Optica Applicata, Vol. XXIV, No. 4, 1994

Specific analytical TM solutions in nonlinearly isotropic Kerr-like media

J. JASIŃSKI

Institute of Physics, Warsaw University of Technology, ul. Koszykowa 75, 00–662 Warszawa, Poland.

In the paper, two specific analytical TM solutions for the three-layered nonlinear Kerr-like waveguide are obtained. The first solution fits the unknown permittivity to the known electromagnetic field. If the field profile is given by sech function, then the permittivity, satisfying the wave equation, is described by hypergeometric function. The second analytical case corresponds to the situation when permittivity remains constant across the nonlinear film. Such condition is satisfied for the TM field because the two electric field components are shifted in the phase by $\pi/2$. The obtained solutions are compared with the numerical Kerr solution and linear TM limit.

1. Introduction

The nonlinear guiding structures are now investigated very extensively. However, such systems exhibit many similarities to the planar linear waveguides. They differ from the linear structures by a few features. Unfortunately, the exact, analytical solution describing propagating electromagnetic fields can be found only for the TE case [1], [2]. For the TM-polarized waves such analysis is far more difficult and until now only a solution in quadrature in an unlimited Kerr-like medium or numerical solution in the three-layered structure have been obtained [3], [4].

In the present work, the analytical solution of the TM wave equation is found due to treating it as an equation for the unknown permittivity adjusted to the assumed field. Under this assumption the wave equation has an exact solution [5]. Taking the field profile the same as in the TE Kerr case, we can obtain the permittivity distribution expressed by an analytical function. Applying this solution to the three-layered structure with the nonlinear substrate, we derive the exact, analytical formulas for the mode equation and the power flow.

The second analytical case describes the fields producing the constant value of the permittivity inside the nonlinear film. This situation may occur only for the TM case because the two electric field components are shifted in the phase by $\pi/2$ and the changes of one of them can be completely compensated by the other. Nevertheless the electromagnetic fields are given by the same expressions as in the linear waveguide, it is essentially nonlinear solution with the resulting dielectric constant different from the linear limit.

2. Hypergeometric solution

Consider an infinite nonlinear dielectric in which a TM-polarized wave of frequency

ω and propagation constant β travels along the z-axis, while the field is homogeneous in the y-direction. For the optically isotropic mechanism of the nonlinearity, we can write two dielectric tensor components in the following form:

$$\varepsilon_{xx} = \varepsilon_{zz} = \varepsilon(E_x^2+E_z^2). \tag{1}$$

The general relation (1) defines Kerr-like dielectric function, while by Kerr permittivity with the linear limit ε_L and the nonlinearity coefficient α we understand $\varepsilon = \varepsilon^{\text{Kerr}} = \varepsilon_L+\alpha(E_x^2+E_z^2)$. Let us express the distance measured along the x-direction in ω/c units. The magnetic component H_y satisfies the wave equation

$$\frac{d}{dx}\left(\frac{1}{\varepsilon}\frac{dH_y}{dx}\right)+(\varepsilon-\beta^2)\frac{H_y}{\varepsilon} = 0. \tag{2}$$

Note that for $\varepsilon = \varepsilon(H_y)$ the wave equation is nonlinear with respect to H_y, but for any known $H_y(x)$ the wave equation treated as the equation for $1/\varepsilon$ is linear. However, the function $\varepsilon(x)$ is not very suitable if we will try to compare it with Kerr permittivity, so let us use the first integral of the TM Maxwell's equations [6] to obtain an equation for $\varepsilon(H_y)$. LEUNG [7] proved that for Kerr-like medium with dielectric permittivity of the form (1) such integral does exist. Writing the first integral in the form

$$\left(\frac{dH_y}{dx}\right)^2 = Q(H_y^2) \tag{3}$$

we can interpret it as a kind of conservation law, so $Q(H_y^2)$ can be called potential. For instance, in the Kerr medium the potential can be obtained by elimination of ε between two relations: $H_y^2 = \varepsilon(\varepsilon^2-\varepsilon_L^2)/(2\alpha(2\beta^2-\varepsilon))$ and $\varepsilon = \varepsilon_L+\alpha((dH_y/dx)^2+H_y^2)/\varepsilon^2$.

