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Analysis of the features of quantum frequency standards given in the forms of maximum of output power vs frequency curve of a single-frequency gas laser and Lamb dip centre

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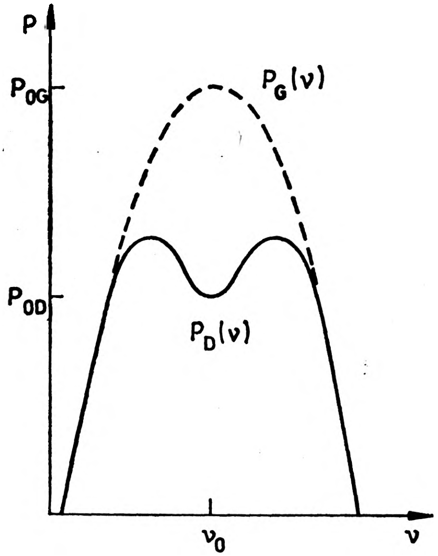
The purpose of this paper is to give an original theoretical explanation of the differences of fundamental features of quantum frequency standards which up till now have been found out only experimentally. It has been proved, that the maximum of the output power vs the frequency curve of single-frequency gas laser (further called the maximum of OPF curve) in the case of approximately Gaussian OPF curve is frequency standard about two times better than in the case of approximately Lorentzian OPF curve. It has been proved also, that Lamb dip centre is frequency standard about ten times better than the maximum of approximately Gaussian OPF curve and about twenty times better than the maximum of approximately Lorentzian OPF curve.

In gas lasers, in which total pressure of gas gain mixture is high (for example, in CO_2-N_2-He laser, pressure of about 2000 Pa or higher, at typical temperature of about 400 K [1]), Lamb dip does not occur. It is so, because homogeneous broadening of vibration-rotation transitions, caused mostly by collision broadening, is then comparable with inhomogeneous broadening, caused by Doppler effect. However, when the total pressure of gas gain mixture is low (for example, in CO_2-N_2-He laser, pressure of about 130 Pa or less, at typical temperature of about 400 K [2]), Lamb dip is very sharply outlined, because then inhomogeneous broadening predominates over the homogeneous one.

An emission line of gain medium can be described by the Lorentzian function [3–5] in high total pressure range. The output power vs frequency curve of single-frequency gas laser, further called OPF curve, resulting from such kind of gain medium, can be approximately described by Lorentzian function as well. In low total pressure range Lamb dip is also described by Lorenztian function, and appears in the background of OPF curve, which is approximately described by the Gaussian function [3–5] (see figure). The background curve may be approximately described by the Gaussian function since due to Doppler effect it is inhomogeneously broadened. Lamb dip appears in the centre of background curve and takes its full width at half the maximum value (further called FWHM value) about ten times less than FWHM value of the background curve.

In mean pressure range (from about 130 Pa to about 2000 Pa in the case of CO_2-N_2-He laser) OPF curve shape results from inhomegeneous broadening, caused by Doppler effect. This curve may be approximately described by Gaussian function, like the background curve in low pressure range.



Output power vs frequency curve of single-frequency gas laser (OPF curve): ν — frequency of laser radiation, ν_0 — central frequency of OPF curve, $P_G(\nu)$ — output power of laser radiation, when OPF curve is approximately described by the Gaussian function, $P_D(\nu)$ — output power of laser radiation, when Lamb dip appears in the background of approximately Gaussian OPF curve, $P_{0D} = P_D(\nu_0), P_{0G} = P_G(\nu_0)$

The FWHM values of OPF curve in mean and high pressure ranges are similar. For example, in CO_2-N_2-He laser these values vary from about 55 MHz [6] to about 150 MHz [1]. For the same type of laser the FWHM value of background curve in low pressure range is about 50 MHz, and the FWHM value of Lamb dip is about 5 MHz [7].

