Optica Applicata, Vol. XII, No. 3-4, 1982

Far field radiation patterns of elliptical apertures and its annuli

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A Fourier transform technique is applied to derive expressions for the diffraction field due to elliptical apertures and its annuli. Perspective plots of the intensity distribution diagrams are generated for various values of the axis-ratio with the help of IBM-370/168 computer. All possible cases are discussed briefly.

1. Introduction

The problem of determining the field radiated by an elliptical aperture has received much attention recently [1-3]. In most of these treatments the analysis is based on different techniques which either impose few limitations on the method used [1] or lead to a complicated derivation [2, 3]. However, with the application of Fourier transform, the solution to these problems becomes much more simple and faster. But only few authors [6, 7] have tried to apply it on elliptical aperture. This communication which uses this Fourier transform technique gives a rapid solution to this problem of many authors [1-3], [10-12].

2. Analysis

Consider the diffraction of a scalar wave field by an aperture as shown in Fig. 1. Then the diffraction field at the observation point p_0 in in the far field region can be written as a two-dimensional Fourier transform [4, 5]:

$$U(x_0,y_0) = C\iint_{-\infty}^{\infty} U_A(x_1,y_1)\exp\left[-\frac{ik}{z}(x_0x_1 + y_0y_1)\right]dx_1dy_1 \qquad (1)$$

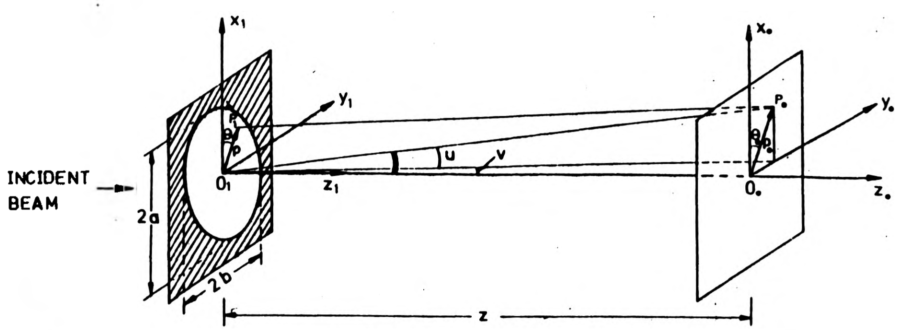

Fig. 1. Block diagram for the diffraction of a scalar wave field by an elliptical aperture. $P_1 = P_1(x_1, y_1, z_1)$, $P_0 = P_0(x_0, y_0, z_0)$

where

$$C = \frac{e^{jkz}}{j\lambda z} \exp\left[\frac{jk}{2z}(x_0^2 + y_0^2)\right],$$

C can be regarded as constant,

$U_A(x_1, y_1)$ - the aperture illumination,

(x_1, y_1); (x_0, y_0) - represent the coordinate system in the plane of the aperture and in the plane of observation,

$z = |z_1 - z_0|$ - distance between the two planes,

$k = 2\pi/\lambda$, λ - wavelength of radiation.

Applying the integral (1) to an elliptical aperture, one obtains

$$U(x_0, y_0) = C \iint_{\text{Ellipse}} U_A(x_1, y_1/\varepsilon) \exp\left[-\frac{jk}{z}(x_0 x_1 + y_0 y_1)\right] dx_1 dy_1,$$

where $\varepsilon = b/a$ - the axial ratio of the ellipse and 2a, 2b are the major and minor axis of the ellipse.

For uniform illumination of the aperture

$$U_A(x_1, y_1/\varepsilon) = 1 \quad \text{inside the aperture,}$$
$$= 0 \quad \text{outside the aperture.}$$

Defining the geometry of the aperture in elliptical coordinate [1, 3]

In the aperture plane:	In the observation plane:
$x_1 = \rho \cos\theta$	$x_0 = \varepsilon\rho_0 \cos\theta_0$
$y_1 = \varepsilon\rho \sin\theta$	$y_0 = \rho_0 \sin\theta_0$
$z_1 = 0$	$z_0 = z$

$$dx_1 dy_1 = \varepsilon\rho d\rho d\theta.$$

Therefore the above integral reduces to

$$U(\rho_0, \theta_0) = C\int_0^a \int_0^{2\pi} \exp\left[-\frac{jk}{z}\varepsilon\rho\rho_0 \cos(\theta - \theta_0)\right]\varepsilon\rho d\rho d\theta. \qquad (2)$$

Since the origin of azimuth can be changed, so that one can replace $\cos(\theta - \theta_0)$ by $\cos\theta$.

