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The influence of wave aberrations on the Wigner distribution*

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We show that the Wigner Distribution Function (WDF) of an optical system with aberrations can be obtained by applying a differential operator to the WDF of the same optical system, but free from aberrations. The first part of this operator predicts a coordinate transformation for the WDF in accordance with the description based on ray aberrations. The remaining part of the operator predicts a new shape of the WDF, as may be expected if one considers the changes suffered by the Point Spread Function. Considered in reverse, we show that the WDF provides a very convenient framework for understanding the impact of aberrations.

1. Introduction

The Wigner Distribution Function (WDF) is an elegant and useful mathematical tool for representing optical signals [1, 2]. It has been used advantageously in the investigation of sound patterns [2], for studying the time-dependence of the physical spectrum of light [3], for testing varifocal lenses [4], and for real time covert communication [5]. The WDF is a signal representation that links geometrical optics concepts with those of Fourier optics [6–9].

In this paper our aim is to show that the WDF, associated with an optical system suffering from aberrations, can be obtained from the WDF for the diffraction limited case by applying a differential operator [10–11]. This operator describes the two following characteristics. It predicts *geometrical optics* modifications as a coordinate transformation of the WDF, which is related to the prediction of transversal ray aberrations. It indicates Fourier optics modifications as changes in the shape of the WDF, which are related to the modifications of the Point Spread Function.

In short: under the influence of the aberrations the rays modify their trajectories, as well as their "amplitudes" These two aspects are described by the WDF.

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In Section 2 we discuss at first briefly the effects of the aberrations on the rays, and later on we analyse the effects of the aberration on the Point Spread Function. In Section 3 we indicate how these two approaches are unified, if one considers the influence of the aberrations on the WDF. For the sake of simplicity our arguments are derived for the 1-dim case.

2. Two approaches for describing the influence of the aberrations

2.1. Geometrical optics approach

Rays are normal to the wavefront. Hence, if the wavefront departs from a sphere at the exit pupil of an optical system (Fig. 1), then the rays do not intersect at focal point (0,0) in the Gaussian image plane, in fact the rays intersect the image plane at different points (x, y), as can be directly seen from a spot diagram which is obtained as follows.

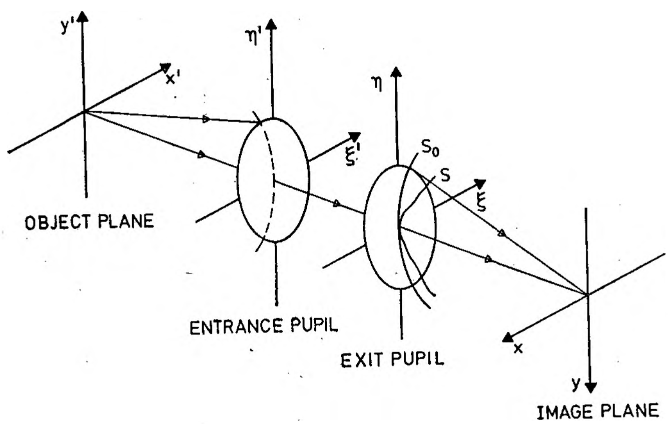

Fig. 1. Schematic diagram of an optical system. The curve $S(\xi, \eta, z)$ represents the surface of the real wavefront at the exit pupil, and the curve $S_0(\xi, \eta, z)$ the reference sphere

The departures from sphericity, at a point (ξ, η) in the exit pupil, are described by a *wave aberration function* $W(\xi, \eta)$ which specifies the optical path differences along the ray between the surface representing the real wavefront $S(\xi, \eta, z)$ and the ideal spherical surface, $S_0(\xi, \eta, z)$, known as the reference sphere [12]. Within the paraxial approximation the aberration function can be written as

$$W(\xi, \eta) = N[S(\xi, \eta, z) - S_0(\xi, \eta, z)] = C_{20}(\xi^2+\eta^2) + C_{31}(\xi^2+\eta^2)\eta + C_{40}(\xi^2+\eta^2)^2 + \text{HIGHER TERMS}. \quad (1)$$

In Eq. (1) we assume rotational symmetry of the optical system, and we denote by N the refractive index of the medium between the exit pupil and the image plane. The direction cosines, within the paraxial approximation [13], are

$$\alpha = X/R = \partial W/\partial \xi, \; \beta = Y/R = \partial W/\partial \eta, \; \nu = -1 \quad (2)$$

where R is the radius of the reference sphere.

