Optica Applicata, Vol. XI, No. 2, 1981

Integral diffraction efficiency of amplitude holograms

TADEUSZ LIPOWIECKI
Technical University of Radom, Radom, Poland.

The problem of Integral Diffraction Efficiency (IDE) of holograms is discussed. It is shown that IDE depends not only the exposure conditions in hologram processing, but also on the statistical properties of the recorded information waves. By application of an adequate large signal approximation of amplitude vs. exposure characteristic $T_a - H$ possibility of achieving maximum IDE is discussed. Experimental IDE measurements of amplitude holograms of randomly diffusing objects are presented.

1. Introduction

The problem, how to achieve an efficient holograms of high quality, is not a new one. There exists a lot of theoretical works in which the nonlinear properties of recording media, the influence of the information to reference power density ratios, and the course of other factors, which eventually can decide upon the final quality parameters of holograms are discussed. Unfortunately, it is difficult to find in this flood of literature some papers treating the influence of the statistical properties of the registered information waves upon diffraction efficiency of holograms.

This report concerns the problem, how the statistical properties of information waves affect the diffraction efficiency of amplitude holograms and gives some practical directions according to proper choice of exposure parameters during processing of the amplitude holograms.

2. Diffraction grating with sinusoidal amplitude profile

Let us consider at first the most simple example: two coherent plane waves with amplitudes u_1 and u_2, wavelength λ and parallel electric vectors, propagate and interfere in free space (fig. 1). A spatial stationary power density distribution because of interference effects arising in the space can be described in the plane S in the direction x, as

$$P(x) = \frac{1}{2z_v}[u_1^2+u_2^2+2u_1u_2\cos(2\pi f_x x+\varphi_{12})], \tag{1}$$

where z_v — wave impedance of the free space,
φ_{12} — relative phase shift between the waves in the point $x = 0$,
f_x — spatial frequency of interference fringes in the plane S, equal to

$$f_x = \frac{\sin\alpha_1 - \sin\alpha_2}{\lambda}, \tag{2}$$

where α_1 and α_2 are relative angles between propagation directions of the waves and the normal vector to the plane S, respectively. In our case, the sign of α_2 is negative.

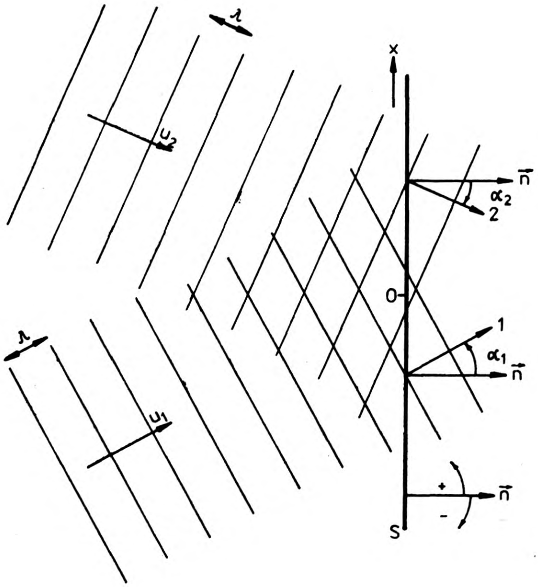

Fig. 1. Two interfering coherent plane waves

If we reduce the distribution (1) with respect to the power density of the wave u_1

$$P_1 = \frac{u_1^2}{2z_v} \tag{3}$$

and introduce the relative value of the amplitude u_2

$$\xi = u_2/u_1, \tag{4}$$

then we get a relative power density distribution in the plane S

$$p(x) = P(x)/P_1 = 1+\xi^2+2\xi\cos(2\pi f_x x+\varphi_{12}). \tag{5}$$

Suppose we have at our disposal an ideal linear medium which registers the power density distributions and satisfies the following conditions:

$$\left.\begin{array}{ll} T_a(H) = 1-\dfrac{H}{H_c} & \text{in the region } 0 \leqslant H \leqslant H_c \\ T_a(H) = 0 & \text{in the region } H > H_c \end{array}\right\}. \tag{6}$$

Because of a linear relation between the exposition $H(x)$ and the power density $P(x)$

$$H(x) = P(x)t, \tag{7}$$

t being the exposure time, it is possible to determine such a value of t that the "cut-off" value H_c would correspond to the defined values of power densities. So, let us determine such a value of the exposure time which allows the best use of the recording possibilities of our ideal medium. As can be seen from fig. 2,

