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Generalized Luneburg lens problem. An analytical solution and its simplifications*

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An analytical representation for the Luneburg's integral is described. It allows to obtain the ideal index profiles for arbitrary generalized Luneburg lenses without use of numerical methods. Following the Rayleigh criterion three simplified formulae are presented. They warrant the computation accuracy within 2×10^{-5}, which is good enough for nearly diffraction-limited performance of the lens and quite sufficient from technological point of view.

1. Introduction

The Luneburg lens is a gradient-index, spherically symmetric refracting structure which performs a perfect focusing of collimated beams. This construction has been well known since 1944, when R.K. Luneburg formulated a simple integral condition for the refractive index profile that provides such an optical operation [1]. This integral has been subsequently solved by Luneburg for the case when light focusing occurs at the lens bondary ($f = 1$). The original Luneburg's solution resulted in the following index distribution: $n = (2-r^2)^{1/2}$, where both the refractive index n and the radial position r are normalized to unit at the edge of the lens region. For many years the solution for the so-called generalized Luneburg lenses, i.e., those having the image surfaces located at distances greater than one lens radius from the centre of the lens ($f > 1$) [2–4], was a troublesome problem. In this case the refractive index profiles were calculated numerically [3, 4]. Recently, however, an analytical series representation has been found for the Luneburg's integral for $f > 1$, which allows to design generalized Luneburg lenses without the use of numerical methods [5, 6].

2. Formulation of the problem. An analytical solution

As it has been proved by LUNEBURG [1], the refractive index distribution $n(r)$ for the lens perfectly focusing a beam of parallel rays may be generally described by the set of equations:

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$$\begin{cases} n = \exp[\omega(\varrho, f)], & \varrho \leqslant 1, \\ n \equiv 1, & \varrho > 1, \end{cases} \tag{1}$$

where

$$\varrho \overset{\text{def}}{=} nr, \tag{2}$$

and

$$\omega(\varrho, f) = \frac{1}{\pi} \int_{\varrho}^{1} \frac{\arcsin x/f}{(x^2 - \varrho^2)^{1/2}} dx. \tag{3}$$

It is assumed here that the radial position r as well as the focal length f are normalized with respect to the radius of the lens region, so that $0 \leqslant r \leqslant 1$ and $f \geqslant 1$.

The original Luneburg's solution for the integral (3) for a lens with $f = 1$ has the following simple form [1]:

$$\omega(\varrho, 1) = \frac{1}{2} \ln[1 + (1 - \varrho^2)^{1/2}], \tag{4}$$

which finally resolves itself into the simple $n(r)$ refractive index distribution presented in the Introduction.

For almost 40 years the integral (3) for the lens with a focal length $f > 1$

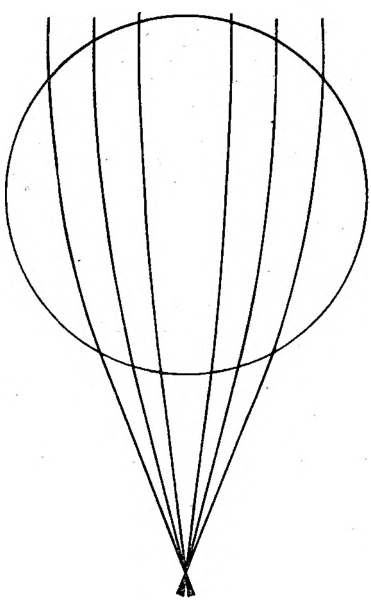

Fig. 1. Generalized Luneburg lens focussing the beam of parallel rays at the distance equal to two lens radii from the centre of the lens ($f = 2$)

(see Fig. 1) had not been evaluated analytically, but in 1981 Colombini presented the exact series representation for the function ω [5]

$$\omega(\varrho, f) = \frac{(1-\varrho^2)^{1/2}}{\pi} \sum_{m=0}^{\infty} a_m f^{-(2m+1)} \sum_{r=0}^{m} b_r \varrho^{2(m-r)} \tag{5}$$

where $a_m = (2m+1)^{-2}$ and $b_r = (2r)!/[4^r(r!)^2]$.

