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Higher order aberrations in holograms*

Grażyna Mulak

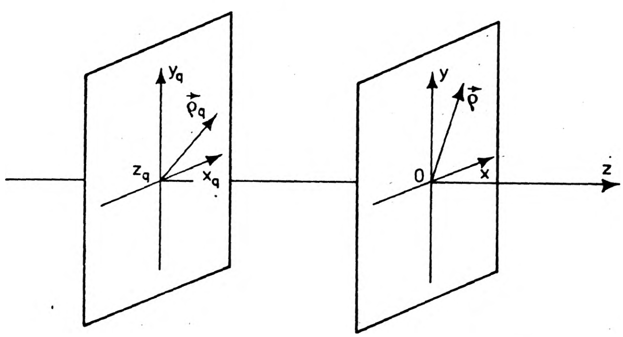
Institute of Physics, Technical University of Wrocław, Wrocław, Poland.

The paper contains an analysis of the higher order aberrations for point-sources of the wavefronts taking part in holographic imaging. The formulae for higher order aberrations are given together with the vanishing conditions for the latters. A simple method allowing to accelerate the convergence of the aberrational expressions within classical binomial expansion as well as enabling to determine the complete aberrations outside this expansions is presented.

Introduction

The sum of aberrations for each of the reconstructed imaging wavefronts Φ_R, and Φ_V is determined by the difference of their respective phases and the phase of the corresponding Gaussian reference sphere

$$\Phi_{R,V} - \Phi_{G_{R,V}} = \varphi_c \mp \varphi_0 \pm \varphi_r - \Phi_{G_{R,V}}. \tag{1}$$



The conscise notation (fig. 1) of each of the wavefront phases φ_q given by

$$\varphi_q = \frac{2\pi}{\lambda_q} z_q \left(\sqrt{1+\left(\frac{\rho-\rho_q}{z_q}\right)^2} - \sqrt{1+\left(\frac{\rho_q}{z_q}\right)} \right) \tag{2}$$

allows to determine quickly the higher order aberrations and their examination. The square roots appearing in the expression (2) may be represented in the form of a series

$$\sqrt{1+\xi} \approx \binom{1/2}{0} + \binom{1/2}{1}\xi + \binom{1/2}{2}\xi^2 + \binom{1/2}{3}\xi^3 + \ldots, \tag{3}$$

which is absolutely convergent for $|\xi| \leqslant 1$.

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Let us consider the first square root in (2). For the sake of convenience we write its expressions accurately to (3) so, that the numerical coefficients following from the expression (3) are ignored, and we restrict our attention to expansion of $\sqrt{1+(\rho-\rho_q)^2}$. The Pascal triangle which is associated with this expansion has the form

$$\begin{array}{c} (\rho^2)-2(\rho\cdot\rho_q)+(\rho_q^2) \\ (\rho^4)-4(\rho^2)(\rho\cdot\rho_q)+[2(\rho^2)(\rho_q^2)+4(\rho\cdot\rho_q)^2]-4(\rho_q^2)(\rho\cdot\rho_q)+(\rho_q^4) \\ (\rho^6)-6(\rho^4)(\rho\cdot\rho_q)+\ldots\ldots \qquad -6(\rho_q^4)(\rho\cdot\rho_q)+(\rho_q^6) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (\rho^{n+1})-(n+1)(\rho^{n-1})(\rho\cdot\rho_p)+\ldots \qquad \ldots\ldots+(\rho_q^{n+1}) \end{array} \tag{4}$$

Due to lack of space the two (3-rd and 4th) subsequent rows of this expansion are writen in extenso below.
The third row:

$$\begin{aligned}(\rho^6)-6(\rho^4)(\rho\cdot\rho_q)&+[3(\rho^4)(\rho_q^2)+12(\rho^2)(\rho\cdot\rho_q)^2]\\ &-[12(\rho^2)(\rho_q^2)(\rho\cdot\rho_q)+8(\rho\cdot\rho_q)^3]\\ &+[3(\rho^2)(\rho_q^4)+12(\rho_q^2)(\rho\cdot\rho_q)^2]-6(\rho_q^4)(\rho\cdot\rho_q)+(\rho_q^6)\end{aligned}$$