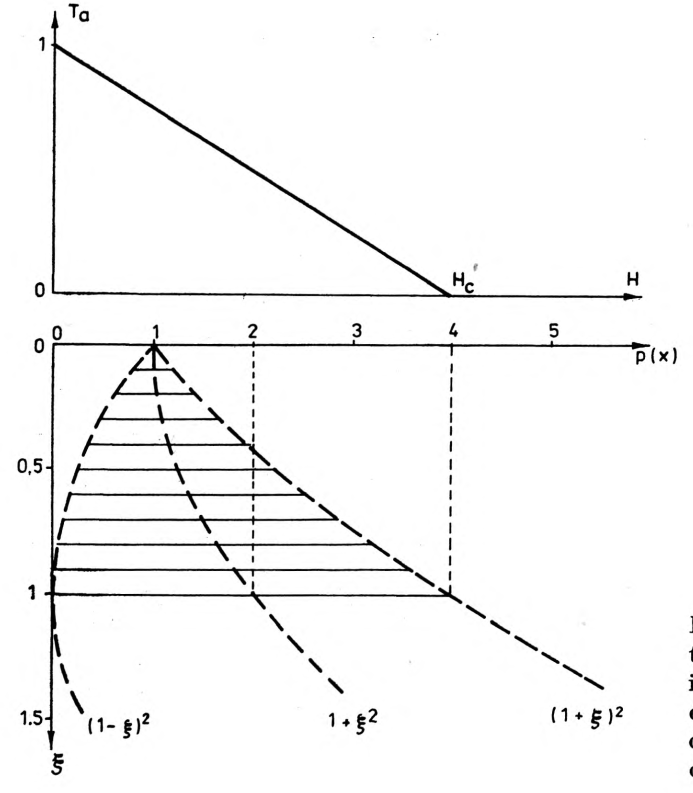

Fig. 2. Graphical illustration of the interaction between the ideal amplitude recording medium and the power density distribution generated by two coherent plane waves

the optimal conditions occur if H_c corresponds to the value of $p(x) = 4$. Now, we can write the relation between T_a and $p(x)$

$$T_a[p(x)] = 1 - \frac{1}{4} p(x). \tag{8}$$

Putting (5) into (8) we get finally

$$T_a(x, \xi) = \frac{3 - \xi^2}{4} - \frac{\xi}{2} \cos(2\pi f_x + \varphi_{12}). \tag{9}$$

Such a periodical structure is called a sinusoidal amplitude diffraction grating. The diffraction on the sinusoidal amplitude structure occurs in 0-order and ± 1-st orders only. The oscillating component in the relation (9) is responsible for the power distribution into the ± 1-st orders of the diffracted waves. The amplitude $\tilde{T}_a$ of this oscillating component consists of two equal amplitudes, responsible for the amplitudes of $+1$-st and -1-st order waves

$$T_{a_{+1}} = T_{a_{-1}} = \frac{1}{2}\tilde{T}_a = \frac{1}{4}\xi. \tag{10}$$

From this relation we can simply define the power diffraction efficiency of an ideal amplitude grating with sinusoidal profile

$$\eta_{+1} = \eta_{-1} = (\xi/4)^2 = \frac{1}{16}\xi^2. \tag{11}$$

If we assume the highest possible value of $\xi_{\max} = 1$, which corresponds to the total utilization of the recording possibilities of the linear medium, we get the maximal possible value of the diffraction efficiency, which can be obtained in the above assumed ideal conditions

$$\eta_{\max} = 1/16 = 6.25\,\%. \tag{12}$$

It is of interest in many publications the value (12) is considered to be as the maximal possible for obtaining the value of diffraction efficiency of amplitude holograms.

Let us look now at the situation in fig. 1 from "holographic" point of view. We can treat the waves u_1 and u_2 as reference and information waves, respectively. The quantity ξ^2 represents the relative power density of the information wave. On the other hand, the power diffraction efficiency of such a sinusoidal profile hologram depends linearly on the same value of ξ^2. In conclusion we see that there is a linear relation between the power of the information wave and the power of the diffracted wave, provided that recording is linear.

3. Local and integral diffraction efficiencies

Let us consider one of the most typical examples of holographic records (fig. 3). A surface of an object O illuminated by coherent radiation diffuses randomly this radiation; hence every point of the surface may be treated as an elementary source of a coherent spherical wave with random amplitude and phase. The whole assembly of the elementary waves generates an information wave u, which is not so homogeneous with respect to the amplitude and phase distributions, as in the case of plane or spherical waves. Power density distribution of the information wave possesses characteristic speckle structure, in which there exist zones both with much higher and much lower power densities than the average power density of the wave. Statistical properties of the power density distribution in coherent waves possessing speckle structure were tested in details and published in [1]. If we neglect the influence of the limited spatial frequency spectrum of the speckle structure on the smallest dimension of speckles, we can tell that the relative integral surface Δs of the hologram H, illuminated by the relative power densities contained in the interval ϱ, $\varrho+\Delta\varrho$, fulfils the relation [1]

$$\Delta s \Big|_{\varrho}^{\varrho+\varrho\Delta} = [1-\exp(-\Delta\varrho)]\exp(-\varrho), \tag{13}$$

where

$$\varrho = P/P_{av}, \tag{14}$$

P — local power density of the information wave in the hologram plane, P_{av} — average power density of the information wave in the hologram plane.

