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Concept of quasi-differentiating and possibilities of its optical implementation*

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In this paper the concept of quasi-differentiating has been presented based on certain properties of the Fourier transform. An analytic example of the quasi-differentiating operation is given. Also, a method of optical implementation by coherent filtering is proposed and the experimental results presented.

1. Introduction

Concept of non-integer order derivative based on certain properties of Fourier transform is mentioned by BRACEWELL [1] and extensively discussed in papers [2, 3], where also the optical realization is analysed. The approach to the non-integer order derivatives is presented in these works, in which the theorem about the Fourier transform of function derivative allows a further generalization of this concept.

In this paper such a generalized operation, called quasi-derivative, is presented. This operation has been illustrated by an analytic example. In the experimental part a coherent processor equipped with binary filters suitable to perform the quasi-differentiating operation has been proposed to illustrate experimentally the concept.

If a function $f(x)$ and its derivatives of an integer n–1-order fulfil the existence conditions for Fourier transform, then the n-th derivative $D^n f(x)$ has the transform $(2\pi ui)^n \cdot F(u)$, where $F(u)$ is the Fourier transform of $f(x)$. More general mathematical operation may be obtained by admitting the non-integer exponent r (in place of natural number n) in the last expression and performing the inverse Fourier transform of the term $(2\pi ui)^r \cdot F(u)$ [3]. The non-integer order derivative of the function $f(x)$ may be defined

$$D^r f(x) = \int_{-\infty}^{+\infty} \left[(2\pi ui)^r \int_{-\infty}^{+\infty} f(x) e^{-i2\pi ux} dx\right] e^{i2\pi ux} du. \tag{1}$$

The above generalization of the derivative to the non-integer orders may be

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illustrated graphically. For this purpose let us use the identity

$$(2\pi ui)^r = |2\pi u|^r e^{i\frac{\pi}{2} r \operatorname{sgn} u} \tag{2}$$

This expression may be treated as a complex vector which may be represented in the complex plane as shown in fig. 1. As it follows from this figure the subse-

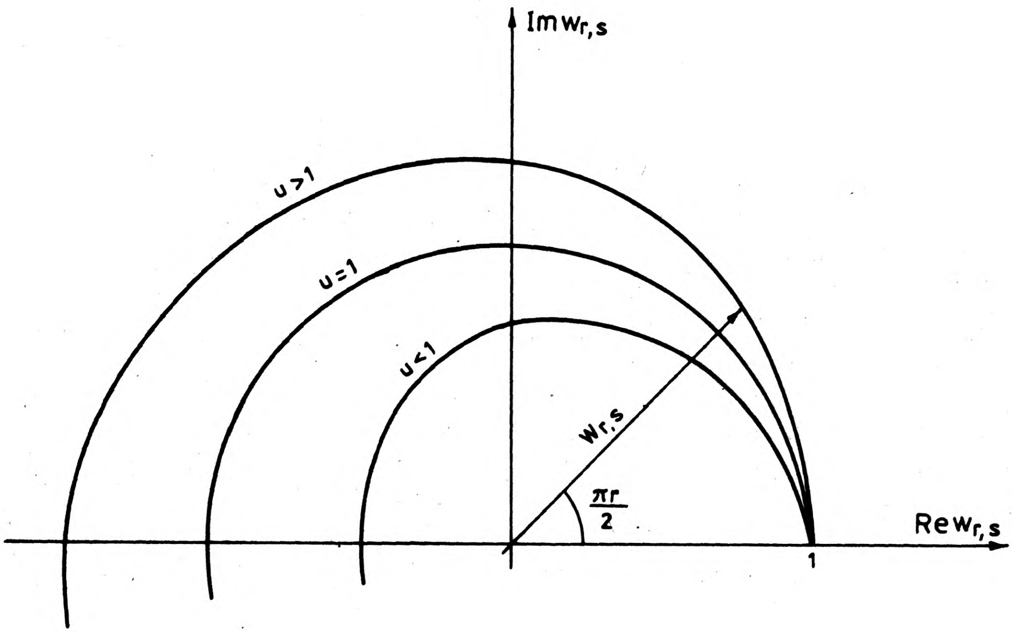

Fig. 1

quent operations of natural-order differentiation correspond to multiplication of its Fourier transform by the suitable complex vectors $(ui)^r$ rotating around the origin of the coordinate system along the spiral curve and overlapping consecutively the coordinate axes. The omitted factor $(2\pi)^r$ may be always taken out in front of the integral.