Using the potential function to get rid of the x-dependence we arrive at

$$2Q(s)\frac{d}{ds}\left(\frac{1}{\varepsilon(s)}\right)+\left(\frac{dQ(s)}{ds}-\beta^2\right)\frac{1}{\varepsilon(s)}+1 = 0 \tag{4}$$

where $s = H^2$. The general solution of the above equation for the arbitrary analytical $Q(s)$ is given in [5], but for the magnetic field of sech profile (the same as in TE case) we can obtain it immediately. Assuming

$$H(x) = \frac{A}{\cosh(\Omega(x-x_0))} \tag{5}$$

with $\Omega^2 = \beta^2-\varepsilon_L$, which corresponds to the potential $Q(H_y^2) = \Omega^2H_y^2-\Omega^2H_y^4/A^2$, we can convince ourselves that Eq. (5) is satisfied by the hypergeometric function

$$\frac{1}{\varepsilon(H^2)} = \frac{1}{2\beta^2-\varepsilon_L}F\left(1,1,1+p;1-\frac{H^2}{A^2}\right) \tag{6}$$

where $p = (\beta^2+\Omega^2)/(2\Omega^2)$. The integration constant has been chosen such that for vanishing magnetic field $H_y \to 0$ the permittivity reaches its linear limit ε_L. As we can see, it means that the maximum value of the dielectric function is $2\beta^2-\varepsilon_L$. Moreover, if the amplitude A of the TM polariton is

$$A^2=\frac{2(\beta^2-\varepsilon_L)}{\alpha}\frac{\varepsilon_L^3}{(2\beta^2-\varepsilon_L)(3\varepsilon_L-2\beta^2)}, \tag{6}$$

then for small magnetic fields the difference between the obtained hypergeometric permittivity and the Kerr dielectric constant is a small value of the second order with respect to H_y^2 [5].

In the linear film and cover with dielectric constants ε_f and $\varepsilon_c<\varepsilon_f$, respectively, the magnetic component H_y can be expressed as

$$H_y=\begin{cases}H_f\cos(k_f x-\varphi), & 0<x<h, \quad \beta^2<\varepsilon_f\\ H_c e^{-\varkappa_c(x-h)}, & x>h\end{cases} \tag{7}$$

where $k_f=(\varepsilon_f-\beta^2)^{1/2}$ and $\varkappa_c=(\beta^2-\varepsilon_c)^{1/2}$ (for $\varepsilon_f>\beta^2$ the cos function in the film region should be replaced by sinh and k_f by $\varkappa_f=(\beta^2-\varepsilon_f)^{1/2}$). The continuity conditions across two interfaces give us the following mode equation (written for the case $\beta^2<\varepsilon_f$):

$$h=\frac{1}{k_f}\left[\arctan\left(\frac{\varepsilon_f\varkappa_c}{\varepsilon_c k_f}\right)+\arctan\left(\frac{\varepsilon_f\Omega T_0}{(2\beta^2-\varepsilon_L)k_f}F(1,1,1,+p;\ T_0)\right)+m\pi\right] \tag{8}$$

where $T_0=\tanh(\Omega x_0)$. The mode equation determines the position of the maximum of the field inside the nonlinear substrate $x_0=1/\Omega\cdot\mathrm{arctanh}(T_0)$ or the boundary value of the magnetic field $H_b=A(1-T_0^2)^{1/2}$ as functions of the waveguide thickness h and propagation constant β.

Having the analytical expressions for the field and permittivity, we can calculate the power flow along the interfaces. By integrating the Poynting vector we obtain the components of the total power P_{tot} supported by the cover, film and the substrate P_c, P_f and P_s:

$$\begin{aligned}P_c&=\frac{\beta}{2\mu_0\omega}\frac{H_c^2}{2\varepsilon_c\varkappa_c},\\ P_f&=\frac{\beta}{2\mu_0\omega}\frac{H_f^2}{2\varepsilon_f}\left(h+\frac{1}{\varkappa_f}(\sin\Phi_c\cos\Phi_c+\sin\Phi\cos\Phi\right),\\ P_s&=\frac{\beta}{2\mu_0\omega}\frac{A^2\Gamma(1+p)}{\Omega(2\beta^2-\varepsilon_L)}\sum_{j=0}^{\infty}\frac{j!(1-T_0^{2j+1})}{(2j+1)\Gamma(1+p+j)}\end{aligned} \tag{9}$$

where $\Phi_c=k_j h-\Phi$. The sum in the formula describing P_s is the result of the integration of the hypergeometric function expressed by power series.