The OPF curve in the case, which can be approximately described by Lorentzian function, has an approximate analytical form

$$P_L(\nu) \simeq P_{0L}\frac{(\Delta\nu_L/2)^2}{(\nu-\nu_0)^2+(\Delta\nu_L/2)^2} \tag{1}$$

where: $P_L(\nu)$ — output power of laser radiation when OPF curve is approximately described by the Lorentzian function; ν_0 — central frequency of OPF curve; $P_{0L} \simeq P_L(\nu_0)$; $\Delta\nu_L$ — FWHM value of OPF curve (width at $P_L(\nu) \simeq P_{0L}/2$).

The OPF curve in the case, which can be approximately described by the Gaussian function, has an approximate analytical form

$$P_G(\nu) \simeq P_{0G}\exp\left[-\left(\frac{\nu-\nu_0}{\Delta\nu_{1/2}/2}\right)^2\right] \tag{2}$$

where: $P_G(\nu)$ — output power of laser radiation, when OPF curve is approximately described by the Gaussian function; $P_{0G} \simeq P_G(\nu_0)$; $\Delta\nu_{1/e}$ — full width at $1/e$ maximum value (width at $P_G(\nu) \simeq P_{0G}/e$), e being the base of a natural logarithm.

In the case of function (2) there is a following interdependence between FWHM value, $\Delta\nu_G$, and $1/e$ — width, $\Delta\nu_{1/e}$:

$$\Delta\nu_G \simeq (\ln 2)^{1/2}\Delta\nu_{1/e} \simeq 0.832\,\Delta\nu_{1/e}. \tag{3}$$

In the process of laser frequency stabilization by means of automatic system, an auxiliary modulation is used [8]. The frequency ν of laser radiation is then described by the formula

$$\nu = \nu_s + D_m \sin(2\pi\nu_m t) \tag{4}$$

where: ν_s — frequency of laser radiation without the auxiliary modulation; D_m — frequency deviation of laser radiation caused by the auxiliary modulation; ν_m — frequency of auxiliary modulation; t — time.

When the detunings $(\nu-\nu_0)$ of laser frequency ν from the central frequency ν_0 of OPF curve are little, having the form

$$|\nu-\nu_0| \lesssim 0.1\frac{\Delta\nu_{1/2}}{2}, \tag{5}$$

(where $\Delta\nu_{1/2}$ — FWHM value of OPF curve), the interdependence between laser output power $P(\nu)$ and laser frequency ν (the OPF curve) may be presented in an approximated form by the expansion in the Taylor series of $P(\nu)$ function in the neighbourhood of the central frequency ν_0 of OPF curve*. This expansion in the Taylor series has a form

$$P(\nu) = P(\nu_0) + \frac{(\nu-\nu_0)^2}{2!}\left.\frac{d^2P(\nu)}{d\nu^2}\right|_{\nu=\nu_0} + \ldots, \tag{6}$$

in which is taken into consideration that

$$\left.\frac{d^nP(\nu)}{d\nu^n}\right|_{\nu=\nu_0} = 0, \tag{7}$$

when n is odd integer. Formula (7) follows from the symmetry of OPF function $P(\nu)$.

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* The condition of little detunings (5) is determined in practice by the difference between the central frequency ν_0 of OPF curve and the frequency of the nearest extremum of OPF curve derivative. This practical condition of little detunings has a form

$$|\nu-\nu_0| \lesssim \varkappa_{\text{lin}}\frac{\Delta\nu_{1/2}}{2} \tag{5a}$$

where the value of linearity coefficient $\varkappa_{\text{lin}}$ is about 0.58 in the case of Lorentzian OPF curve or of Lamb dip and about 0.85 in the case of Gaussian OPF curve.