By using the identity

$$J_n(x) = \frac{j^{-n}}{2\pi}\int_0^{2\pi} e^{jx\cos\alpha'} e^{jn\alpha'} d\alpha'$$

the integral (2) reduces to

$$U(\rho_0,\theta_0) = 2\pi\varepsilon C\int_0^a J_0\left(k\rho\frac{\rho_0}{z}\varepsilon\right)\rho d\rho$$

with its solution

$$U(\rho_0, \theta_0) = 2\pi\varepsilon a^2 C\frac{J_1\left(\dfrac{k\varepsilon a\rho_0}{z}\right)}{k\varepsilon a\dfrac{\rho_0}{z}}. \qquad (3)$$

Introducing the dimensionless parameters

$$u = x_0/z, \quad v = y_0/z$$

so that

$$\frac{\rho_0}{z} = \left[\left(\frac{u}{\varepsilon}\right)^2 + v^2\right]^{1/2}$$

The equation (3) simplifies to

$$U(\rho_0, \theta_0) = abC\frac{2J_1(k\sqrt{a^2u^2 + b^2v^2})}{k\sqrt{a^2u^2 + b^2v^2}} \qquad (4)$$

and the normalized intensity is obtained as

$$I = \left\{ \frac{U(\rho_0, \theta_0)}{U(0,0)} \right\}^2 = \left\{ \frac{2J_1(k\sqrt{a^2u^2 + b^2v^2})}{k\sqrt{a^2u^2 + b^2v^2}} \right\}^2 . \tag{5}$$

This is the far field intensity distribution due to an elliptical aperture.

Let us now consider below a few interesting cases.

A. Circular aperture

In this case $a = b$, i.e. $\varepsilon = 1$ and the expression (5) reduces to standard Airy's formula [5], i.e.

$$I = \left\{ \frac{2J_1\left(ka\frac{\rho_0}{z}\right)}{ka\frac{\rho_0}{z}} \right\}^2 \tag{6}$$

B. Elliptic annulus (similar ellipses)

Here we apply equation (4) to an elliptic annulus made from two concentric but similar ellipses having the same axis-ratio, i.e., $\varepsilon = b/a = b'/a' = \varepsilon'$ with the different length of the axes, i.e., $a' = \alpha a$, where α is the obstruction ratio. Then the diffraction field due to inner ellipse can be written as

$$U'(\rho_0, \theta_0) = 2\pi\varepsilon a'^2 C \frac{J_1\left(k\varepsilon a'\frac{\rho_0}{z}\right)}{k\varepsilon a'\frac{\rho_0}{z}}$$

which can be simplified to give

$$U'(\rho_0, \theta_0) = 2\pi\varepsilon a'^2 C \frac{J_1\left(k\varepsilon a'\frac{\rho_0}{z}\right)}{k\varepsilon a'\frac{\rho_0}{z}} .$$

Therefore the resultant diffraction field becomes

$$U_R(\rho_0, \theta_0) = U(\rho_0, \theta_0) - U'(\rho_0, \theta_0) =$$

$$= \pi abC \left\{ \frac{2J_1(k\sqrt{a^2u^2 + b^2v^2})}{k\sqrt{a^2u^2 + b^2v^2}} \right.$$

$$- \alpha^2 \frac{2J_1(k\alpha\sqrt{a^2u^2 + b^2v^2})}{k\alpha\sqrt{a^2u^2 + b^2v^2}} \Bigg] ,$$

with the normalized intensity

$$I = \left[\frac{U_R(\rho_0, \theta_0)}{U_R(0,0)}\right]^2 = \frac{1}{(1-\alpha^2)^2}\left[\frac{2J_1(k\sqrt{a^2u^2 + b^2v^2})}{k\sqrt{a^2u^2 + b^2v^2}} - \alpha^2 \frac{2J_1(k\alpha\sqrt{a^2u^2 + b^2v^2})}{k\alpha\sqrt{a^2u^2 + b^2v^2}}\right]^2 \qquad (7)$$

C. Elliptic annulus (confocal ellipses)

Similarly one can extend the above analysis to an elliptic annulus made from two confocal ellipses. In that case the relation

$$a^2 - b^2 = a'^2 - b'^2$$

with $a' < a$, and $b' < b$ must be satisfied.