Recently, another series representation for the Luneburg's integral has been independently found [6]

$$\omega(\varrho, f) = \frac{(1-\varrho^2)^{1/2}}{\pi} \sum_{k=0}^{\infty} s_k(f) \varrho^{2k} \tag{6}$$

where

$$s_k(f) = \sum_{l=0}^{\infty} \frac{(2l)!}{(2^l l!)^2 [2(k+l)+1]^2 f^{2(k+l)+1}} \tag{7}$$

It should be noted that both solutions (5) and (6) are similar, i.e., an appropriate renumeration allows us to pass from each representation to the other one.

Expression (5) seems to be more applicable to incorporate it into a computer programme, since its form requires at least several computational steps to obtain good accuracy. The representation (6) seems to be more convenient for fast handy calculations and, additionally, it leads to some analytical conclusions. The coefficients $s_k(f)$ which depend on the focal length are readily obtained using even a pocket calculator. Their sum yields

$$\sum_{k=0}^{\infty} s_k(f) = \arcsin 1/f, \tag{8}$$

what is easy to prove applying the following order of summation ($s_{kl}(f)$ denotes l-th component of k-th coefficient)

$$\sum_{k=0}^{\infty} s_k(f) = s_{01}(f) + [s_{02}(f) + s_{11}(f)] + [s_{03}(f) + s_{12}(f) + s_{21}(f)] + \dots$$

$$= \frac{1}{f} + \frac{1}{2\cdot 3 f^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5 f^5} + \dots = \arcsin 1/f. \tag{9}$$

The described property proves that the series (6) is convergent and its upper bound is

$$\lim_{\varrho\to 1} \omega(\varrho, f) = \frac{\arcsin 1/f}{\pi} (1-\varrho^2)^{1/2} \cong \frac{\sqrt{2}\arcsin 1/f}{\pi} (1-\varrho)^{1/2}. \tag{10}$$

Such a behaviour of ω-function has been pointed out by other authors [3, 4].

Table 1. Values of coefficients $s_0(f),\ldots, s_7(f)$ for lenses with focal lengths $f = 2,\ldots, 10$, cal-
sum of these coefficients is in good agreement with the respective values of the function

	s_0	s_1	s_2	s_3	s_4
$f = 2$	0.5074707	0.0145824	0.0013403	0.0001733	0.0000265
$f = 3$	0.3354557	0.0042012	0.0001695	0.0000097	0.0000007
$f = 4$	0.2508831	0.0017561	0.0000397	0.0000013	0.
$f = 5$	0.2004493	0.0008954	0.0000129	0.0000003	0.
$f = 6$	0.1669257	0.0005170	0.0000052	0.0000001	0.
$f = 7$	0.1430199	0.0003251	0.0000024	0.	0.
$f = 8$	0.1251089	0.0002176	0.0000012	0.	0.
$f = 9$	0.1111875	0.0001528	0.0000007	0.	0.
$f = 10$	0.1000557	0.0001113	0.0000004	0.	0.

Relation (8) may be useful to control the computation accuracy in the preliminary calculations of $s_k(f)$-coefficients. This accuracy ought to be rather high, but is easy to achieve, especially for $f \geqslant 2$. The values of coefficients $s_0(f), \ldots, s_7(f)$ for lenses with $f = 2, 3, \ldots, 10$ are shown in Table 1. Computations have been performed with accuracy better than 10^{-7}. These values become smaller and smaller successively, hence, the sum of the first several coefficients is nearly equal to $\arcsin(1/f)$.

In view of all the above properties, it is reasonable to expect that the simplification of the general solution (6), consisting in an appropriate truncation of the series, will not substantially influence the lens performance. This problem, however, needs several words of comment.

3. Computation accuracy. Short discussion

Consider in the Luneburg lens a central ray the trace of which coincides with the lens diameter d. Assuming that the lens has an exact index profile, let the optical path of the central ray inside the lens area be denoted by S. Let S^* denote the optical path of an analogical ray but inside the lens with an approximate index profile. Following the well-known Rayleigh criterion, such a lens will possess good image characteristics if

$$|\Delta S| = |S-S^*| = \left|\int_0^d (n-n^*)\,ds\right| = \left|\int_0^d \Delta n ds\right| \leqslant \lambda/4 \tag{11}$$

where λ is the wavelength of light coupled into the lens, n and n^* are the exact and approximate refractive indices, respectively, and Δn is the computation error.