In the case of approximately Lorentzian OPF curve (formula (1)) we get

$$\left.\frac{d^2P_L(\nu)}{d\nu^2}\right|_{\nu=\nu_0} \simeq -\frac{8P_{0L}}{(\Delta\nu_L)^2}. \tag{8}$$

In the case of approximately Gaussian OPF curve (formula (2)) we get

$$\left.\frac{d^2P_G(\nu)}{d\nu^2}\right|_{\nu=\nu_0} \simeq -\frac{2P_G(\nu_0)}{(\Delta\nu_{1/e}/2)^2} = -\frac{8P_{0G}\ln 2}{(\Delta\nu_G)^2}. \tag{9}$$

It is possible to show that in the case of Lamb dip we get

$$\left.\frac{d^2P_D(\nu)}{d\nu^2}\right|_{\nu=\nu_0} \simeq \frac{8P_{0D}}{(\Delta\nu_D)^2}(1-2\zeta_D^{-2}\ln 2) \tag{10}$$

where: P_{0D} — output power of laser radiation in Lamb dip centre ν_0; ζ_D — coefficient of Lamb dip narrowing.

The coefficient ζ_D of Lamb dip narrowing is defined

$$\zeta_D = \Delta\nu_G/\Delta\nu_D \tag{11}$$

where: $\Delta\nu_D$ — full width at the half depth value of Lamb dip; $\Delta\nu_G$ — FWHM value of background curve $P_G(\nu)$ (see figure). This coefficient has in practice the following values:

$$\zeta_D \simeq 5-10. \tag{12}$$

When ζ_D increases then Lamb dip contrast decreases, and vice versa.

The approximated form of functions $P_L(\nu)$, $P_G(\nu)$, $P_D(\nu)$ under little detunings condition (5) results from the formulae (6)–(10) and is expressed by the respective relations:

$$P_L(\nu) \simeq P_{0L}\{1-(\delta_L/2)^{-2}[(\nu-\nu_0)/\nu_0]^2\}, \tag{13}$$

$$P_G(\nu) \simeq P_{0G}\{1-(\delta_G/2)^{-2}[(\nu-\nu_0)/\nu_0]^2\ln 2\}, \tag{14}$$

$$P_D(\nu) \simeq P_{0D}\ \{1+(\delta_D/2)^{-2}[(\nu-\nu_0)/\nu_0]^2(1-2\zeta_D^{-2}\ln 2)\}, \tag{15}$$

in which

$$\delta_L = \Delta\nu_L/\nu_0, \tag{16}$$

$$\delta_G = \Delta\nu_G/\nu_0, \tag{17}$$

$$\delta_D = \Delta\nu_D/\nu_0. \tag{18}$$

Substituting (4) to (13)–(15) we get a general formula

$$P(\nu) \simeq A+B\sin(2\pi\nu_m t)+C\cos(4\pi\nu_m t), \tag{19}$$

in which the coefficients A, B, C in Lorentzian curve, Gaussian curve and Lamb dip, are defined accordingly by the following formulae:

$$A_{\mathrm{L}} = P_{0\mathrm{L}}\left\{1-\left(\frac{\delta_{\mathrm{L}}}{2}\right)^{-2}\left[\left(\frac{\nu_{\varepsilon}-\nu_0}{\nu_0}\right)^2+\frac{1}{2}\left(\frac{D_{\mathrm{m}}}{\nu_0}\right)^2\right]\right\}, \tag{20}$$

$$A_{\mathrm{G}} = P_{0\mathrm{G}}\left\{1-\left(\frac{\delta_{\mathrm{G}}}{2}\right)^{-2}\left[\left(\frac{\nu_{\varepsilon}-\nu_0}{\nu_0}\right)^2+\frac{1}{2}\left(\frac{D_{\mathrm{m}}}{\nu_0}\right)\right]\ln 2\right\}, \tag{21}$$

$$A_{\mathrm{D}} = P_{0\mathrm{D}}\left\{1+\left(\frac{\delta_{\mathrm{D}}}{2}\right)^{-2}\left[\left(\frac{\nu_{\varepsilon}-\nu_0}{\nu_0}\right)^2+\frac{1}{2}\left(\frac{D_{\mathrm{m}}}{\nu_0}\right)^2\right](1-2\zeta_{\mathrm{D}}^{-2}\ln 2\right\}, \tag{22}$$