The fourth row:

$$\begin{aligned}&(\rho^8)-8(\rho^6)(\rho\cdot\rho_q)+[4(\rho^6)(\rho_q^2)+24(\rho^4)(\rho\cdot\rho_q)^2]\\ &\quad-[24(\rho^4)(\rho_q^2)(\rho\cdot\rho_q)+32(\rho^2)(\rho\cdot\rho_q)^3]\\ &\quad+[6(\rho^4)(\rho_q^4)+48(\rho^2)(\rho_q^2)(\rho\cdot\rho_q)^2+16(\rho\cdot\rho_q)^4]\\ &\quad-[24(\rho^2)(\rho_q^4)(\rho\cdot\rho_q)+32(\rho_q^2)(\rho\cdot\rho_q)^3]\\ &\quad+[4(\rho^2)(\rho_q^6)+24(\rho_q^4)(\rho\cdot\rho_q)^2]-8(\rho_q^6)(\rho\cdot\rho_q)+(\rho_q^8).\end{aligned}$$

The structure of the next row will be obtained multiplying the previous one by $[(\rho^2)-2(\rho\cdot\rho_q)-(\rho_q^2)]$. The correctness of the performed operations may be easily verified. After rearrangement of the polynomials according to the power of ρ the sum of the numerical coefficients occurring within parentheses – grouping the terms of the same power of ρ – should be equal to the respective $\binom{n}{r}$ of the Newton binomial.

After performing all operations required by (2) in the triangle (4), the terms with the expansion coefficient $\binom{n+1}{n+1}$ will be cut out, while the phase differences defined n (1) will give in the first row

$$\sum_q \varphi_q^I = \frac{1}{2}\sum_q \frac{2\pi}{\lambda_q}\frac{(\rho^2)-2(\rho\cdot\rho_q)}{z_q} = \frac{1}{2}\frac{2\pi}{\lambda_c}\left(\frac{(\rho^2)-2(\rho\cdot\rho_c)}{z_c}\right.$$

$$\left.\mp\mu\frac{(\rho'^2)-2(\rho'\cdot\rho_o)}{z_o}\pm\mu\frac{(\rho'^2)-2(\rho'\cdot\rho_r)}{z_r}-\frac{(\rho^2)-2(\rho\cdot\rho_{G_{R,V}})}{z_{G_{R,V}}}\right),$$

if the possible changes in both the scaling ($\rho = m\rho'$), and the wavelength ($\mu = \lambda_c/\lambda_o$) which may occur during the reconstruction are admitted. This expression represents the Gaussian imaging. When compared to zero it allows to determine the coordinates $x_{G_{R,V}}$, $y_{G_{R,V}}$, $z_{G_{R,V}}$ of the Gaussian sphere. The phases corresponding to the second row ordered in a similar way $\sum_q \varphi_q^{\mathrm{III}}$ define the third order aberrations, while those given in the fourth row $\sum_q \varphi_q^{V}$ represent the aberrations of fifth order etc.

The structure of aberrations

The aberrations defined in the way explained above are determined along the ray. Each of the components of the aberration of given order describes the wavefront deviation from the sphere and constitutes a defined aberrational surface. With the increase of the aberration order the number of types of such surfaces increases, beside those existing in the previous row there appear some new ones. Fig. 2 presents symbollically the Pascal triangle with the marked directions of summing the contributions to the particular kinds of aberrations.