In accordance with this interpretation the generalization of the differentiation to comprise also the non-integer orders consists in taking account of all the intermediate vectors at the positions different from those on the coordinate axis. The vectors at consecutive positions taken during their clockwise ($u < 0$) or counter-clockwise ($u > 0$) rotation change their moduli and arguments in a strictly defined way depending on differentiation order r.

2. Quasi-differentiation

Basing on the above graphical interpretation it is possible to further generalize the operator of differentiation. Let us, namely, assume that the vector representing the complex factor in front of the Fourier transform rotates as shown in fig. 2 changing both its modulus and argument like in (2), but independently of one another. Such a vector may be represented as

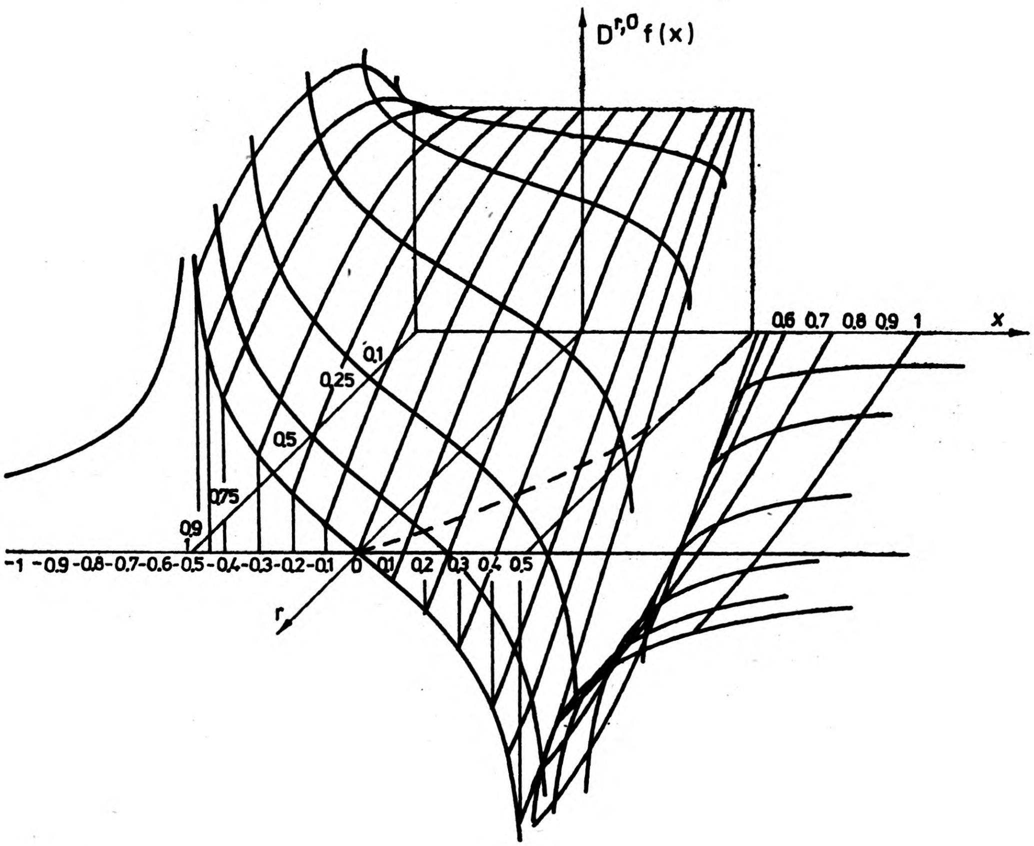

Fig. 2

$$w_{r,s} = (2\pi)^s \cdot |u|^s e^{i\frac{\pi}{2} r \operatorname{sgn} u}. \tag{3}$$