$$B_{\mathrm{L}} = -K_{\mathrm{L}}\frac{D_{\mathrm{m}}}{\nu_0}\frac{\nu_{\varepsilon}-\nu_0}{\nu_0}, \tag{23}$$

$$B_{\mathrm{G}} = -K_{\mathrm{G}}\frac{D_{\mathrm{m}}}{\nu_0}\frac{\nu_{\varepsilon}-\nu_0}{\nu_0}, \tag{24}$$

$$B_{\mathrm{D}} = +K_{\mathrm{D}}\frac{D_{\mathrm{m}}}{\nu_0}\frac{\nu_{\varepsilon}-\nu_0}{\nu_0}, \tag{25}$$

$$C_{\mathrm{L}} = +\frac{K_{\mathrm{L}}}{4}\left(\frac{D_{\mathrm{m}}}{\nu_0}\right)^2, \tag{26}$$

$$C_{\mathrm{G}} = +\frac{K_{\mathrm{G}}}{4}\left(\frac{D_{\mathrm{m}}}{\nu_0}\right)^2, \tag{27}$$

$$C_{\mathrm{C}} = -\frac{K_{\mathrm{D}}}{4}\left(\frac{D_{\mathrm{m}}}{\nu_0}\right)^2. \tag{28}$$

Coefficients K_{L}, K_{G}, K_{D} in formulae (23)–(28) are respectively defined by the following formulae:

$$K_{\mathrm{L}} = \frac{8P_{0\mathrm{L}}}{\delta_{\mathrm{L}}^2}, \tag{29}$$

$$K_{\mathrm{G}} = \frac{8P_{0\mathrm{L}}\ln 2}{\delta_{\mathrm{G}}^2}, \tag{30}$$

$$K_{\mathrm{D}} = \frac{8P_{0\mathrm{D}}}{\delta_{\mathrm{D}}^2}(1-2\zeta_{\mathrm{D}}^{-2}\ln 2). \tag{31}$$

Synchronous detection applied in the process of laser frequency stabilization by means of automatic system [8] to a signal of the frequency ν_{m}, yields the output voltage of synchronous detector. This voltage is proportional to the value of coefficient B_{L} or B_{G} or B_{D} (formulae (23)–(25)) in the case of frequency standards given in the approximate form of Lorentzian curve, Gaussian curve or Lamb dip, respectively. This output voltage is independent of the values of coefficients A_L, A_G, A_D (formulae (20)–(22)) and of C_L, C_G, C_D (formulae (26)–(28)). Therefore, this voltage is proportional to the relative detuning value $[(\nu_\varepsilon - \nu_0)/\nu_0]$ of laser from the central frequency ν_0 of frequency standard, and to the value of relative frequency deviation (D_m/ν_0) of laser subject to auxiliary modulation. This conclusion is true when the condition (5), in which ν is described by formula (4), is satisfied.

One of the parameters characterizing the system of automatic frequency stabilization of gas laser (further called AFSL system) is the so-called relative threshold detuning [9], which may be also called a relative threshold error. It is a minimal relative detuning $(|\nu_\varepsilon - \nu_0|/\nu_0)$ corrected automatically. It is possible to show that the value of relative threshold detuning (further called RTD value) is directly proportional to the value of coefficient $K_L^{-1/2}$ in the case of frequency standard given in the approximate form of Lorentzian curve, or to $K_G^{-1/2}, K_D^{-1/2}$ – in the case of frequency standards given respectively in the approximate form of Gaussian curve or of Lamb dip. Since the RTD value should be minimized, it is necessary to maximize the value of coefficients K_L, K_G, K_D (formulae (29)–(31)).