Let us assume now that Δn has a constant value along the lens diameter. This assumption sharpens the requirements referring to the computation accuracy (in general the value of Δn will change being equal to zero at the edge and culated using Eq. (7). The zeros denote the values smaller than 5×10^{-8}. Note, that the arc sin $(1/f)$

s_5	s_6	s_7	sum	arcsin $(1/f)$
0.0000045	0.0000008	0.0000002	0.5235987	0.5235987
0.	0.	0.	0.3398368	0.3398368
0.	0.	0.	0.2526802	0.2526802
0.	0.	0.	0.2013579	0.2013579
0.	0.	0.	0.1674480	0.1674480
0.	0.	0.	0.1433474	0.1433475
0.	0.	0.	0.1253277	0.1253278
0.	0.	0.	0.1113410	0.1113410
0.	0.	0.	0.1001674	0.1001674

the centre of the lens), but is needed to convert the condition (11) into a more convenient form. We have

$$\left|\int_0^d \Delta n ds\right| = \left|\Delta n \int_0^d ds\right| = |\Delta n d| \leqslant \lambda/4, \tag{12}$$

whence

$$|\Delta n| \leqslant \lambda/4d. \tag{13}$$

As it is well known, the Luneburg lenses are being produced exclusively for the planar-waveguide optics use up to now. For this reason, the meaning of the parameters in relation (13) must be adjusted to waveguide optics formalism. Therefore, by Δn we will understand the difference between the exact and approximate normalized effective refractive indices (not bulk) and by λ the effective wavelength of a mode guided throughout the planar lens area, which is equal to the product of the wavelength of light coupled into the waveguide, λ_c, and the effective refractive index n' inside the lens. If we assume typical values of these parameters: $n' \sim 2$, $\lambda_c \sim 0.4$ μm (in practice > 0.4 μm), and the waveguide Luneburg lens diameter $d \sim 1$ cm, then from the inequality (13) we finally derive the following condition:

$$|\Delta n| \leqslant 2 \times 10^{-5}. \tag{14}$$

From the calculations of the planar Luneburg lens index profile performed with such an accuracy, the lens with performance fulfilling the Rayleigh criterion is derived.

4. Simplification of the general solution

Having established the accuracy criterion (14), we are now in position to simplify the solution (6) by an appropriate truncation of the infinite series. As it results from Eqs. (6) and (1), the approximate value of the refractive index n^* derived owing to such a simplification will be always smaller than the exact value n. Thus, we can rewrite the condition (14) in the form

$$0 \leqslant n - n^* \leqslant 2\times 10^{-5}, \tag{15}$$

from which

$$1 - \frac{2\times 10^{-5}}{n} \leqslant \frac{n^*}{n} \leqslant 1 \tag{16}$$

and

$$\ln\left(1 - \frac{2\times 10^{-5}}{n}\right) \leqslant \omega^* - \omega \leqslant 0. \tag{17}$$

Here $\omega(\varrho, f)$ and $\omega^*(\varrho, f)$ are the exact and approximate values, respectively, of the function (6) for an arbitrary ϱ. Taking with good approximation $\ln(1-2\times 10^{-5}/n) \cong -2\times 10^{-5}/n$, we obtain

$$0 \leqslant \Delta\omega \leqslant \frac{2\times 10^{-5}}{n} \tag{18}$$

where $\Delta\omega = \omega - \omega^*$. For security we should place in the relation (18) the maximum possible value $n = \sqrt{2}$ (central value of refractive index for the classical Luneburg lens), to derive finally

$$0 \leqslant \Delta\omega \leqslant \sqrt{2}\times 10^{-5}. \tag{19}$$

Such an accuracy is required in calculation of $\omega(\varrho, f)$ if the Luneburg lens having good image quality is to be obtained.

In connection with this, let the following condition for truncation of the series described by Eq. (6) be suggested: we examine successive terms having the form $(1/\pi)(1-\varrho^2)^{1/2} s_k(f)\varrho^{2k}$ ($(k+1)$-th term, $k = 0, 1, 2, \ldots$) and if the maximum value of any term appears to be less than 1×10^{-5} then this term and all the subsequent ones may be omitted. This condition can be expressed explicitly in form of the following inequality:

$$\frac{s_k(f)}{(2k+1)\pi}\left(\frac{2k}{2k+1}\right)^k < 1\times 10^{-5},\ k = 0, 1, 2, \ldots \tag{20}$$

where the left-hand side represents the maximum value of $(k+1)$-th term from Eq. (6) (easy to prove after differentiation of this term over ϱ).