Now using equation (4) we write the diffraction field due to the inner ellipse as

$$U'(\rho_0, \theta_0) = \pi a'b'C \frac{2J_1(k\sqrt{a'^2u^2 + b'^2v^2})}{k\sqrt{a'^2u^2 + b'^2v^2}} .$$

Therefore the resultant field due to elliptic annulus becomes

$$U_R(\rho_0, \theta_0) = U(\rho_0, \theta_0) - U'(\rho_0, \theta_0),$$

$$U_R(\rho_0, \theta_0) = \pi C \left[ab \frac{2J_1(k\sqrt{a^2u^2 + b^2v^2})}{k\sqrt{a^2u^2 + b^2v^2}} - a'b' \frac{2J_1(k\sqrt{a'^2u^2 + b'^2v^2})}{k\sqrt{a'^2u^2 + b'^2v^2}} \right].$$

If one expands the Bessel functions into series and simplifies the whole relation, one shall get equation (18) of reference [2] in which the author arrived at the same results based on a conformal mapping technique that uses the complex variables.

The normalized intensity becomes

$$I = \left[\frac{U_R(\rho_0, \theta_0)}{U_R(0,0)} \right]^2 =$$

$$= \left[\frac{ab \dfrac{2J_1(k\sqrt{a^2u^2 + b^2v^2})}{k\sqrt{a^2u^2 + b^2v^2}} - a'b' \dfrac{2J_1(k\sqrt{a'^2u^2 + b'^2v^2})}{k\sqrt{a'^2u^2 + b'^2v^2}}}{a\,b - a'\,b'} \right]^2 \tag{8}$$

D. Annular circular aperture

In this case $b = a$, $b' = a' = \alpha a$, and $\varepsilon = \varepsilon' = 1$, so that the equations (7) and (8) reduce to

$$I = \frac{1}{(1-\alpha^2)^2} \left\{ \frac{2J_1\left(k\,a\,\frac{\rho_0}{z}\right)}{k\,a\,\frac{\rho_0}{z}} - \alpha^2 \frac{2J_1\left(\alpha\,k\,a\,\frac{\rho_0}{z}\right)}{\alpha\,k\,a\,\frac{\rho_0}{z}} \right\}^2 \tag{9}$$

which agrees with the results of LINFOOT and WOLF [8] given many years ago.

3. Computational results and discussion

The results of the intensity patterns computed from equations (5), (7), (8) and (9) are displayed in Figs. 2–5. It is interesting to notice from Figs. 2a–c that as the axis-ratio (b/a) decreases from unity to some smaller values, i.e., for $\varepsilon = b/a = 1$, 0.5, and 0.25 the intensity pattern changes from circular symmetric to elliptical and finally to a slit type pattern. Figs. 3a–3c show the corresponding intensity patterns due to an elliptic annulus (case B) with $\varepsilon = b/a = b'/a' = \varepsilon' = 0.5$ for the obstruction ratios $\alpha = a'/a = 0.1$, 0.4, and 0.7, respectively. From these three cases it is observed that with the increasing obstruction ratio, the central maxima tends to become narrow and there is a flow of energy to the secondary rings which results in an increase in the intensity of the secondary maxima. The same argument holds for the Figs. 5a–5c which represent the similar diagrams for