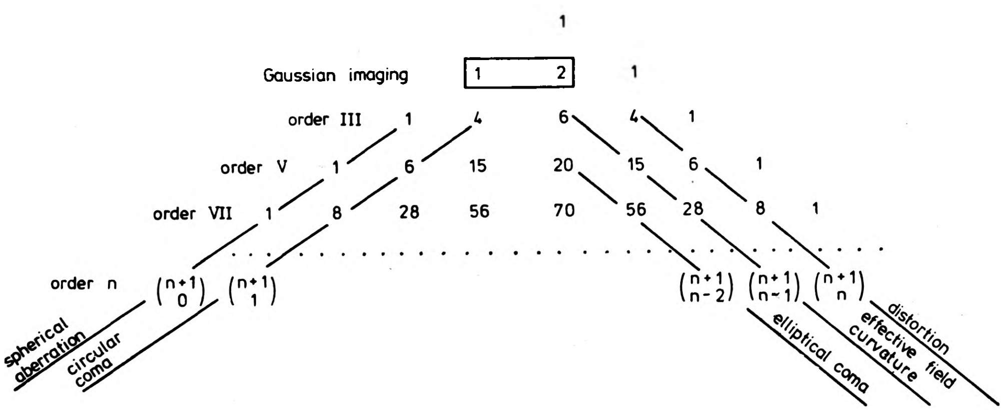

Fig. 2

The nomenclature connected with the higher order aberrations and used in the literature is poor. The names of effective field curvature, circular coma, elliptical coma taken from the lens system terminology and cited after BUCHDAHL [1] do not exhaust all the possibilities. The effective field curvature — the aberration with the coefficients $\binom{n+1}{n-1}$ — is composed of field curvature and astigmatism. Their mutual relation may be fixed on the base of the relation (4).

Due to the central position of the term describing the astigmatism and the field curvature of the third order (fig. 2) the expressions lying symmetrically with respect to the effective field curvature with the coefficients

$\binom{n+1}{2}$, and those lying on the symmetry axis of the traingle in every second row starting with the third row, i.e.

$$[2(\rho^2)(\rho_q^2)+4(\rho\cdot\rho_q)^2]$$

$$[3(\rho^4)(\rho_q^2)+12(\rho^2)(\rho\cdot\rho_q)^2]\ldots\ldots\ldots[3(\rho^2)(\rho_q^4)+12(\rho_q^2)(\rho\cdot\rho_q)^2]$$

$$[4(\rho^6)(\rho_q^2)+24(\rho^4)(\rho\cdot\rho_q)^2]\ldots\ldots[6(\rho^4)(\rho_q^4)+\ldots\ldots+16(\rho\cdot\rho_q)^4]\ldots[4(\rho^2)(\rho_q^6)+24(\rho_q^4)(\rho\cdot\rho_q)^2]$$

$$\binom{n+1}{2} \qquad\qquad \binom{n+1}{\frac{n+1}{2}} \qquad\qquad \binom{n+1}{n-1}$$

may be called the astigmatism and field curvature of the second and third kind as it is suggested by their structures.

The summation aiming at determination of the complete aberrations of given kind and realized by transposing the infinite number of terms of the series is here admissible. This transposition will not change the sum of the series, since the series (3), which is the basis for the definition of aberrations is absolutely convergent. Some doubts may arise due to quick increase of the $\binom{n+1}{k}$ values, while approaching the symmetry axis of the Pascal triangle. On the axis of symmetry the coefficient of Newton binomials expansion is $\binom{n+1}{\frac{n+1}{2}}$. As it may be easily shown the radius of convergence of the series is equal to 1, similarly as for the remaining directions of summation. The physical realizability of the aberrations imposes the requirement of convergences for the series representing those aberrations, our formalism fulfills this condition.

Some aberrations of higher orders and their coefficients

Let $n = 3, 5, 7 \ldots$ denote the order of aberrations, a_n — the numerical coefficient of the given order of aberration following from the development into series (3). The aberrations and their coefficients will be defined according to the convention proposed by MEIER [2] for the third order aberrations*. The discussion will be carried out for the wavefront Φ_R. All the aberrations are expressed in the $2\pi/\lambda_c$ units.

