{\omega}$. Hence, our further considerations may be limited to the study of the intensity changes with the frequency of magnitude $\Delta\omega$ only. Now, on account of (31), the relation (40) may be reduced to the form

$$I_3(\bar{A}) = \sum_n I_3^{(n)}(\bar{A}) \tag{41}$$

where $I_3^{(n)}(\bar{A})$ is the intensity generated by all points sources with labels m_1, m_2 fulfilling the relation $n = m_1 + m_2$. The points with the constant values of n are marked in Fig. 9. For example, the component of the intensity $I_3^{(1)}(\bar{A})$ is given by the interference of light emitted by the points with labels (3, −2), (2, −1), (1, 0), (0, 1), (−1, 2), (−2, 3), and so on.

The above equations may be used to analyse moiré phenomenon with the aid of a computer. In our case, we shall present an analytical method study a double-diffraction system with Ronchi gratings (equal widths of the bright and dark lines). The coefficients of Fourier series representing both gratings can be found from the relation $c_m = 0.5$ sinc $(0.5\pi m)$. They may be expressed in a more general form, taking into account the grating shifts with comparison to their symmetrical positions. Beside the complication of the analysis, it does not introduce any new elements except for the respective shifts of moiré fringes. This means that a shift of one of the gratings, measured in the fractional part of the grating period, introduces the equivalent shift of moiré fringes, measured in the fractional part of the period of fringes. For this reason, the influence of the shifts of gratings is not considered.

Analogically to Fig. 9, the central part of the Fourier spectrum of a double-diffraction system, composed of Ronchi gratings, is shown in Fig. 10. The field amplitude distribution is symbolized by the diameters of the spots (that equal $|c_{m_1} c_{m_2}|$), and the power distribution – by their area. As $c_m = 0$ for m even, in every section with n odd ($n = m_1 + m_2$) there are two spots only [$(0, m_2)$ and $(m_1, 0)$ for $m_1 = m_2 = m_2 = \pm 1, \pm 3, \ldots$] with equal amplitudes. On the other hand, for every section with n even, except for $n = 0$, we have the equidistant and infinite set of spots with the period equal to $2\Delta\omega$. For the section $n = 0$, additionally to the situation described for n even and $n \neq 0$, the spot (0, 0) is observed.

To determine the intensity distribution of moiré fringes on the sphere Σ_3 (see Fig. 8) it is sufficient, according to (41), to sum up the partial intensity distributions generated by the spots of every section with n constant. The partial distributions are periodical with the frequencies equal to module $\Delta\omega$. It means that every partial intensity distribution, and consequently, the intensity distribution of the moiré fringes, can be described by the following series:

$$I_3(\bar{A}) = I_0 \sum_p j_p \cos(p\Delta\bar{\omega}\bar{A} + f_p), \tag{42}$$

$I_0 = (z_1 V_0/z_3)^2$; j_p, f_p designate the amplitude and the phase, respectively, of p-th intensity harmonic for the intensity normalized to $I_0 = 1$ ($p = 0, 1, 2, \ldots$).

For example, let us consider the section with $n = 1$, where the spots (1, 0) and (0, 1) are located. On account of the relations (31), (33b) and (34) the partial field distribution will be in the form

$$V_3^{(1)}(\bar{A}) = I\frac{z_1 V_0}{z_3}[c_1 c_0 \exp(\mathrm{ik}p_{1,3}\,\omega_1^2)\exp(i\bar{A}\bar{\omega}_1) + c_0 c_1 \exp(ikp_{2,3}\,\omega_2^2)\exp(i\bar{A}\bar{\omega}_2)]. \tag{43}$$

For the partial intensity distribution we can write

$$I_3^{(1)}(\bar{A}) = V_3^{(1)}(\bar{A})\, V_3^{(1)}(\bar{A})^* = 2I_0 c_0^2 c_1^2 [1+\cos(\Delta\bar{\omega}\bar{A}+f_1)] \tag{44}$$

where, as previously, $\Delta\bar{\omega} = \bar{\omega}_2 - \bar{\omega}_1$, and

$$f_1 = kp_{2,3}\,\omega_2^2 - kp_{1,3}\,\omega_1^2. \tag{45}$$

The distribution $I_3^{(1)}(\bar{A})$ contains two intensity harmonics with the frequencies equalling zero and $\Delta\bar{\omega}$. Moreover, according to the general relation (42), $j_0 = j_1 = 2c_0^2 c_1^2$ and $f_0 = 0$, $j_p = 0$ for $p > 1$.