Application of the condition (20) leads us readily to the conclusion that the following simplified expressions for $\omega(\varrho, f)$

$$\omega^*(\varrho, f) = \begin{cases} \dfrac{(1-\varrho^2)^{1/2}}{\pi}\,[s_0(f) + s_1(f)\varrho^2 + s_2(f)\varrho]^4 & \text{for } f \geqslant 2 \\[2ex] \dfrac{(1-\varrho^2)^{1/2}}{\pi}\,[s_0(f) + s_1(f)\varrho^2] & \text{for } f \geqslant 3 \\[2ex] \dfrac{(1-\varrho^2)^{1/2}}{\pi}\,s_0(f) = (1-\varrho^2)^{1/2}\omega(0, f) & \text{for } f \geqslant 10 \end{cases} \tag{21}$$

Table 2. Computational errors committed due to the application of approximate expressions (21) to the index profile calculations. Δ denotes the difference between the exact and approximate refractive indices, f is focal length, and ϱ is a variable defined by Eq. (2)

	$f=2$	$f=3$	$f=10$
ϱ	$\Delta\times10^5$	$\Delta\times10^5$	$\Delta\times10^5$
0.00	0.00	0.00	0.00
0.05	0.01	0.01	0.01
0.10	0.01	0.01	0.03
0.15	0.01	0.01	0.06
0.20	0.01	0.01	0.13
0.25	0.01	0.03	0.22
0.30	0.01	0.06	0.30
0.35	0.02	0.09	0.40
0.40	0.40	0.15	0.52
0.45	0.06	0.23	0.65
0.50	0.10	0.33	0.78
0.55	0.16	0.47	0.91
0.60	0.25	0.63	1.04
0.65	0.38	0.83	1.15
0.70	0.57	1.04	1.25
0.75	0.80	1.26	1.33
0.80	1.07	1.47	1.38
0.85	1.36	1.64	1.37
0.90	1.58	1.71	1.27
0.91	1.60	1.69	1.22
0.92	1.62	1.66	1.19
0.93	1.62	1.62	1.13
0.94	1.60	1.57	1.08
0.95	1.56	1.50	1.01
0.96	1.49	1.40	0.93
0.97	1.37	1.27	0.82
0.98	1.19	1.08	0.68
0.99	0.90	0.81	0.49
1.00	0.00	0.00	0.00

are sufficient to perform the index profiles for planar Luneburg lenses with the accuracy resulting from the Rayleigh criterion. The values of the computational errors due to the use of approximate expressions (21) for determining the $n(\varrho)$ index profile are presented in Table 2. As we can see, the maximum errors for lenses with $f=2,3$ and 10 are less than 2×10^{-5}, what is in agreement with the condition (14). Note, that the proposed simplifications preserve correct values of refractive index for the centres and edges of lenses.

Finally, it is worth to mention the simple way in which the computation accuracy can be improved significantly. To this end it suffices to add to each of expressions (21) the term consisting of an appropriate power of variable ϱ multiplied by the factor, the value of which is equal to difference between arcsin $(1/f)$ and the sum of the coefficients taken into account. For instance, for lenses with $f\geqslant 2$ we have

$$\left.\begin{aligned} \omega^*(\varrho,f) &= \frac{(1-\varrho^2)^{1/2}}{\pi}\,[s_0(f) \\ &+ s_1(f)\varrho^2 + s_2(f)\varrho^4 + c(f)\varrho^6], \\ c(f) &= \arcsin\frac{1}{f} - [s_0(f) + s_1(f)s_2(f)]. \end{aligned}\right\} \tag{22}$$

Possibility of such an improvement results directly from Eq. (8).