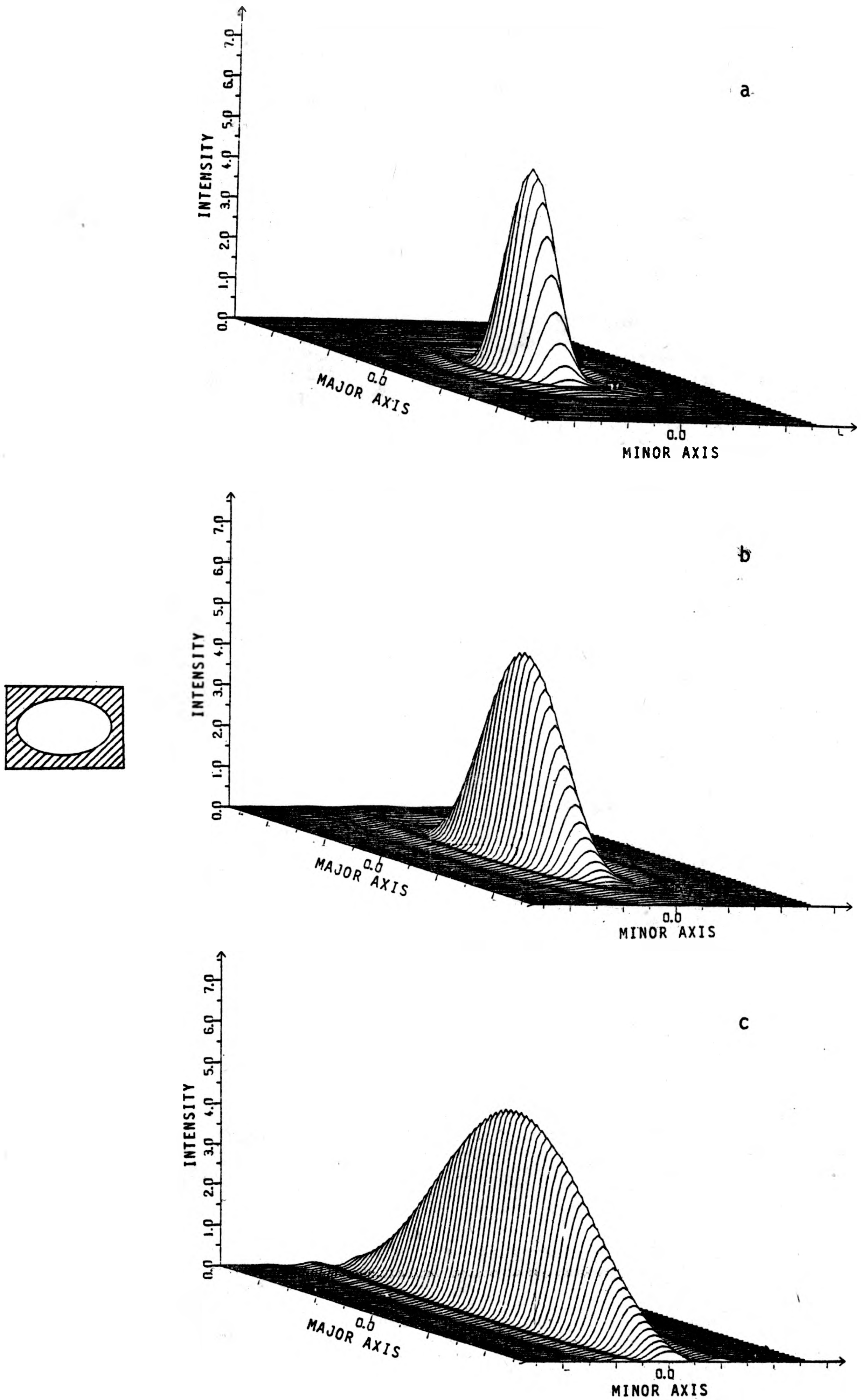

Fig. 2. Normalized intensity in arbitrary units vs. normalized co-ordinates $(x_0/a, y_0/b)$ along the major- and the minor-axis for elliptical aperture with: a. $\varepsilon = 1$, b. $\varepsilon = 0.5$, c. $\varepsilon = 0.25$