Analogically, we can show that the spots $(0, -1)$ and $(-1, 0)$ – the section $n = -1$, see Fig. 10 – give the partial intensity distribution $I_3^{(-1)}(\bar{A})$, which differs from the distribution $I_3^{(1)}(\bar{A})$ defined by the Eq. (44) by the sign of the phase f_1 of the first intensity harmonic, only.

Both intensity distributions of fringes $I_3^{(1)}(\bar{A})$ and $I_3^{(-1)}(\bar{A})$ may be observed in the system shown in Fig. 11. Let G_1 and G_2 be the Ronchi gratings with equal

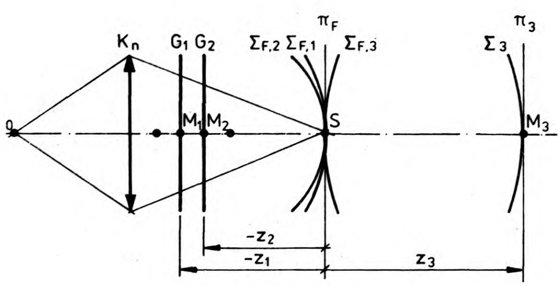

Fig. 11. Double-diffraction system of the gratings G_1 and G_2 demonstrating the moiré fringe phenomenon with filtering of the spectrum. The system is illuminated by the point source S_0 through the condenser K_n (π_F – filter plane, π_3 – observation plane)

linear periods. Moreover, let the gratings be illuminated by a point source S_0 through a condenser K_n. If the point S is the image of S_0 given by K_n, then the spectra of both gratings arise successively on the spheres $\Sigma_{F,1}$ and $\Sigma_{F,2}$. The centres of the spheres are at M_1 and M_2, respectively. The phase shifts between the spheres $\Sigma_{F,1}$ and $\Sigma_{F,2}$ are related to the field propagation between the gratings G_1 and G_2. The moiré fringes arise on the sphere Σ_3 with the centre S, and its spectrum is on the sphere $\Sigma_{F,3}$ with the centre M_3. The proposed configuration is convenient from the experimental point of view because all spectra and the sphere Σ_3 are real in one space. In this manner, the spectrum can be easily filtered with respective diaphragm located at the plane π_F, and the result of filtering can be registered in the plane π_3. If the diaphragm transmits the spots from the section $n = 1$ or $n = -1$, the fringes observed in the plane π_3 have the same period and direction, alternatively. However, in the general case, they are not coincident. The changes of the distance z_3 between the plane π_3 and the spectrum plane introduce the shifts of both fringe images, but in the directions mutually opposite. The shifts of fringes are related in Eq. (45) to the change of the phase f_1, induced by the change of z_3, the opposite directions concern the opposite signs of the phase f for both distributions.

If the spots of the sections $n = 1$ and $n = -1$ are transmitted simultaneously, the stationary fringes will be observed with the contrast changes during the change of the distance z_3. This fact results immediately from the relation

$$I_3^{(1,-1)}(\bar{A}) = I_3^{(1)}(\bar{A}) + I_3^{(-1)}(\bar{A}) = 4I_0 c_0^2 c_1^2 [1 + \cos(f_1)\cos(\Delta\bar{\omega}\bar{A})]. \tag{46}$$

The same conclusions about fringe shifts and fringe contrast can be deduced directly from the Fourier distribution. For this purpose, it is sufficient to remark that the direction of the linear fringes coincides with the dashed line shown in Fig. 10.

According to (46) the fringes for $\cos(f_1) = 1$ and $\cos(f_1) = 0$ are shown in Fig. 12. The fringes of the component intensity distributions may be observed at the edge of Fig. 12b because both areas of the fringes are not exactly coincident. The