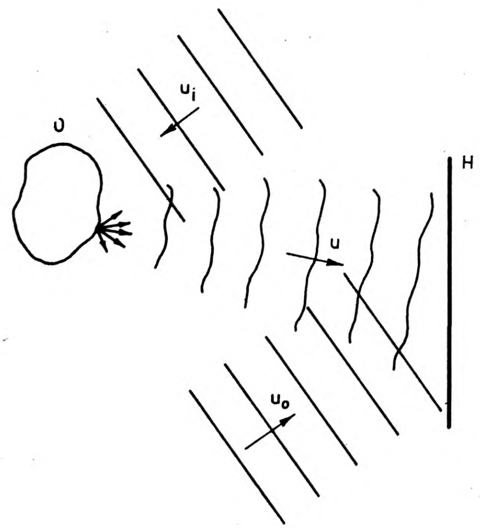

Fig. 3. A scheme of a holographic record of a randomly scattering object: O — object, H — hologram plane, u_1 — illuminating wave, u — information wave, u_0 — reference wave

The relation (13) enables to answer at once the following exemplary questions: what part of the relative integral hologram surface is illuminated by the information wave in two intervals of power densities, i.e. in 0–10 $\mu W/cm^2$ and 10–30 $\mu W/cm^2$, if the hologram plate is illuminated by randomly diffused coherent wave with average power density equal to 10 $\mu W/cm^2$? For the above data the corresponding values of ϱ and $\Delta\varrho$ are equal to 0, 1 and 1, 3, respectively. From the relation (13) we obtain at once the values of Δs equal to 63.2 % and 31.8 %.

Let us consider the realistic situation in the hologram plane, when we record the information wave with high dynamics of amplitude changes, like in the case when the illuminating wave is scattered randomly by an object surface. There will exist zones in the recording plane where local values of the relative amplitude ξ are higher than unity and nonlinear effects will appear. The situation like in fig. 2 is shown in fig. 4, but the possible values of ξ taken into consideration are higher. Becuase of ξ three regions can be distinguished:

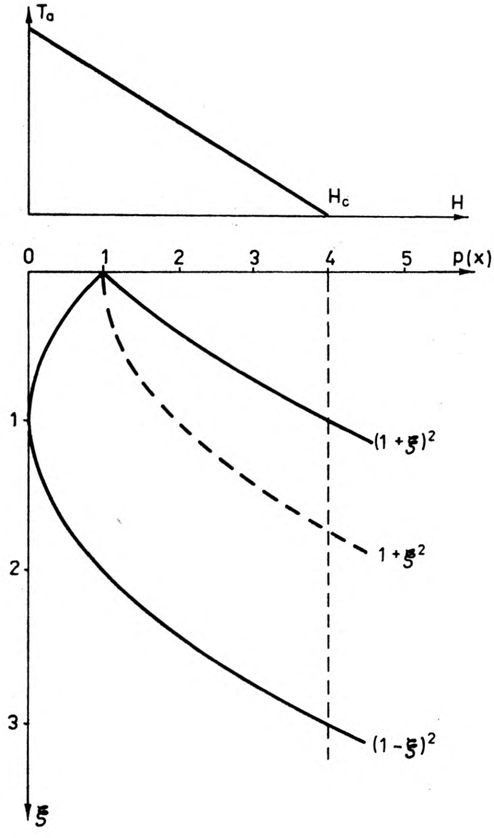

Fig. 4. Interaction between characteristic $T_a - H$ and the relative power distribution in the hologram plane at higher levels of signals

1. The region, where $0 \leqslant \xi \leqslant 1$, i.e., as it was mentioned above, the linear region of recording in which local power densities of the information wave depend linearly on local diffraction efficiencies.

2. The region, where $1 < \xi < 3$, here there occur nonlinear effects, i.e. the local values of the diffraction efficiency at first slightly increase with ξ, then they tend to zero.

3. The region, where $\xi > 3$; the recording does not occur and the corresponding zones of the hologram represent black spots, limiting the effective hologram aperture.