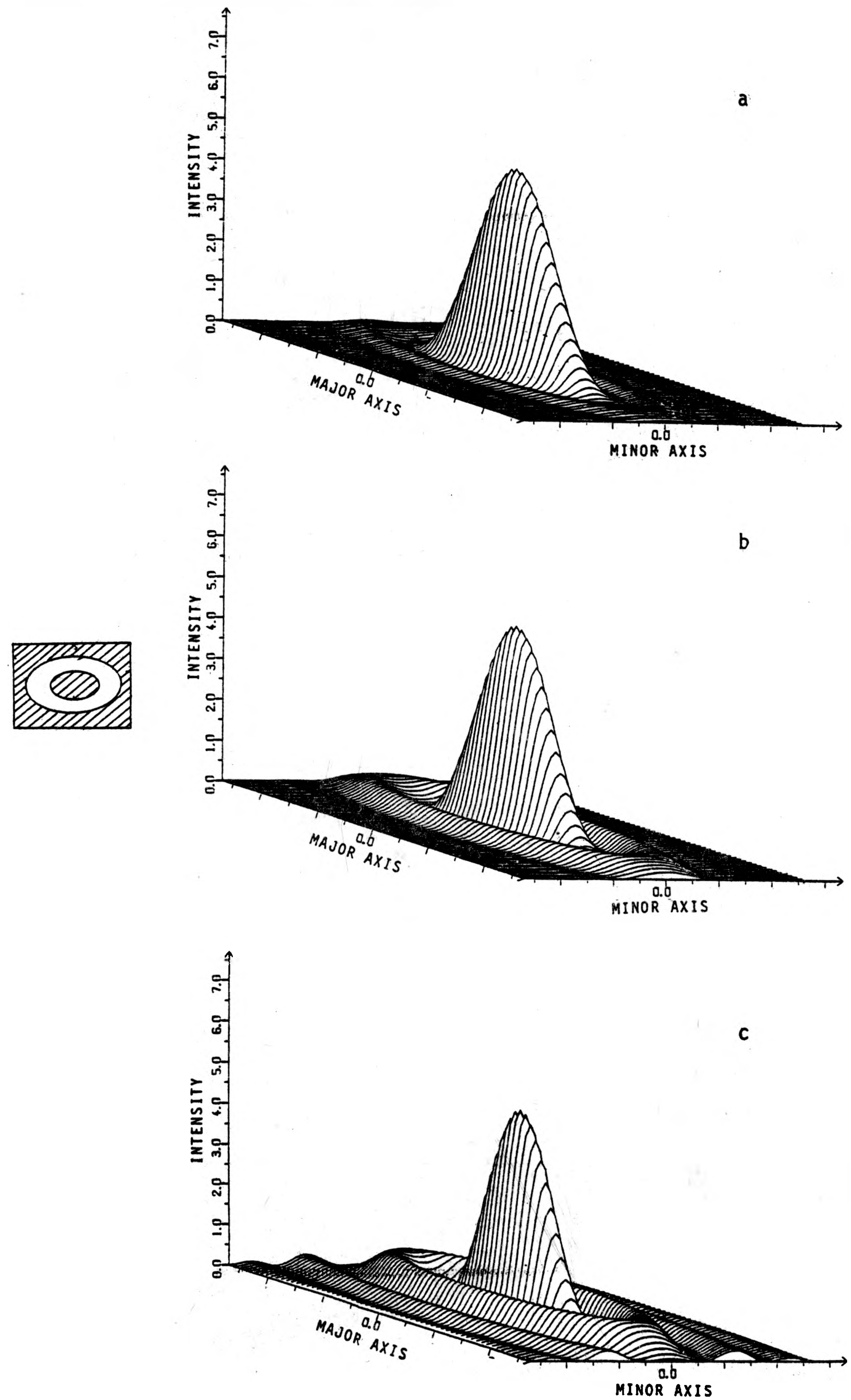

Fig. 3. Normalized intensity in arbitrary units vs. normalized co-ordinates $(x_0/a, y_0/b)$ along the major- and the minor-axis for elliptic annulus (Case B) with: a. $\alpha = 0.1$, $\varepsilon = 0.5$, b. $\alpha = 0.4$, $\varepsilon = 0.5$, c. $\alpha = 0.7$, $\varepsilon = 0.5$