In Figures 1 and 2, we compare the field profiles and the power flow of the Kerr solution [7] with the hypergeometric solution obtained above. As we should expect, for the small powers (for β^2 close to ε_L) the differences are insignificant. They increase with increasing β, but because of material limitations, this region is physically inaccessible.

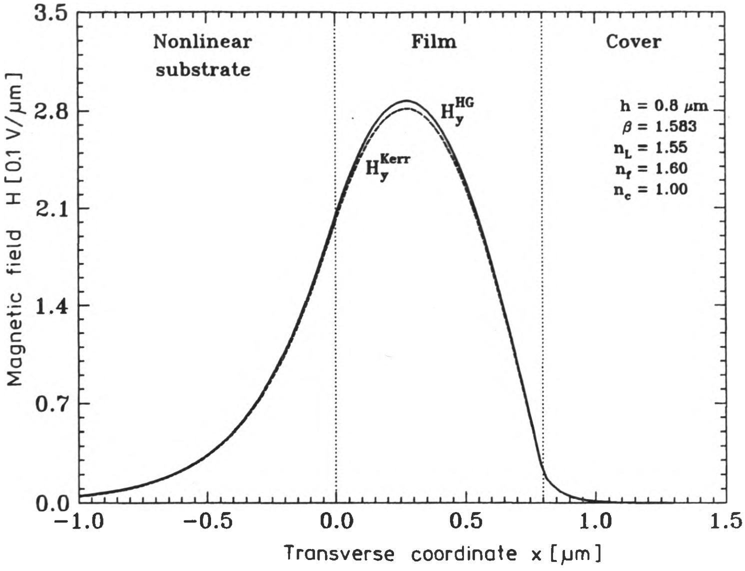

Fig. 1. Magnetic field profile of the hypergeometric and Kerr solution for 0-th order mode

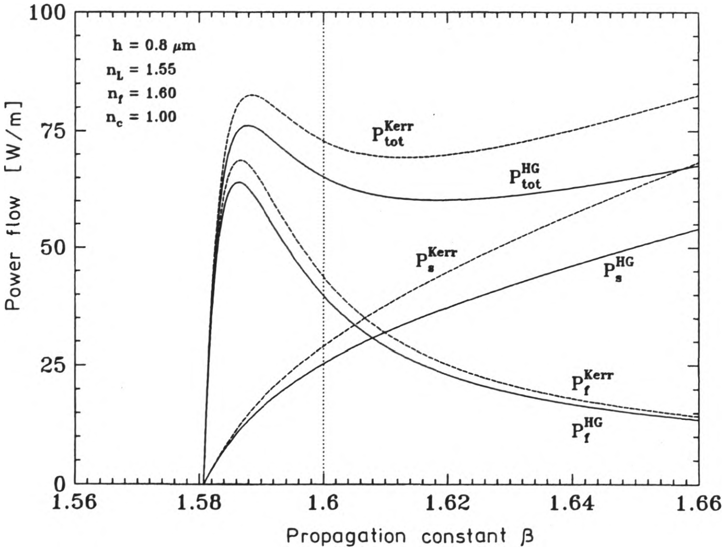

Fig. 2. Comparison of the power flows carried by the 0-th order mode between the hypergeometric and Kerr solution. $P_{tot} = P_s + P_f + P_c$. Power flows carried by the cover P_c are very low and are not shown. For $\beta = (\varepsilon_s)^{1/2}$ (vertical dotted line) the trigonometric solution in the film turns into the hyperbolic solution

3. Balanced Kerr solution

Now let us assume the linear substrate with dielectric constant ε_s and nonlinear Kerr film $\varepsilon = \varepsilon_L + \alpha(E_x^2 + E_z^2)$. For convenience we express the magnetic component H_y in the electric field units Vm^{-1}. The condition $\varepsilon = \text{const.}$, however impossible for TE modes, can be satisfied in TM case, since permittivity depends on the sum of two components. For the fields generating such permittivity the solution takes the same form as in linear case

$$\begin{aligned} H_y &= H_s e^{k_s x}, && x < 0, \\ H_y &= H_f \cos(k_f x - \Phi), && 0 < x < h, \\ H_y &= H_c e^{-\varkappa_c (x-h)}, && x > h. \end{aligned} \tag{10}$$