It is evident that each of the above regions corresponds to an appropriate part of the hologram surface. Relative values of these parts depend not only on the exposure conditions, like the exposure time and the ratio M, but on the statistical properties of the information wave, especially on its power density distribution.

Let us discuss the question which parameters affect the component of T_a, which decides upon the first order diffraction. As we can see from fig. 2, one of these parameters is the slope of the characteristic $T_a - H$. This slope depends on the relation between coordinates H and $p(x)$, and the latter, in turn, depends on the product of the reference wave power density P_0 and the exposure time t_e, i.e.

$$H_0 = P_0 t_e. \tag{15}$$

If we want to realize the conditions as in fig. 2, the value H_0 should be chosen in such a way that at the exposition with the reference wave only we ought to get

$$T_e(H)_0 = [T_a(H_0)]^2 = 0.5625. \tag{16}$$

Finally, we can treat the value H_0 as the first argument, which decides upon the first-order diffraction amplitude component of the amplitude transmittance, i.e. T_{a_1}.

Other parameter which affects the value of T_{a_1} is the relative amplitude ξ of the information wave. In linear region T_{a_1} depends linearly on ξ, in nonlinear region T_{a_1} decreases with ξ, tending to zero. The function $T_{a_1}(H_0, \xi)$ can be written explicitely, if we choose the value H_0 and assume a mathematical description of the characteristic $T_a - H$. The local value of the diffraction efficiency of the hologram will be then

$$\eta(H_0, \xi) = [T_{a_1}(H_0, \xi)]^2. \tag{17}$$

Now we must take into account the statistical properties of the information wave. If the function $F(\varrho)$ describing the statistical power density distribution of the information wave is known, then the relative integral surface of the hologram, illuminated by the information wave in the range of relative power density ϱ, $\varrho + d\varrho$, is

$$ds = F(\varrho)\, d\varrho. \tag{18}$$

Let us denote the value M by the ratio of the average power density of the information wave P_{av} to the power density of the information wave P_0

$$M = P_{av}/P_0. \tag{19}$$

If we notice that

$$\xi^2 = (u/u_0)^2 = P/P_0, \tag{20}$$

then we can write (14) as

$$\varrho = \xi^2/M. \tag{21}$$

Becuase

$$d\varrho = \frac{2}{M}\,\xi d\xi, \tag{22}$$

we get

$$ds = \frac{2}{M}\,F(\xi^2/M)\,\xi d\xi. \tag{23}$$

Finally, the integral diffraction efficiency for the amplitude hologram will be given by the relation

$$\eta_{\text{int}}(H_0, M) = \frac{2}{M}\int_0^{\infty} [T_{a_1}(H_0, \xi)]^2 F(\xi^2/M)\,\xi d\xi. \tag{24}$$

4. Results of mathematical calculations of an ideal example

There were made mathematical calculations of an ideal example. The following assumptions were made:

1. The recording medium is an ideal one and fulfils the condition (6).

2. The relation between coordinates H and $p(x)$ obeys the condition shown in figs. 2 and 4.

3. The information wave being produced by an ideal randomly scattering object, the power density distribution of the wave fulfils the relation

$$ds/d\varrho = F(\varrho) = \exp(-\varrho). \tag{25}$$

As an additional condition it has been assumed that the resolving power of the recording medium does not limit the spatial frequency range.

At first the relative surface distribution of the hologram vs. the ratio M and the relative amplitude ξ were calculated. The results are listed in table 1.

Table 1. The relative distribution in percentages of the hologram surface vs. the ratio M and the relative amplitude of the information wave ξ

M \ ξ	0–1	1.0–1.2	1.2–1.4	1.4–1.6	1.6–1.8	1.8–2.0	2.0–2.2	2.2–2.4	2.4–2.6	2.6–2.8	2.8–3.0	1–3	3–∞
0.0625	100.00												
0.1111	99.988	0.011	0.001									0.012	
0.2500	98.170	1.517	0.276	0.036								1.829	
0.5000	86.467	7.920	3.649	1.387	0.444	0.026						13.426	
0.6667	77.687	10.781	6.246	3.137	1.374	0.527	0.178	0.053	0.014			22.310	
0.8000	71.350	12.121	7.901	4.553	2.334	1.068	0.438	0.098	0.053	0.016		28.582	
1.0000	63.212	13.095	9.607	6.355	3.814	2.085	1.041	0.514	0.199	0.077		36.787	0.001
1.2500	55.067	13.333	10.754	7.947	5.412	3.411	1.995	1.085	0.549	0.259	0.114	44.859	0.074
1.5000	48.691	13.031	11.226	8.922	6.608	4.587	2.978	1.818	1.105	0.566	0.289	51.130	0.179
1.6000	46.474	12.869	11.281	9.186	6.990	4.991	3.353	2.123	1.270	0.718	0.384	53.165	0.361
2.0000	39.347	11.978	11.144	9.727	8.014	6.256	4.641	3.279	2.209	1.421	0.873	59.542	1.111
3.0000	28.323	9.799	9.830	9.444	8.654	7.565	6.440	5.268	4.152	3.177	2.353	66.682	4.979
4.0000	22.120	8.112	8.505	8.535	8.243	7.698	6.968	6.127	5.241	4.366	3.546	67.339	10.540
6.0000	15.380	5.957	6.554	6.862	7.705	6.950	6.705	6.330	5.889	5.337	4.750	63.039	22.313
8.0000	11.750	4.728	5.257	5.656	5.917	6.045	6.046	5.932	5.719	5.425	5.066	55.791	32.465