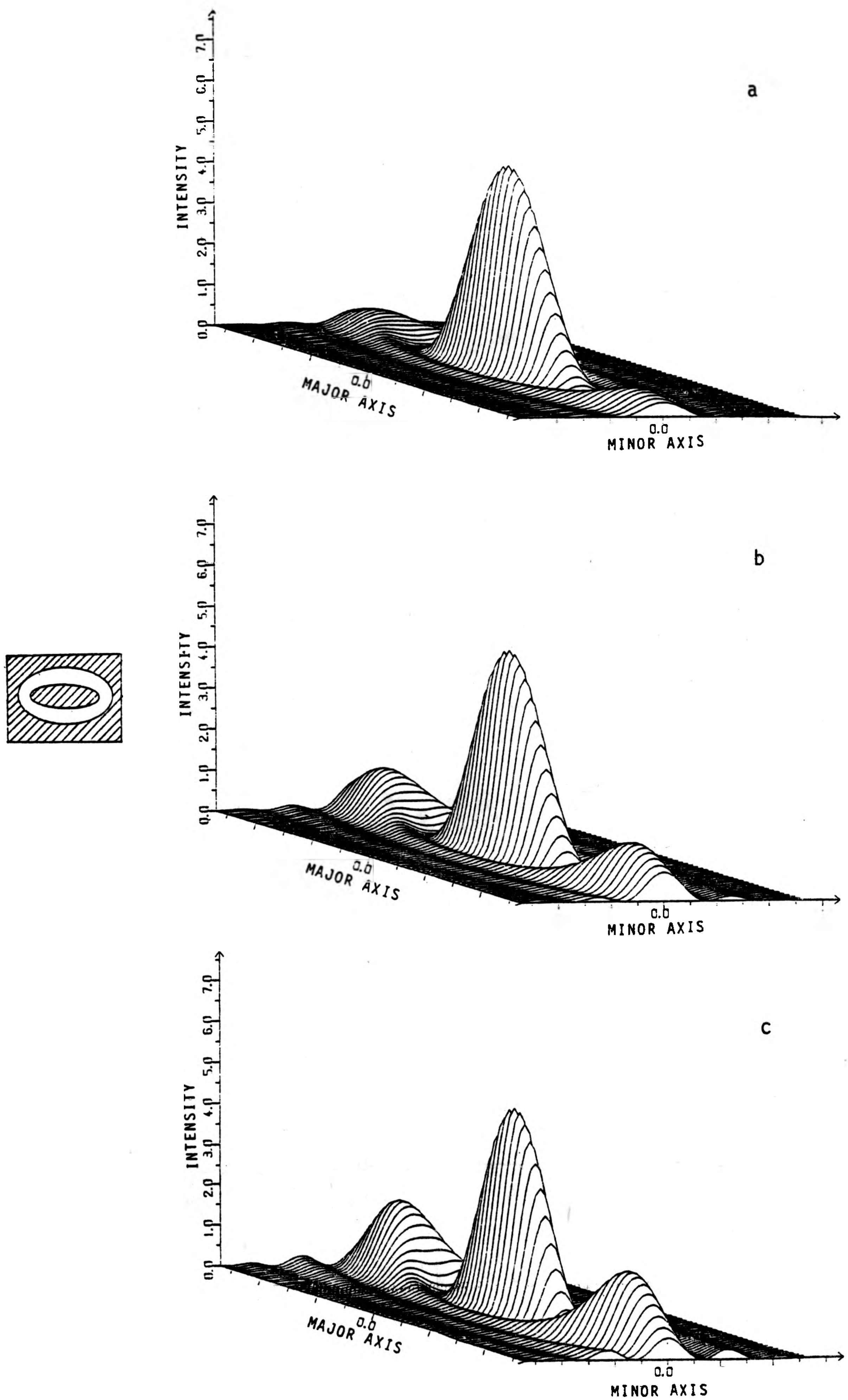

Fig. 4. Normalized intensity in arbitrary units vs. normalized co-ordinates (x_0/a, y_0/b) along the major- and the minor-axis for elliptical annulus (Case C) with: a. $\varepsilon = 0.5$, $\varepsilon' = 0.1$, b. $\varepsilon = 0.5$, $\varepsilon' = 0.2$, c. $\varepsilon = 0.5$, $\varepsilon' = 0.3$