A. The spherical aberration

The complete spherical aberration is defined by the sum

$$\sum_{n=3}^{\infty}\sum_{q} a_n \binom{n+1}{0}\frac{\rho^{n+1}}{z_q^n} = \sum_{n=3}^{\infty} a_n \varrho^{n+1} S_n^{R,V}, \tag{5}$$

* Higher order aberrations in Champagne convention [3] are obtained easily by expanding $R_q\sqrt{1+\frac{\rho^2-2\rho\rho_q}{R_q^2}}$ into series (3) instead of (2) and writting the structure of binomial expansion $(\rho^2-2\rho\cdot\rho_q)^n$ instead of $(\rho-\rho_q)^{2n}$ in a way analogical to (4).

where

$$S_n^{R,V} = \frac{1}{z_c^n} \mp \frac{\mu}{m^{n+1}} \frac{1}{z_o^n} \pm \frac{\mu}{m^{n+1}} \frac{1}{z_r^n} - \frac{1}{z_{G_{R,V}}^n} \tag{6}$$

is the coefficient of spherical aberration of n-th order. For the wavefront Φ_R

$$S_n^R = \frac{1}{z_c^n} - \frac{\mu}{m^{n+1}} \frac{1}{z_o^n} + \frac{\mu}{m^{n+1}} \frac{1}{z_r^n} - \left(\frac{1}{z_c} - \frac{\mu}{m^2} \frac{1}{z_o} + \frac{\mu}{m^2} \frac{1}{z_r}\right)^n. \tag{6a}$$

If $z_c = z_r = \infty$ (keeping in mind that n is an odd number):

$$S_n^R = \frac{\mu}{m^{n+1}} \left(\left(\frac{\mu}{m}\right)^{n-1} - 1\right) \frac{1}{z_o^n} \tag{6b}$$

and disappears for $\mu = m$.

If, however, in the face of (6a) $z_r = z_o$, then the spherical aberration is always equal to zero independently of z_c, μ, and m.

B. Circular coma

The complete circular coma is determined by

$$-\sum_{n=3}^{\infty} \sum_q a_n \binom{n+1}{1} \frac{\rho^{n-1}}{z_q^n} (\rho \cdot \rho_q) = -\sum_{n=3}^{\infty} a_n (n+1) \varrho^n \times (C_{n_x}^{R,V} \cos\theta + C_{n_y}^{R,V} \sin\theta), \tag{7}$$

where $C_{n_x}^{R,V}$, $C_{n_y}^{R,V}$ – the coefficients of the circular coma of n-th order of the form

$$C_{n_x}^{R,V} = \frac{x_c}{z_c^n} \mp \frac{\mu}{m^n} \frac{x_o}{z_o^n} \pm \frac{\mu}{m^n} \frac{x_r}{z_r^n} - \frac{x_{G_{R,V}}}{z_{G_{R,V}}^n}. \tag{8}$$

For the wavefront Φ_R:

$$\begin{aligned} C_{n_x}^R &= \frac{x_c}{z_c^n} - \frac{\mu}{m^n} \frac{x_o}{z_o^n} + \frac{\mu}{m^n} \frac{x_r}{z_r^n} - \left(\frac{x_c}{z_c} - \frac{\mu}{m} \frac{x_o}{z_o} + \frac{\mu}{m} \frac{x_r}{z_r}\right) \\ &\times \left(\frac{1}{z_c} - \frac{\mu}{m^2} \frac{1}{z_o} + \frac{\mu}{m^2} \frac{1}{z_r}\right)^{n-1} = \frac{x_c}{z_c} \left[\frac{1}{z_c^{n-1}} - \left(\frac{1}{z_c} - \frac{\mu}{m^2} \frac{1}{z_o} + \frac{\mu}{m^2} \frac{1}{z_r}\right)^{n-1}\right] \\ &- \frac{\mu}{m} \frac{x_o}{z_o} \left[\frac{1}{(m z_o)^{n-1}} - \left(\frac{1}{z_c} - \frac{\mu}{m^2} \frac{1}{z_o} + \frac{\mu}{m^2} \frac{1}{z_r}\right)^{n-1}\right] \\ &+ \frac{\mu}{m} \frac{x_r}{z_r} \left[\frac{1}{(m z_r)^{n-1}} - \left(\frac{1}{z_c} - \frac{\mu}{m^2} \frac{1}{z_o} + \frac{\mu}{m^2} \frac{1}{z_r}\right)^{n-1}\right]. \end{aligned} \tag{8a}$$