In the case when $r = s$ we obtain the previously defined non-integer order differentiation. However, for $r \neq s$ we obtain the final operation for the function $f(x)$, which may be represented in the form

$$D^{r,s}f(x) = (2\pi)^s \int_{-\infty}^{+\infty} |u|^s F(s) e^{i\left(\frac{\pi}{2} r \operatorname{sgn} u + 2\pi ux\right)} du. \tag{4}$$

After performing the respective transformation the above expression may be put in the form

$$D^{r,s}f(x) = 2(2\pi)\left\{\cos\frac{\pi r}{2} \int_0^\infty u^s [P_c(u)\cos 2\pi ux - N_s(u)\sin 2\pi ux]\,du \right.$$
$$\left. - \sin\frac{\pi r}{2} \int_0^\infty u^s [P_c(u)\sin 2\pi ux + N_s(u)\cos 2\pi ux]\,du, \right. \tag{5}$$

where $P_c(u)$ is the Fourier transform of cosine type of the even part of the function $f(x)$, while $N_s(u)$ is the sine Fourier transform of the odd part of the function $f(x)$. The functions $D^{r,s}(x)$ will be called quasi-derivatives of s-order with respect to the modulus and of r-order with respect to the argument. Let us consider the following cases:

i) For $r = 0$, $s = 1$ we obtain the factor in the form $|u|^s$. The vector $w_{r,s}$ lies on the horizontal axis, which may be called the axis of the function, but its modulus will be equal to the modulus of the vector corresponding to the first derivative.

ii) For $r = 1$, $s = 0$ we obtain the factor in the form $e^{i\pi 2 \operatorname{sgn} u}$. The vector $w_{r,s}$ lies on the vertical axis, which may be called the axis of the first derivative, but its modulus is equal to unity as it is the case for the vector corresponding to the original function.

3. An example

As an example of the operation (4) let us consider the result of action of such an operator on the rectangular function $\Pi(x)$. The Fourier transform of the function $\Pi(x)$ is

$$F(u) = \frac{\sin \pi u}{\pi u}. \tag{6}$$

By substituting the above transform to the formula (5) and taking advantage of the fact that the original function is odd we obtain

$$D^{r,s}f(x) = 2(2\pi)^s \left(\cos\frac{\pi r}{2}\int_0^\infty u^s \frac{\sin \pi u}{\pi u}\cos 2\pi u x\, du \right.$$
$$\left. -\sin\frac{\pi r}{2}\int_0^\infty u^s \frac{\sin \pi u}{\pi u}\sin 2\pi u x\, du\right). \tag{7}$$

After calculating the above integral we get finally

$$D^{r,s}f(x) = \begin{cases} \dfrac{2^s\Gamma(s)}{\pi}\sin\dfrac{\pi}{2}(s-r)\left[\dfrac{1}{(1-2x)^s} - \dfrac{1}{(-1-2x)^s}\right], & \text{for } x < -\dfrac{1}{2}, \\[2ex] \dfrac{2^s\Gamma(s)}{\pi}\left[\dfrac{\sin\frac{\pi}{2}(s+r)}{(1+2x)^s} + \dfrac{\sin\frac{\pi}{2}(s-r)}{(1-2x)^s}\right], & \text{for} -\dfrac{1}{2} \leqslant x < \dfrac{1}{2}, \\[2ex] \dfrac{2^s\Gamma(s)}{\pi}\sin\dfrac{\pi}{2}(s+r)\left[\dfrac{1}{(1+2x)^s} - \dfrac{1}{(2x-1)^s}\right], & \text{for } x \geqslant \dfrac{1}{2}. \end{cases} \tag{8}$$

As may be noted in the above relations the derivative $D^{r,s}f(x)$ for $s = 0$ becomes indetermined. Therefore the limit

$$D^{r,0}f(x) = \lim_{s \to 0} D^{r,s}f(x). \tag{9}$$

must be evaluated. The calculation of the above limit based on the de L'Hospital theorem leads to the formula

$$D^{r,0}f(x) = \begin{cases} \dfrac{1}{\pi}\sin\dfrac{\pi r}{2}\ln\left(\dfrac{1-2x}{-1-2x}\right), & \text{for } x < -\dfrac{1}{2}, \\ \cos\dfrac{\pi r}{2} + \dfrac{1}{\pi}\sin\dfrac{\pi r}{2}\ln\left(\dfrac{1-2x}{2+2x}\right), & \text{for } -\dfrac{1}{2} \leqslant x < \dfrac{1}{2}, \\ \dfrac{1}{\pi}\sin\dfrac{\pi r}{2}\ln\left(\dfrac{2x-1}{2x+1}\right), & \text{for } x \geqslant \dfrac{1}{2}. \end{cases} \tag{10}$$