Thus, the intersection coordinates at the image plane are

$$(x, y) = R(\partial W/\partial \xi, \partial W/\partial \eta). \tag{3}$$

The loci of intersections are known as the spot diagram (see Fig. 2), and it is a way for representing the influence of aberrations from a geometrical point of view.

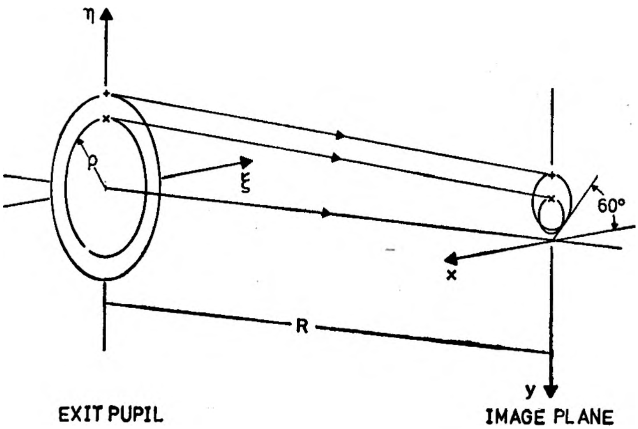

Fig. 2. Loci of ray intersections or spot diagram of an optical system suffering from coma, $W(\xi, \eta) = C_{31}\varrho^2\eta$

2.2. Fourier optics approach

From the viewpoint of Fourier optics, the presence of aberrations in an optical system is represented by phase distortions. Hence the generalized pupil function is

$$\tilde{p}(\xi, \eta) = \exp\big(jkW(\xi\, \eta,)\big)\tilde{p}_0(\xi, \eta) \tag{4}$$

where the finite size of the exit pupil is described by

$$\tilde{p}_0(\xi, \eta) = 1 \text{ if } \xi^2+\eta^2 < \varrho^2$$
$$= 0 \text{ if } \xi^2+\eta^2 > \varrho^2. \tag{5}$$

In the paraxial approximation the complex amplitude in the image plane is the Fourier transform of Eq. (4), namely

$$p(x, y) = \int\int_{-\infty}^{\infty} \exp\big(jkW(\xi, \eta)\big)\tilde{p}_0(\xi, \eta)\exp\big(j2\pi(x\xi+y\eta)/\lambda R\big)\,d_\xi\, d\eta. \tag{6}$$

Now, we know by the derivative theorem of the Fourier transform that multiplication in the Fourier domain by $j2\pi\xi/\lambda R$ is equivalent to $\partial/\partial x$ in the space domain. Thus, any power series in ξ and η can be represented in the space domain as a power series in $(-j\lambda R/2\pi)\partial/\partial x$ and $(-j\lambda R/2\pi)\partial/\partial y$. In particular, the power series of $\exp(jkW(\xi, \eta))$ can be transformed in a differential operator $\exp(jkW(-jR/k\partial/\partial x,\ -jR/k\partial/\partial y)$ [10, 11]. Therefore, Eq. (6) can be written as

$$p(x, y) = \exp(jkW(-j\lambda R/2\pi\partial/\partial x,\ -j\lambda R/2\pi\partial/\partial y)p_0(x, y) \tag{7}$$

where $p_0(x, y)$ is the Point Spread Function for the diffraction limited case

$$p_0(x, y) = \int\int_{-\infty}^{\infty} \tilde{p}_0(\xi, \eta)\exp\{j2\pi(x\xi + y\eta)/\lambda R\}\, d\xi d\eta. \tag{8}$$