The maximization of coefficients K_L, K_G, K_D can be obtained by maximizing of the power P_{0L}, P_{0G}, P_{0D}, respectively. In this case, however, there appear some restrictions which result from the fact, that the detector of laser output power can not be charged with a too great radiation power. Therefore, if the latter is too great, then before being delivered to power detector it must be attenuated (using attenuating systems being on the outside of the laser) to such a level, that its level value, averaging in period $(1/\nu_m)$ of modulating signal, is not greater than continuous power limit of the applied detector. As a consequence, the new values of the coefficients K_L, K_G, K_D are equal to the former ones divided by the value of the attenuation factor of the laser output power.

The reduction of the laser output power is necessary in the case of molecular lasers, because their output power can be many times greater than the continuous power limit of laser power detector. But in the general case of gas lasers the output power can be much smaller than the continuous power limit of laser power detector. Thus the laser output power should not be weakened but maximized.

In accordance with the presented considerations, coefficients P_{0L}, P_{0G}, P_{0D} in formulae (29)–(31) in the case of molecular laser should be interpreted as the values of laser output power (when generation frequency equals the central frequency ν_0 of OPF curve) supplied to power detector in AFSL system, that is the values of power attenuated to the level of continuous power limit of laser power detector used in AFSL system. Therefore, in order to maximize the coefficients K_L, K_G, K_D, in the AFSL system the laser power detector should have the possibly highest continuous power limit and be supplied with laser output power attenuated to the value equal to the continuous power limit of this detector.

The maximization of coefficients K_L, K_G, K_D may be also obtained by respective minimization of the relative FWHM values $\delta_L, \delta_G, \delta_D$ (formulae (16)–(18)). However, the minimization possibilities of these values are very limited, because the relative FWHM values $\delta_L, \delta_G, \delta_D$ of frequency standards are determined by the features of gain medium. Due to an admissible reduction of this medium pressure, the values of parameters $\delta_L, \delta_G, \delta_D$ may be reduced at most two times with respect to their maximal values.

The three frequency standards (Lamb dip and approximately Lorentzian curve and Gaussian curve) subject to analysis may be compared by comparing the RTD values resulting from those standards. In accordance with the presented considerations we may apply such values of coefficients K_L, K_G, K_D, which result from the typical values of coefficients $\delta_L, \delta_G, \delta_D$, and assume that laser power detector used in AFSL system has for each analysed frequency standard the same continuous power limit $(P_{\lim})_{\max}$. The last assumption may be written as

$$P_{0L} \simeq P_{0G} \simeq P_{0D} \simeq (P_{\lim})_{\max}. \tag{32}$$

The analysed frequency standards may be compared by using the coefficients

$$Q_{GL} = \left(\frac{K_G}{K_L}\right)^{1/2}, \tag{33}$$

$$Q_{DG} = \left(\frac{K_D}{K_G}\right)^{1/2} \tag{34}$$

where: Q_{GL} — reduction coefficient of the RTD value of AFSL system in which the maximum of approximately Gaussian OPF curve is used, as compared with the corresponding value in this system in which the maximum of approximately Lorentzian OPF curve is used; Q_{DG} — defined like Q_{GL} for the centre of Lamb dip and the maximum of approximately Gaussian OPF curve.

After substituting (29)–(32) to (33) and (34) we get

$$Q_{GL} = \frac{\delta_L}{\delta_G}(\ln 2)^{1/2}, \tag{35}$$

$$Q_{DG} = \left(\frac{\zeta_D^2}{\ln 2} - 2\right)^{1/2}. \tag{36}$$

In the case of molecular CO_2-N_2-He laser the typical values of parameters $\Delta\nu_G$ and $\Delta\nu_L$ are about 55 MHz [6] and about 100–150 MHz [1], respectively. From these values and from formulae (16) and (17) it follows

$$\delta_L \simeq (2-3)\,\delta_G. \tag{37}$$

After substituting (37) to (35) and (12) to (36), we get the following evaluation of Q_{GL} and Q_{DG} parameters

$$Q_{GL} \simeq 1.7-2.5, \tag{38}$$

$$Q_{DG} \simeq 5.9-12. \tag{39}$$

Analogically to (33) and (34) we may define coeffcient Q_{DL} (relating to the Lamb dip and approximately Lorentzian OPF curve)

$$Q_{\mathrm{DL}} = \left(\frac{K_{\mathrm{D}}}{K_{\mathrm{L}}}\right)^{1/2}. \tag{40}$$