In the fig. 2 the derivative $D^{r,0}f(x)$ is presented as a function of the variables r and x. As it follows from this figure $D^{r,0}f(x)$ changes from the even-type function for $r = 0$ to the odd type one for $r = 1$. The derivative $D^{0,s}f(x)$ is shown in fig. 3. It is even for arbitrary value of s. In fig. 3 the run of $D_{0,s}f(x)$ is shown for $x \in \left(-\dfrac{1}{2}, 0\right)$ and $x \in \left(\dfrac{1}{2}, \infty\right)$ to make the graph more readable.

4. Optical implementation

In order to realize the operation of quasi-differentiation by optical means a coherent optical processor equipped with the suitable binary filter may be used [4]. The amplitude transmittance of the binary holographic filter may be calculated from the Kogelnik formula [5] defining diffraction efficiency of the binary holographic filter

$$\eta = \frac{l}{d}\left(\frac{\sin\dfrac{\pi l}{d}}{\pi l/d}\right)^2, \tag{11}$$

where l — width of the dark fringe,

d — distance between the midpoints of two neighbouring fringes.

After transforming the formula and taking account of the required amplitude transmittance, in accordance with (3), we obtain

$$l/d = \frac{1}{\pi}\arcsin(u/u_{\max})^s, \tag{12}$$

where $u_{\max}$ is the running coordinate at the filter edge. Since the relation (12)

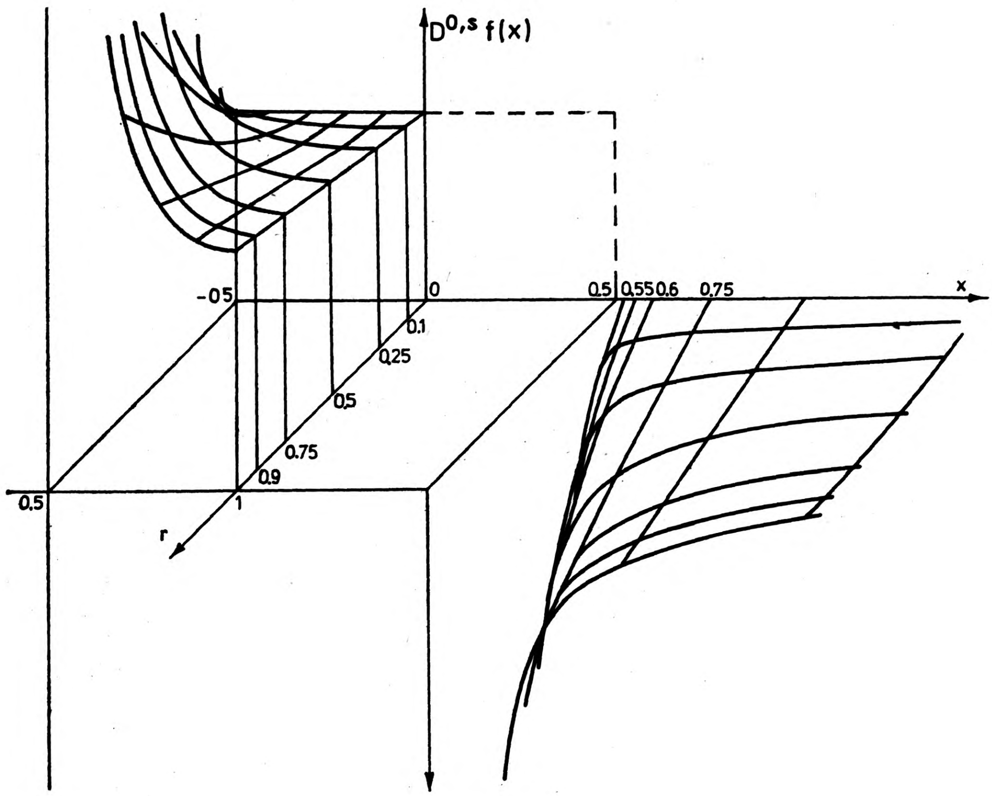

Fig. 3

describes only the changes of the absolute value of the amplitude, ignoring the phase, it is necessary to shift suitably the midpoints of the dark fringes on the left and right hand sides of the filters, repectively, so that the phase changes $\pi r/2$ be realized. This shift is equal to $r/4$ of the interfringe distance.