The calculations of IDE were realized in two regions, separately: in the whole linear region and in the subregions of nonlinear region. Taking account of the 2-nd assumption we can treat the parameter H_0 as to be fixed (known), and therefore the IDE may be considered as the function of the ratio M only. The expression for IDE can be written as a sum of two integrals

$$\eta_{\text{int}}(M) = \frac{2}{M}\int_0^1 [T_{a_1}(\xi)]^2 \exp(-\xi^2/M)\,\xi d\xi + \frac{2}{M}\int_1^3 [T_{a_1}(\xi)^2 \exp(-\xi^2/M)\,\xi d\xi. \quad (26)$$

The first integral corresponds to the linear region, the second — to the nonlinear one. Introducing the variable (21) and the relation (10) we get the relation for the partial IDE of the linear region

$$\eta_{\text{int}_{0-1}}(M) = \frac{M}{16}\int_0^{1/M} \varrho \exp(-\varrho)\,d\varrho = \frac{M}{16}\left[1-\left(1+\frac{1}{M}\right)\exp\left(-\frac{1}{M}\right)\right]. \quad (27)$$

It is worth to notice that

$$\lim_{M\to 0} \eta_{\text{int}_{0-1}}(M) = \lim_{m\to 0} \eta_{\text{int}}(M) = M/16. \quad (28)$$

As it may be seen from table 1, the integral relative surface of the nonlinear region does not exceed 2% even at the ratio $M = 0.25$, and therefore the relation (28) represents a total IDE at sufficiently small values of the ratio M.

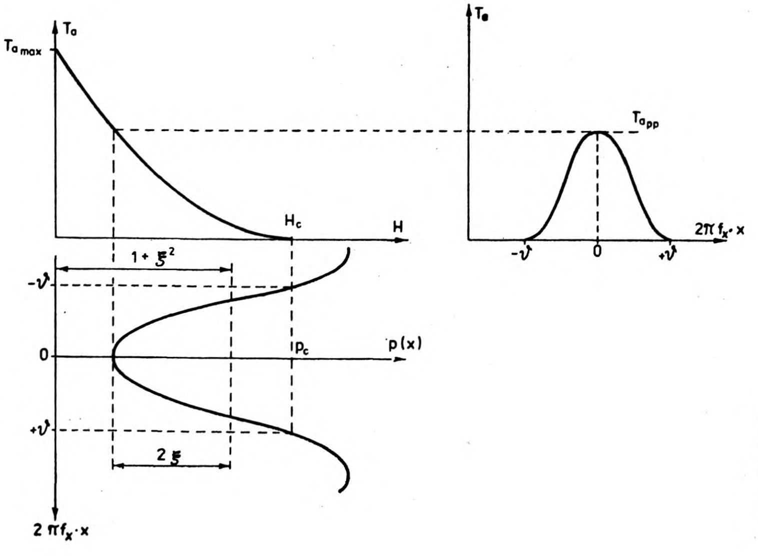

Fig. 5. Determination of the cut-off angle ϑ and the peak value $T_{a_{pp}}$ in nonlinear recording conditions

The procedure of calculations of the partial IDE values in nonlinear region is shown in outlines. Let us look at fig. 5, where the characteristic $T_a - H$ is approximated by the left branch of the m-order parabole

$$\left.\begin{aligned} T_a &= \frac{T_{a_{\max}}}{H_c^m}(H_c - H)^m, & \text{if } 0 \leqslant H \leqslant H_c \\ T_a &= 0, & \text{if } H > H_c \end{aligned}\right\}. \tag{29}$$