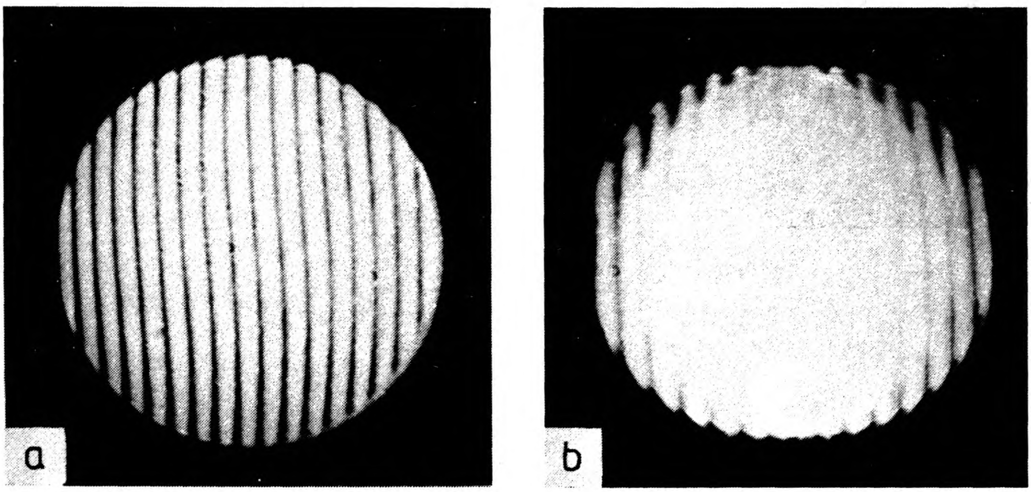

Fig. 12. Photographs of the moiré fringes obtained in the configuration shown in Fig. 11 for two different distances z_3: **a** – maximum contrast of the fringes, **b** – minimum contrast of the fringes

perceptible curvature of fringes is connected with the aberrations of the condenser. The spectrum of the double-diffraction system registered in the plane π_F, is shown in Fig. 13. It can be seen that the used gratings differ from Ronchi ones. Because of that the experimental demonstration of the relation (46) shown in Fig. 12 was possible only in the case the filtering of the sections and the filtering op proper spots in the sections $n = 1$ and $n = -1$. In spite of the equal linear periods of both gratings, the fundamental angular frequencies ω_1 and ω_2 are slightly different because the gratings G_1 and G_2 are at the different distances from the point S. On the base of the spectrum image of two Ronchi gratings shown in Fig. 10 we can conclude that the first intensity harmonic, beside the sections $n = 1$ and $n = -1$, can be generated by the spots $(1, -1)$, $(0, 0)$ and $(-1, 1)$ of the section $n = 0$ only. For these spots the partial intensity distribution will be given by the following

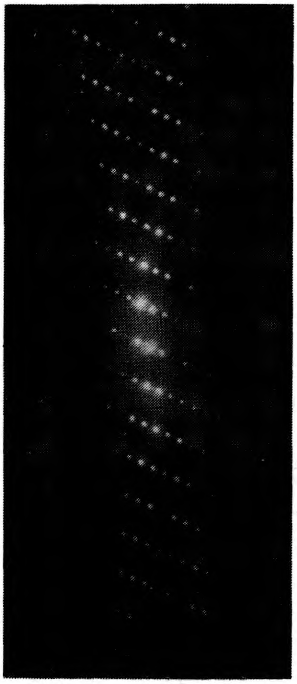

Fig. 13. Intensity distribution in the Fourier plane for two inclined diffraction gratings

expression

$$I_3^{(0)}(\bar{A}) = I_0\,[c_0^4 + 2c_1^4 + 4c_0^2 c_1^2 \cos(\Delta\varphi)\cos(\Delta\bar{\omega}\bar{A}) + 2c_1^2 \cos(2\Delta\bar{\omega}\bar{A}) \tag{47}$$

where

$$\Delta\varphi = kp_{1,3}\,\omega_1^2 + kp_{2,3}\,\omega_2^2 - 2kp_{2,3}\,\bar{\omega}_1\,\bar{\omega}_2 . \tag{48}$$

In this case we have three intensity harmonics, the contrast of the first harmonic depending on the position of the sphere Σ_3 only.

Summing up the Eqs. (46) and (47) the intensity distribution on the sphere Σ_3 for 7 central spots of Fig. 10 will be obtained. It is worth remarking that the even harmonics of the intensity ($p = 2, 4, \ldots$ in the Eq. (42)) will be generated by the sections with n even only.

So far, as the first intensity harmonic of moiré fringes for the Ronchi gratings can be studied analytically (this conclusion may be extended to the odd harmonics), the second or every higher even harmonics are composed of the infinite terms, and the computer analysis is necessary. The last inference concerns the odd intensity harmonics not only in the case of Ronchi gratings.

6. Conclusion

The general equations for the Fresnel field of a dffraction gratings system illuminated by a point source have been derived. The simplicity of the procedure results from the analysis of the Fourier spectrum of the field propagating through the system. The advantages of the proposed approach can be seen particularly well during the analysis of the moiré fringes with filtering of their spectrum.

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Область Френеля для системы дифракционных решеток освещенной точечным источником

Определены общие формулы в области Френеля для системы состоящей из произвольного количества дифракционных решеток. Анализ основан на исследовании изменений спектра Фурье во время распространения волны через дифракционные решетки. Обнаружено применение анализа для изучения полос Моры возникающих в двойной дифракционной системе.