In figure 4 the binary filter to perform the one-dimensional quasi-differentiation operation for $r = 1/2$, $s = 0$ is shown. For such a case the diffraction efficiency of the filter is constant and the middle points of the fringes on the left and right hand parts of the filters are mutually shifted by the quantity equal to $1/4$ of the interfringe distance. After reducing the filter sizes to obtain the carrier frequency of the filter sufficiently high to separate the subsequent diffraction orders and to realize the filtering we obtain at the system output the image presented in figs. 5 and 6.

These figures present the effect of quasi-differentiation of an object in $+1$ and -1 orders of diffraction. In order to check the accuracy of the performed operation the photometering of the image in the first-order of diffraction has been carried out and the results were compared with the squared modulus

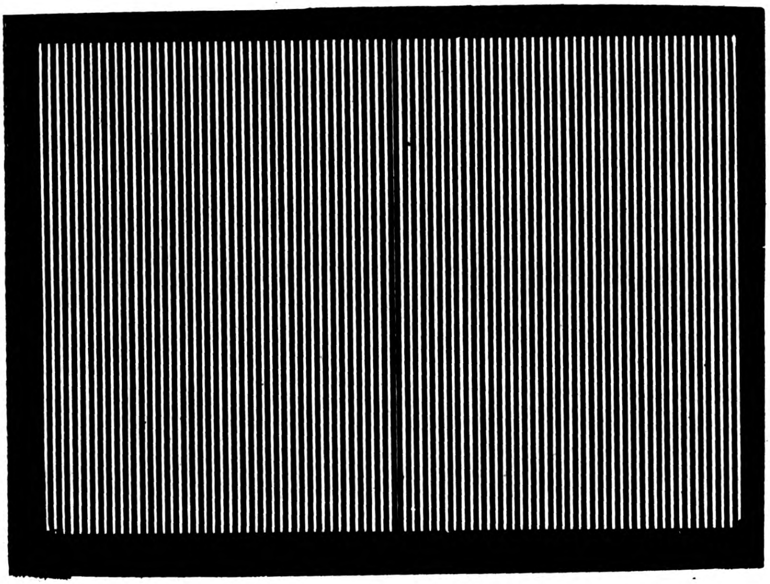

Fig. 4

of the quasi-derivative of the rectangle function calculated theoretically. The results are shown in fig. 7.

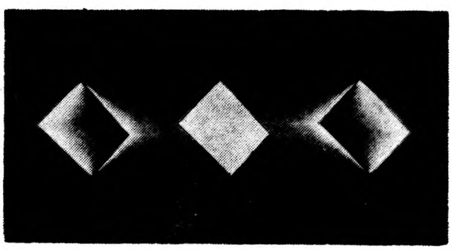

Fig. 5

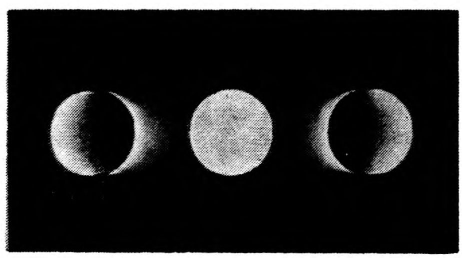

Fig. 6

The solid curve presents the light intensity in the image in the first order of diffraction and the broken curve the respective theoretical results. The author believes that the consistence of the curve obtained experimentally with the analytic curve is good. The differences are caused mainly by the coherent noises restricted by (small) sizes of the filter and inaccuracies of the filter implementation.

5. Concluding remarks

The purpose of this work was to define a new differentiating operation, i.e., the quasi-differentiating, and to illustrate the concept by a numerical example and to check the possibilities of its experimental realization with the satisfying accuracy. As it is clear from the above this purpose seems to be successfully achieved.