The cut-off angle ϑ is

$$\vartheta = \arccos \frac{1+\xi^2 - p_c}{2\xi}. \tag{30}$$

Applying the conventional Fourier procedure we can determine the first-order diffraction component of the amplitude transmittance pulse $T_{a_{pp}}$

$$a_{1(m)}(\vartheta) = T_{a_1}/T_{a_{pp}}. \tag{31}$$

In our case, corresponding to $m = 1$,

$$a_{1(1)}(\vartheta) = \frac{2\vartheta - \sin 2\vartheta}{\pi(1-\cos\vartheta)}. \tag{32}$$

The value of $T_{a_{pp}}$ can be determined by substituting $(1-\xi)^2$ into normalized eq. (29). In our case such normalization is obtained when $H_c = 4$, and $T_{a_{\max}} = 1$. Finally

$$T_{a_1}(\xi) = T_{a_{pp}}(\xi)\, a_{1(1)}[\vartheta(\xi)]. \tag{33}$$

Substituting (33) into the second integral of (26) and taking appropriate limits of the integral, we get the expressions for the partial IDE-s in the subregions of the nonlinear recording region. All numerical results are gathered in table 2.

Graphical representations of the results given in table 2 are shown in fig. 6. The maximal value of the total IDE, equal to 4.20 %, corresponds to $M_{opt} = 1.54$.

Table 2. The distribution of the partial IDE-s in percentages vs. the ratio M and the relative amplitude ξ of the information wave

	$m = 1$												
M \ ξ	0–1	1.0–1.2	1.2–1.4	1.4–1.6	1.6–1.8	1.8–2.0	2.0–2.2	2.2–2.4	2.4–2.6	2.6–2.8	2.8–3.0	1–3	0–3
0.0625	0.3906												0.3906
0.1111	0.6944												0.6944
0.2500	1.4194	0.1086	0.0190	0.0022								0.1298	1.5492
0.5000	1.8563	0.5668	0.2516	0.0841	0.0220	0.0009						0.9254	2.7817
0.6667	1.8424	0.7715	0.4306	0.1902	0.0680	0.0190	0.0041	0.0007	0.0001			1.4842	3.3266
0.8000	1.7768	0.8674	0.5447	0.2761	0.1155	0.0386	0.0102	0.0013	0.0003			1.8541	3.6309
1.0000	1.6515	0.9371	0.6623	0.3854	0.1888	0.0753	0.0242	0.0067	0.0011	0.0001		2.2810	3.9325
1.2500	1.4938	0.9541	0.7414	0.4819	0.2679	0.1231	0.0464	0.0141	0.0030	0.0003		2.6322	4.1260
1.5000	1.3580	0.9325	0.7739	0.5410	0.3271	0.1656	0.0693	0.0236	0.0060	0.0008		2.8398	4.1978
1.6000	1.3020	0.9209	0.7777	0.5570	0.3460	0.1802	0.0780	0.0276	0.0069	0.0010		2.8953	4.1973
2.0000	1.1275	0.8572	0.7682	0.5898	0.3967	0.2258	0.1079	0.0426	0.0120	0.0019	0.0001	3.0022	4.1297
3.0000	0.8352	0.7012	0.6777	0.5727	0.4284	0.2731	0.1498	0.0685	0.0226	0.0043	0.0001	2.8984	3.7363
4.0000	0.6625	0.5805	0.5863	0.5174	0.4081	0.2779	0.1620	0.0796	0.0285	0.0059	0.0002	2.6464	3.3089
6.0000	0.4682	0.4263	0.4518	0.4161	0.3814	0.2509	0.1559	0.0823	0.0320	0.0072	0.0002	2.2041	2.6723
8.0000	0.3596	0.3383	0.3624	0·3430	0.2929	0.2182	0.1406	0.0771	0.0311	0.0073	0.0003	1.8112	2.1708