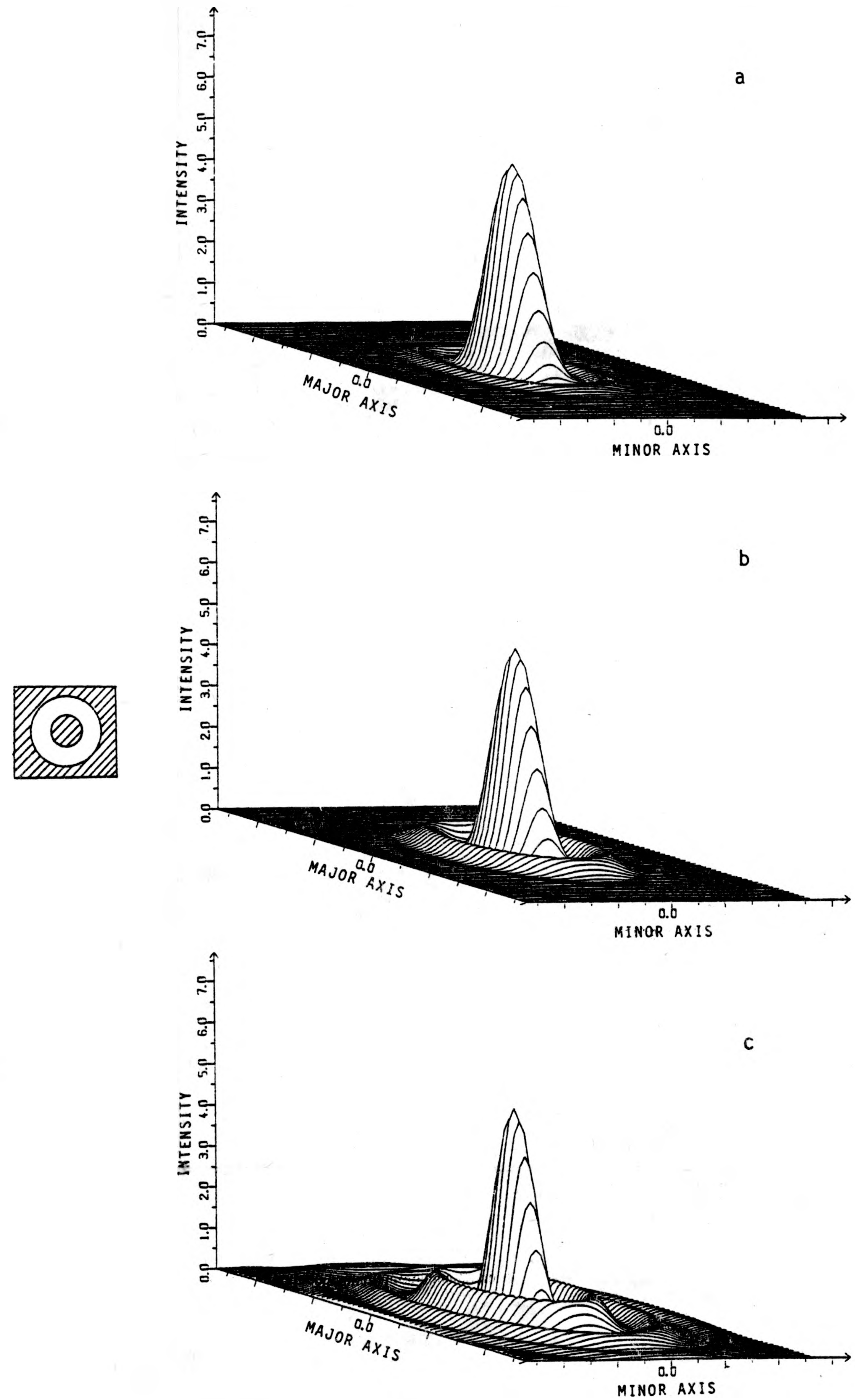

Fig. 5. Normalized intensity in arbitrary units vs. normalized co-cordinates (x_0/a, y_0/b) along the major- and the minor-axis for annular circular aperture (Case D) with: a. $\alpha = 0.1$, b. $\alpha = 0.4$, c. $\alpha = 0.7$

annular circular aperture (case D) with $\varepsilon = b/a = b'/a' = \varepsilon' = 1$ for $\alpha = 0.1$, 0.4, and 0.7, respectively [9]. But on the other hand, in case of confocal elliptic annulus (case C), this flow of energy is towards the side lobes. This is illustrated in Figs. 4a–4c for $\varepsilon = b/a = 0.5$, $\varepsilon' = b'/a' = 0.1$, 0.2, and 0.3, respectively. This way one can generate a family of intensity patterns of the elliptical apertures and its annuli by varying the parameters ε, α and ε' [3].

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Received April 3, 1982

ДИФРАКЦИОННЫЕ ФИГУРЫ ДАЛЕКОГО ПОЛЯ ЭЛЛИПТИЧЕСКИХ ОТВЕРСТИЙ И КОЛЕЦ

Техника преобразований Фурье была применена для выведения выражения для дифракционного поля, образованного на эллиптических отверстиях и кольцах. Приведены перспективные графики распределения интенсивностей для различных значений отношений оси эллипса, полученные с помощью ЭВМ ИБМ-370/168. Кратко обсуждены все возможные случаи.