We illustrate the result in Eq. (7) with a simple example. For a 1-dim pupil suffering from comatic aberration, Eq. (4) becomes

$$\tilde{p}(\eta) = \exp(jkC_{31}\eta^3)\tilde{p}_0(\eta) \tag{9}$$

where $\tilde{p}_0(\eta) = 1$ only if $|\eta| < \varrho$. For this example, Eq. (7) reduces to

$$p(y) = \exp[jkC_{31}(-jR/k)^3 d^3/dy^3]p_0(y) = \sum_{n=0}^{\infty} [-C_{31}(R^3/k^2)]^n/n_3!]\frac{d^{3n}}{dy^{3n}}p_0(y), \tag{10}$$

and Eq. (8) reduces to

$$p_0(y) = 2\varrho[\sin(\varrho ky/R)/(\varrho ky/R)]. \tag{11}$$

This means that the comatic line spread function $p(y)$ is obtained as an "incompleted" Taylor series expansion of the diffraction limited line spread function $p_0(y)$. If in addition the coma coefficient $C_{31} \ll 1/k^3$ then Eq. (10) can be approximated up to $n = 2$. That is

$$p(y) \cong p_0(y) - C_{31}(R^3/k^2)\frac{d^3}{dy^3}p_0(y) + 0.5(C_{31}R^3/k^2)^2\frac{d^6}{dy^6}p_0(y). \tag{12}$$

From Eq. (12) it is apparent that the line spread function is modified by the special functions $p_n(y) = d^{3n}/dy^{3n}p_0(y)$ with $n = 0, 1, 2$, which are shown in

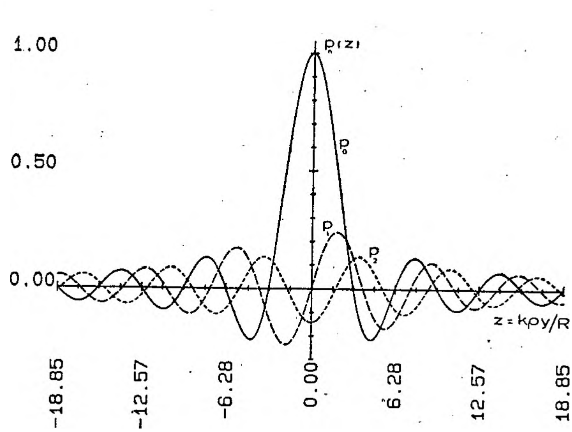

Fig. 3. In an optical system suffering from coma, the diffraction limited line spread fuction, p_0, changes in shape by adding new functions p_1, p_2, p_3,... where $p_n = \partial^{3n}/\partial x^{3n}p_0$

Fig. 3. It is to be noted that for $n = 1$ the special function is no longer peaked at the origin. Thus, the resulting line spread function $p(y)$ in Eq. (12) will no longer be symmetric around the origin, as one should expect for comatic aberration.

In conclusion from the point of view of Fourier optics, the influence of the aberrations can be represented as distortions on the Point Spread Function, which are produced by a differential operator.

3. The WDF approach

The aberrations change the WDF in two ways. Firstly there is an area preserving coordinate transformation of the (y, η) — Wigner domain — which predicts exactly the geometrical optics result in Eq. (2b) of ray displacement, and additionally, there is an operator which changes the shape of the WDF. Hence, in a lax formulation we can say that there is a change of the "amplitude of the rays" due to diffraction. Therefore, the advantage of the WDF approach is that it connects the simple (but inexact) geometrical optical treatment with the more complicated wave optical treatment in an elegant fashion.

The WDF for the 1-dim version of the pupil in Eq. (4) is

$$F(y, \eta) = \int_{-\infty}^{\infty} \tilde{p}(\eta+\eta'/2)\tilde{p}^*(\eta-\eta|/2)\exp(j2\pi y\eta'/\lambda R)\,d\eta'$$

$$= \int_{-\infty}^{\infty} \exp(jk[W(\eta+\eta'/2)-W(\eta-\eta|'/2)])$$

$$\tilde{p}_0(\eta+\eta'/2)\tilde{p}_0^*(\eta-\eta'/2)\exp(j2\pi y\eta'/\lambda R)\,d\eta'. \tag{13}$$