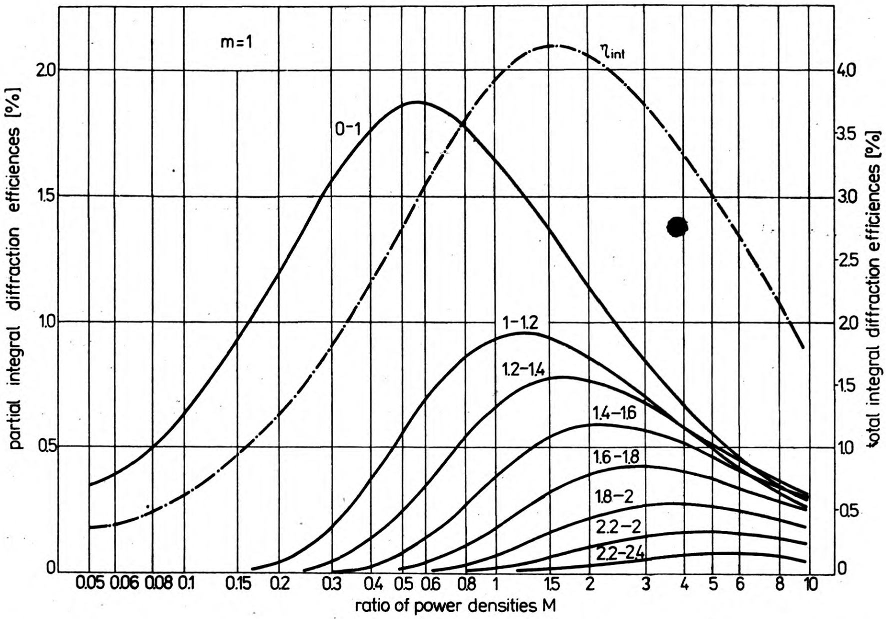

Fig. 6. Relation of the partial and total IDE-s vs. the ratio M by linear approximation of the $T_a - H$ characteristic: $m = 1$

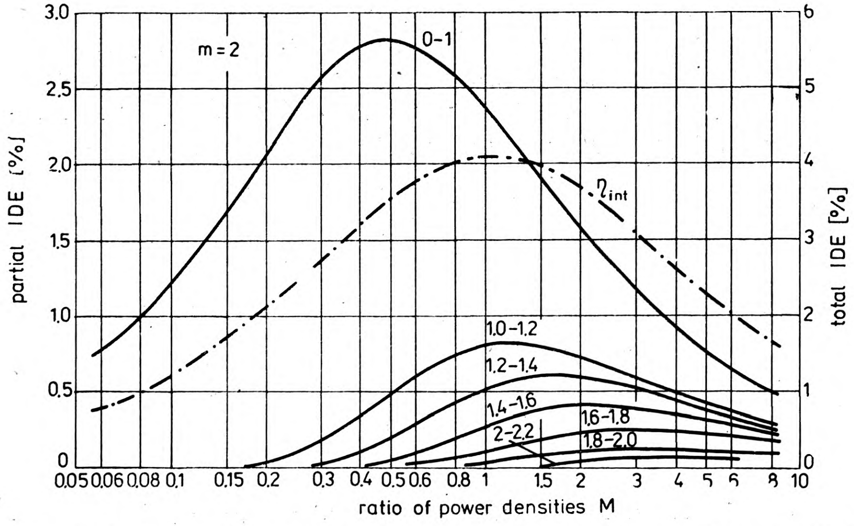

Fig. 7. Relation of the partial and total IDE-s vs. the ratio M by parabolic approximation of the $T_a - H$ characteristic: $m = 2$

The calculations were performed for the approximation of the $T_a - H$ characteristic at $m = 2$ and at the same normalization conditions, as in the case $m = 1$. Then the IDE reaches its maximal value equal to 4.13 % at $M_{opt} = 1.00$. Graphical illustrations of the calculation results at $m = 2$ are shown in fig. 7.

5. Experimental IDE optimalization of amplitude holograms

Mathematical considerations presented in chapter 4 show the physical sense of the phenomena which occur in the recording plane, but the calculated values of IDE and M are related to the assumed idealistic shapes of $T_a - H$ characteristics. The real values of IDE-s and M ratios can be obtained only experimentally. In this chapter the results of experimental IDE optimization of amplitude holograms are presented.

In order to obtain high reproducibility of measurements and eliminate outside influences, the following conditions were assumed:

1. The information wave was obtained by application of a scattering plate, fulfilling the Lambert law.
2. The geometric configuration of the recording assembly assured the condition, that the maximal space frequency of the recorder interference field was at leat 10 times lower than the resolving power of the recording medium.
3. Special antishock and antivibration pneumatic suspension was applied.
4. Temperature and humidity in the laboratory room were stabilized.
5. Uniform conditions in processing of recording plates were strickly kept.
6. All the recording plates had the same serial production number.

As a recording medium the Agfa-Gevaert 10E75 plates were applied. As an object a scattering glass plate limited by a 6 mm circular aperture was used. In IDE measurement the reverse configuration of the reference wave was applied and in the image plane of the object the integrating photosensitive element was placed, its aperture being limited by the same 6 mm diaphragm. Moreover, the ratio of the average power transmittance T_e to the power transmittance of a non-exposed plate T_{e0} was measured.