5. A $n(r)$ refractive index distribution. Effect of a partial compensation of the computational error

Our preceding considerations concerned the accuracy problem connected with $n(\varrho)$ index profile computations. However, it is usually interesting to know something about the error committed while performing an approximate $n(r)$

index distribution. Such a distribution may be obtained using the set of parametric equations resulting from Eqs. (1) and (2)

$$\begin{cases} n^* = \exp[\omega^*(\varrho, f)], \\ r^* = \varrho/n^*. \end{cases} \tag{23}$$

For an arbitrarily given ϱ we have

$$n^*(\varrho) = n(\varrho) - \Delta n(\varrho) \tag{24}$$

(notation the same as previously), and

$$r^*(\varrho) = \varrho/n^*(\varrho) = \varrho/[n(\varrho) - \Delta n(\varrho)] = \frac{\varrho}{n(\varrho)}\left[1 + \frac{n(\varrho)}{n(\varrho)} + \frac{\Delta n^2(\varrho)}{n^2(\varrho)} + \dots\right]$$

$$\cong r(\varrho) + \Delta r(\varrho) \tag{25}$$

where $r(\varrho) = \varrho/n(\varrho)$ is the exact value of radial position, and

$$\Delta r(\varrho) = r(\varrho)\frac{\Delta n(\varrho)}{n(\varrho)} = \frac{\varrho \Delta n(\varrho)}{n^2(\varrho)} \tag{26}$$

is the error resulting from the application of approximate computation procedure. Note (following the data from Table 2), that Δr is of the same range as Δn in the regions where Δn has considerable values.

Let us consider now an exact $n(r)$ index profile (see Fig. 2). Let the point $A(r(\varrho), n(\varrho))$ correspond to the exact values of n and r calculated for an arbitrarily given ϱ. As it results from Eqs. (24) and (25), the approximate values n^* and r^* corresponding to the same value of ϱ will be represented by the point B. The function $n(r)$ is monotonically decreasing and for this reason the point B will

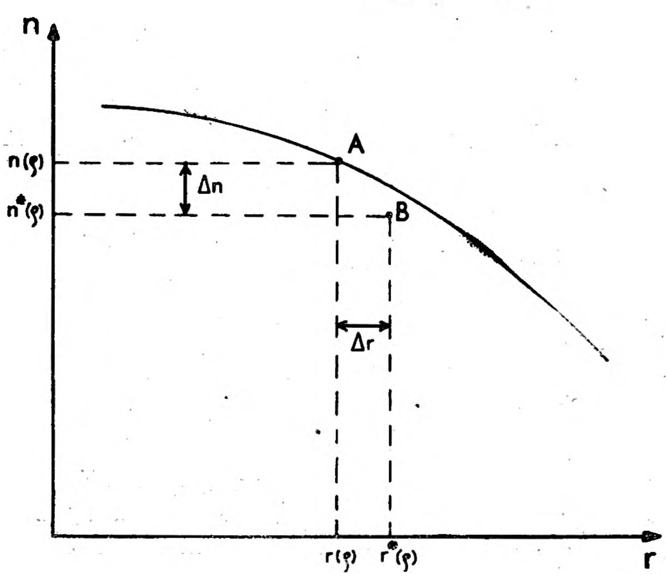

Fig. 2. Refractive index n vs. radial position r (example curve). Point $A(n, r)$ corresponds to the exact values of n and r derived for arbitrarily chosen ϱ; n^* and r^* denote approximate values of refractive index and radial position, respectively (represented by point B)

be placed considerably nearlier to the ideal curve than it could result from the value of the computation error $\Delta n(\varrho)$ (however, far enough from the ideal point A). Thus, performing the index profile $n(\varrho)$ with the assumed accuracy, we finally obtain reasonably better accuracy in fitting of the $n(r)$ distribution.

6. Summary

In order to simplify the computation procedure in the design of generalized Luneburg lenses, the approximate solutions have been presented. They warrant the accuracy better than 2×10^{-5}. From the technological point of view, we can only wish that such accuracies be obtainable in practice.

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References

[1] Luneburg R. K., *Mathematical theory of optics*, Brown University, Providence, Rhode Island 1944.
[2] Stettler R., Optik **12** (1955), 529.
[3] Morgan S. P., J. Appl. Phys. **29** (1958), 1358.
[4] Southwell W. H., J. Opt. Soc. Am. **67** (1977), 1010.
[5] Colombini E., J. Opt. Soc. Am. **71** (1981), 1403.
[6] Sochacki J., J. Opt. Soc. Am. **73** (1983), 789.

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