Now, since the aberration function for the 1-dim case is of the form

$$W(\eta) = \sum_{m=1}^{M} C_m \eta^m, \tag{14}$$

then the first term in the integral in Eq. (13) can be written as

$$\exp(jk[W(\eta+\eta'/2)-W(\eta-\eta'/2)]) = \exp\left(jk\sum_{m=0}^{[M/2]} a_m(\eta)\eta'^{2m+1}\right) \tag{15}$$

where

$$[M/2] = M/2-1 \text{ if } M \text{ is even}$$
$$= (M-1)/2 \text{ if } M \text{ is odd,} \tag{16a}$$

and the coefficients a_m are related to the coefficients C_m as follows

$$a_m(\eta) = [1/(4^m(2m+1)!)]\frac{d^{2m+1}}{d\eta^{2m+1}}W(\eta). \tag{16b}$$

Consequently, the WDF in Eq. (13) can be written as

$$F(y,\eta) = \int_{-\infty}^{\infty} \exp\left(jk \sum_{m=0}^{[M/2]} a_m(\eta)\eta'^{2m+1}\right)$$
$$\tilde{P}_0(\eta+\eta'/2)\tilde{P}_0^*(\eta-\eta'/2)\exp(j2\pi\eta' y/\lambda R)\,d\eta'. \tag{17}$$

Here again, as in obtaining Eq. (7) from Eq. (6), by invoking the derivative theorem of the Fourier transform, the WDF in Eq. (17) can be written as

$$F(y,\eta) = \exp\left(-\sum_{M=0}^{[M/2]} b_m(\eta)\partial^{2m+1}/\partial y^{2m+1}\right) F_0(y,\eta). \tag{18}$$

In Eq. (18) F_0 denotes the WDF for the diffraction limited case

$$F_0(y,\eta) = \int_{-\infty}^{\infty} \tilde{P}_0(\eta+\eta'/2)\tilde{P}_0^*(\eta-\eta'/2)\exp(j2\pi y\eta'/\lambda R)\,d\eta', \tag{19}$$

and

$$b_m(\eta) = [R^{2m+1}/((2k)^{2m}(2m+1)!)]\frac{d^{2m+1}}{d\eta^{2m+1}} W(\eta). \tag{20}$$

The meaning of the differential operator in Eq. (18) is discussed in what follows. It is perfectly valid to split the operator in Eq. (18) in two parts. The first part includes only the term $m = 0$ in the sum within the argument of the exponential function. The second part contains the remaining terms $m = 1, 2, 3 \ldots$. Hence, Eq. (18) is equivalent to

$$F(y,\eta) = \exp\left[-\sum_{m=1}^{[M/2]} b_m(\eta)\frac{d^{2m+1}}{dy^{2m+1}}\right]\exp\left[-R\frac{dW}{d\eta}\frac{d}{dy}\right]F_0(y,\eta). \tag{21}$$

But, according to Taylor's series theorem any continuous function $g(x)$ derivable n times (with $n\to\infty$) can be approximated as

$$g(x-x_0) = \left[1+\frac{(-x_0)}{1!}\frac{d}{dx}+\frac{(-x_0)^2}{2!}\frac{d^2}{dx^2}+\ldots\right]g(x) = \exp\left(-x_0\frac{d}{dx}\right)g(x). \tag{22}$$

This means that

$$\exp\left(-R\frac{dW}{d\eta}\frac{d}{dy}\right)F_0(y,\eta) = F_0\left(y-R\frac{dW}{d\eta},\eta\right), \tag{23}$$

that is, the first term of the operator only indicates a displacement of the WDF, which is precisely the displacement predicted by geometrical optics (see Eq. (2b)).

Moreover, the second term of the operator in Eq. (21) does not contain "completed" Taylor's series, as in Eq. (23), they are rather "incompleted" series in the same sense as those discussed in Eqs. (7)–(12). In other words, the WDF will modify its shape in a similar fashion as the Point Spread Function modifies its shape under the influence of aberrations. This analogy can be further discussed making use of the projection theorem

$$|P(y)|^2 = \int_{-\infty}^{\infty} F(y, \eta)\, d\eta. \tag{24}$$

This discussion goes, however, beyond our present scope.

4. Conclusion

We have shown that the influence of aberrations on the WDF can be expressed by means of a differential operator of exponential type. The first part of this operator predicts a coordinate transformation of the WDF in accordance with the formulae for transversal ray aberrations from geometrical optics. The second part of the operator describes the functional variations of the WDF, occuring in a similar fashion to the changes suffered by the Point Spread Function.

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