From formulae (33), (34) and (40) we get the equation

$$Q_{\mathrm{DL}} = Q_{\mathrm{DG}} Q_{\mathrm{GL}}. \tag{41}$$

This equation and the evaluations (38) and (39) yield the evaluation of coefficient

$$Q_{\mathrm{DL}} \simeq 10-30. \tag{42}$$

From (38) it follows, that the maximum of approximately Gaussian OPF curve gives the RTD value of AFSL system about two times smaller than that for the maximum of approximately Lorentzian OPF curve. Therefore, the frequency standard given by the maximum of approximately Gaussian OPF curve is about two times better than that expressed by the maximum of approximately Lorentzian OPF curve. Analogically, from the evaluations (39) and (42) it results, that the centre of Lamb dip is frequency standard about ten times better than the maximum of approximately Gaussian OPF curve and about twenty times better than the maximum of approximately Lorentzian OPF curve. Furthermore, as it follows from the formula (36), Lamb dip centre is the better frequency standard the higher is the value of narrowing coefficient ζ_{D}.

Frequency standards curves apart from the possibly low relative FWHM values required for minimizing the RTD values, should be characterized by central frequency ν_0 of high constancy and reproducibility. The above requirements with respect to ν_0 result from the fact, that the condition

$$\Delta\nu_s = \Delta\nu_0 \tag{43}$$

must be fulfilled, since then coefficients B_{L}, B_{G}, B_{D} (formulae (23)–(25)) remain constant. In (43) $\Delta\nu_s$ and $\Delta\nu_0$ are the frequency increments of ν_s and ν_0, respectively. Thus, and error signal in AFSL system (output voltage of synchronous detector [8]) does not change when the condition (43) is fulfilled. Therefore, AFSL system does not stabilize the changes of frequency ν_0 but follows it. Just for this reason high constancy and reproducibility of the central frequency ν_0 of frequency standard are required.

The value of standard frequency ν_0 inconstancy in the case of Lamb dip centre or of approximately Gaussian or Lorentzian OPF curves maximum is of the same order of magnitude. This value changes due to the changes of the following gas gain mixture parameters:

i) temperature,

ii) composition and total pressure, and

iii) electric discharge conditions.

Therefore, the constancy and the reproducibility of standard frequency ν_0 may increase when the following parameters of gas gain mixture are stabilized:

1. Temperature (for example, by stabilization of water temperature cooling a laser discharge tube).
2. Composition and total pressure.
3. Intensity of current flowing through the laser discharge tube.

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Анализ свойств квантовых стандартов частоты в виде максимума кривой зависимости выходной мощности от частоты одночастотного газового лазера, а также в виде центра провала Лэмба

Проводится оригинальное теоретическое объяснение констатированных до сих пор только экспериментально различий в свойствах фундаментальных квантовых стандартов частоты. В работе доказано, что максимум кривой зависимости выходной мощности от частоты одночастотного газового лазера (названное дальше максимум кривой ВМОЛ) является стандартом частоты приблизительно в два раза лучшим в случае кривой ВМОЛ, описанной функцией Гаусса, чем функцией Лоренца. Доказано тоже, что центр провала Лэмба является стандартом частоты приблизительно десять раз лучшим, чем максимум кривой ВМОЛ, описанной функцией Гаусса, а также приблизительно двадцать раз лучшим, чем максимум кривой ВМОЛ, описанной функцией Лоренца.