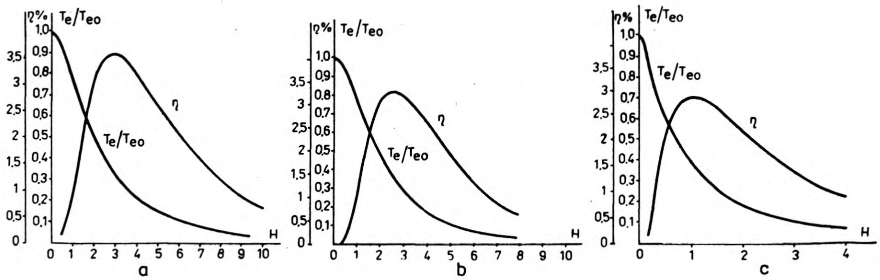

Fig. 8. Dependences of hologram dfifraction efficiencies η and relative power transmittances T_e/T_{e0} vs. exposures H (arbitrary units), for following values of the ratio M: $M = 1:2$ (a), $M = 1:1$ (b), $M = 2:1$ (c)

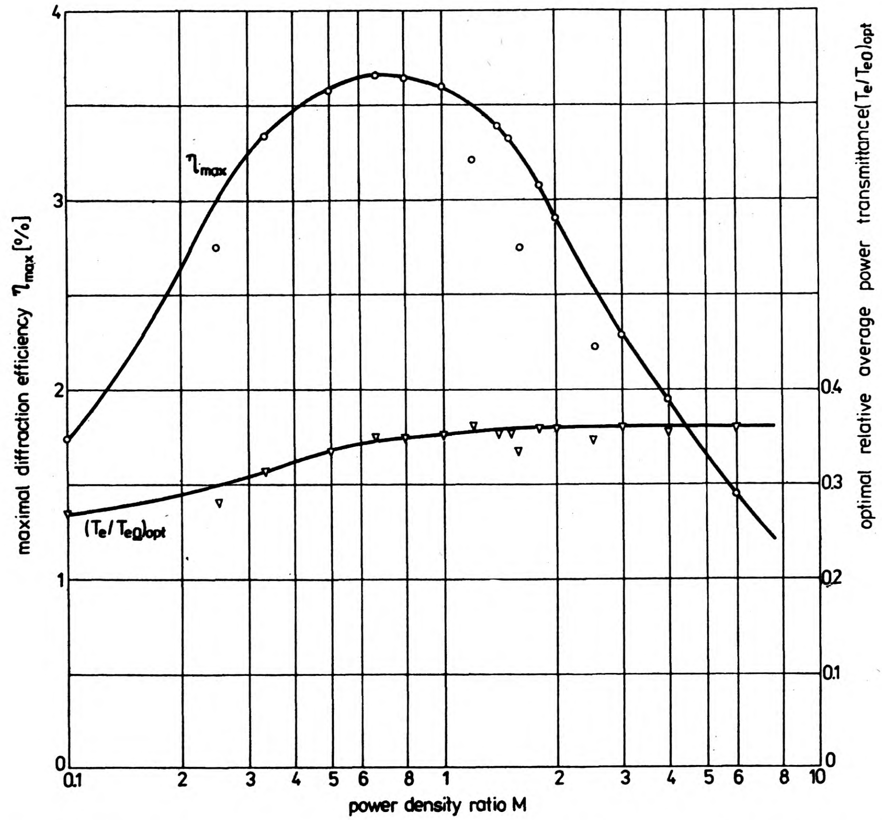

Fig. 9. The diagram vs. the power density ratio M, composed of the values of η_{max} and $(T_e/T_{e0})_{opt}$, interpolated from seventeen diagrams like the ones presented in fig. 8

The results obtained for some values of the parameter M are shown in fig. 8. The H coordinates in the diagrams are relative ones. The value of η_{max} and the corresponding value $(T_e/T_{e0})_{opt}$ were determined from each of seventeen diagrams $\eta - H$. These values were plotted vs. the ratio M – final diagrams are shown in fig. 9. The maximum-maximorum of the η value is equal to 3.67 % and occurs, when $M_{opt} = 0.7$, the relative power transmittance $(T_e/T_{e0})_{opt} = 0.35$. The results obtained are in excellent agreement with everyday holographic practice.

References

[1] LIPOWIECKI T., Optica Applicata V (1976), 3–9.

Received March 3, 1980
in revised form August, 28, 1980

Интегральный дифракционный коэффициент полезного действия амплитудных голограмм

В статье обсуждены проблемы, связанные с интегральным коэффициентом полезного действия амплитудных голограмм. Доказано, что к.п.д. зависти не только от условий экспозиции голограммы, но также от статистических свойств регистрируемой информационной волны. Применяя соответствующую для больших сигналов аппроксимацию характеристики $T_a - H$, было доказано наличие максимального значения интегрального дифракционного коэффицента полезного действия. Приведены экспериментальные результаты, касающиеся измерений интегрального дифракционного коэффициента полезного действия амплитудных программ дифузно рассеивающих объектов, в функции условий экспозиции.