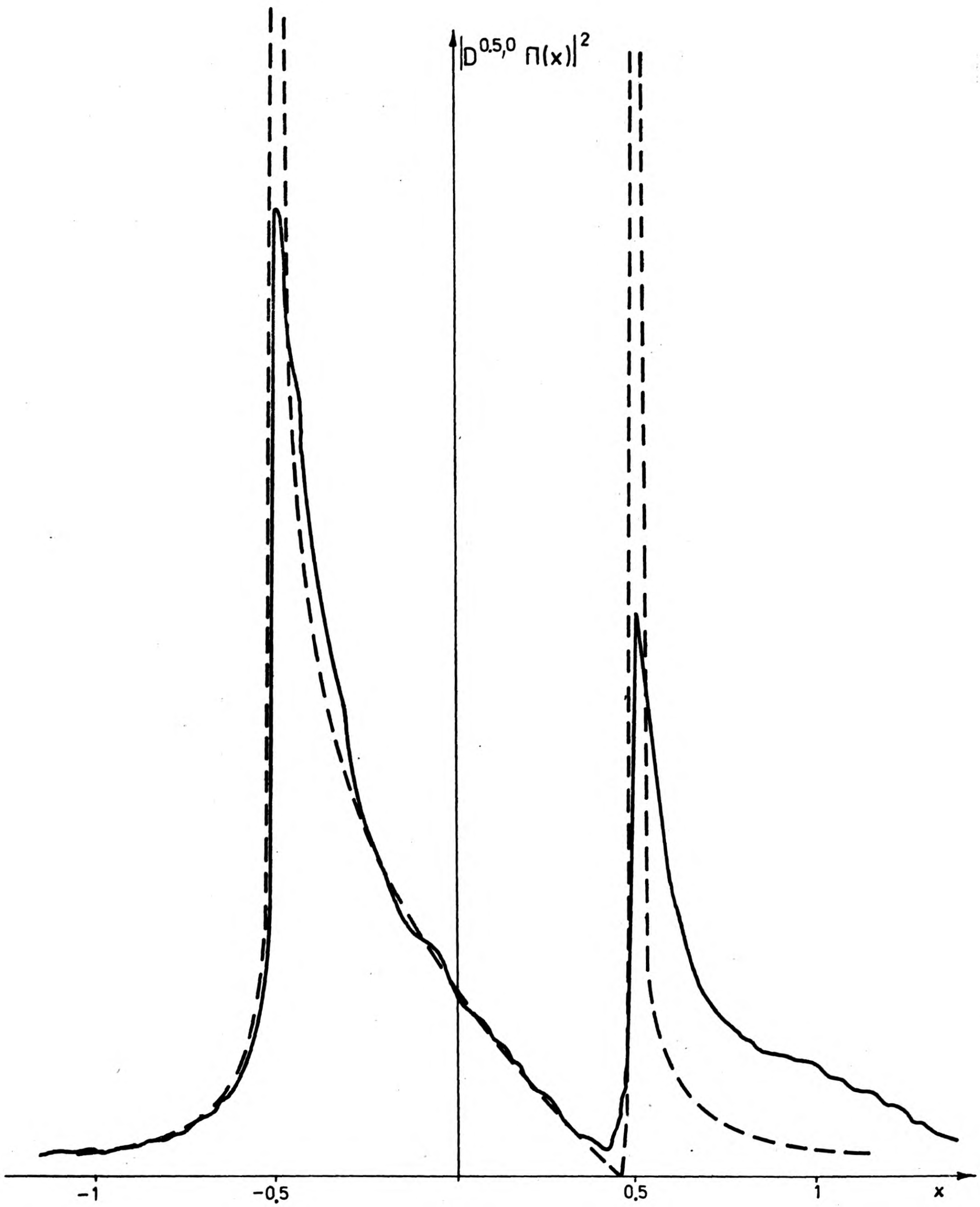

Fig. 7

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Концепция квазидифференцирования и возможность его оптической реализации

Представлена концепция производных функций нецелого порядка, определённых на основе некоторых свойств преобразования Фурье. Приведён аналитический пример такой операции, а также возможность выполнения оптического синтетического фильтра для практической реализации такой операции.