The equality $E_x^2 + E_z^2 = \text{const.}$ is satisfied at any point inside the nonlinear film if both maxima of the electric components E_{x0} and E_{z0} equal each other $E_{x0} = E_{z0} = \beta H_f/\varepsilon = ((\varepsilon - \varepsilon_L)/\alpha)^{1/2}$, what gives the dielectric constant across the Kerr film

$$\varepsilon = 2\beta^2 = 2\Omega^2. \tag{11}$$

The continuity conditions at both interfaces lead to the mode equation, which, however looks similar to the linear mode equation, describes completely different modes

$$h = \frac{1}{\beta}\left(\arctan\frac{2\beta\sqrt{\beta^2 - \varepsilon_c}}{\varepsilon_c} + \arctan\frac{2\beta\sqrt{\beta^2 - \varepsilon_s}}{\varepsilon_s} + m\pi\right). \tag{12}$$

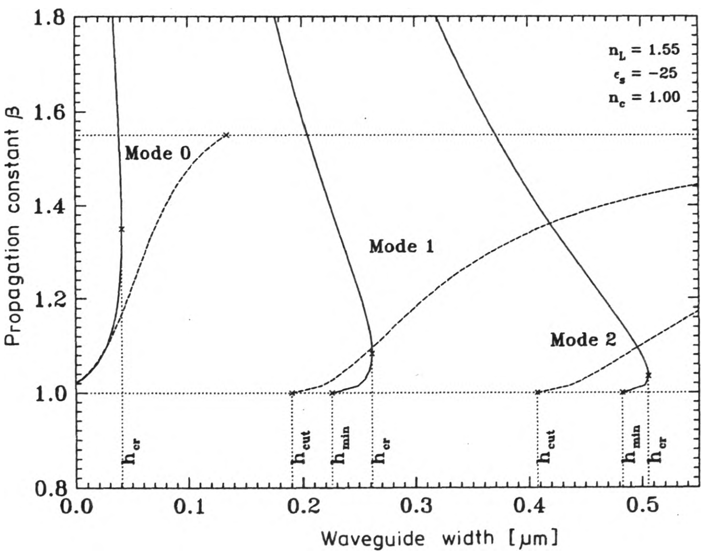

Fig. 3. Propagation constants β of the balanced modes (solid lines) and linear modes (dashed lines) as functions of the waveguide width for negative permittivity of the cover

In order to obtain the materially accessible values of the dielectric constant ($\varepsilon < 1.3\varepsilon_L$) enabling the appearance of the balanced modes, we should apply the Kerr film embedded between two media with dielectric constants less than $\varepsilon/2$. This requirement can be realized without any trouble, for instance, in a waveguide based on a metallic substrate $\varepsilon_s < 0$ (with no attenuation, which means $\text{Im}\,\varepsilon_s = 0$). In Figure 3, we illustrate the lines $\beta(h)$ following from the mode equation (8) and the appropriate linear mode equation. As we can see, the lines are quite different. Note that lower order modes appear in very thin waveguides, but the higher order balanced modes are possible to obtain.

4. Conclusions

The presented results describe specific cases of the TM solution. The first one can be treated as an analytical approximation of the Kerr TM numerical solution. For small guided power, or β close to its linear limit, this approximation is quite good. The second solution simply marks certain points in the whole spectrum of the numerically obtained Kerr TM modes at which analytical representation is possible. Although, it exists for any ε_L, ε_c and ε_s, the physically reasonable values of the nonlinear permittivity ε result only for large difference between ε and dielectric constants of the linear substrate and cover ε_s and ε_c.

References

[1] BOARDMAN A. D., EGAN P., IEEE J. Quant. Electron. **22** (1986), 319.
[2] CHEN W., MARADUDIN A. A., J. Opt. Soc. Am. B **5** (1988), 529.
[3] BOARDMAN A. D., MARADUDIN A. A., STEGEMAN G. I., TWARDOWSKI T., WRIGHT E. M., Phys. Rev. A **35** (1987), 1159.
[4] LEUNG K. M., Phys. Rev. B **32** (1985), 5093.
[5] JASIŃSKI J., GNIADEK K., Opt. Quant. Electron. 1994 (in print).
[6] BOARDMAN A. D., EGAN P., MIHALACHE D., LANGBEIN U., LEDERER F., [In] *Modern Problems in Condensed Matter Physics*, North-Holland, Amsterdam 1987, p. 160.
[7] GNIADEK K., RUSEK M., Tech. Digest Series J. Opt. Soc. Am., *Nonlinear Guided Wave Phenomena*, Conf. Ed. Cambridge 1991, p. 100.