If $z_c = z_r = \infty$, x_c/z_c, and x_r/z_r need not be equal to zero in the general case, and $n-1$ is an even number, then we have

$$C_{n_x}^R = -\left(\frac{\mu}{m^2} \frac{1}{z_o}\right)^{n-1} \left[\frac{x_c}{z_c} - \frac{\mu}{m} \left(\left(\frac{m}{\mu}\right)^{n-1} - 1\right) + \frac{\mu}{m} \frac{x_r}{z_r}\right]. \tag{8b}$$

The aberration disappears if the slopes of the reconstructing and reference beams fulfil the conditions $(x_c/z_c) = -(\mu x_r/m z_r)$, and $(y_c/z_c) = -(\mu y_r/m z_r)$, for additionally satisfied $\mu = m$. On the other hand, for $z_r = z_o$ (8a) is transformed into

$$C^R_{n_x} = \frac{\mu}{m}\left[\frac{1}{(mz_o)^{n-1}} - \frac{1}{z_c^{n-1}}\right]\left(\frac{x_r}{z_r} - \frac{x_o}{z_o}\right). \tag{8c}$$

The circular coma disappears at $x_r = x_o$, $y_r = y_o$ or, independently of this condition, at $z_c = \pm m z_o$.

C. Elliptical coma

Elliptical coma is defined by

$$-\sum_{n=5}^{\infty}\sum_{q} a_n\left\{\frac{n-1}{2}(n+1)\frac{\varrho^2}{z_q^n}\varrho_q^{n-3}(\rho\cdot\rho_q) + \left[\binom{n+1}{3} - \frac{n-1}{2}(n+1)\right]\right.,$$

$$\left.\frac{\varrho_q^{n-5}}{z_q^n}(\rho\cdot\rho_q)^3\right\} = -\sum_{n=5}^{\infty} a_n\left\{\frac{n^2-1}{2}\varrho^3(C'_{e_x}\cos\theta + C'_{e_y}\sin\theta)\right.$$

$$+\left[\binom{n+1}{3} - \frac{n^2-1}{2}\right]\varrho^3(C''_{e_{x^3}}\cos^3\theta + 3C''_{e_{x^2y}}\cos^2\theta\sin\theta$$

$$\left.+3C''_{e_{xy^2}}\cos\theta\sin^2\theta + C''_{e_{y^3}}\sin^3\theta)\right\}.$$

The indices n, R, V are here omitted for the sake of convenience. For example, two from six coefficients of this aberration are of the form

$$\begin{aligned} C'_{e_x} &= \frac{\varrho_c^{n-3}}{z_c^n}x_c \mp \frac{\mu}{m^3}\frac{\varrho_o^{n-3}}{z_o^n}x_o \pm \frac{\mu}{m^3}\frac{\varrho_r^{n-3}}{z_r^n}x_r - \frac{\varrho_{G_{R,V}}^{n-3}}{z_{G_{R,V}}^n}x_{G_{R,V}}, \\ C''_{e_{x^3}} &= \frac{\varrho_c^{n-5}}{z_c^n}x_c^3 \mp \frac{\mu}{m^3}\frac{\varrho_o^{n-5}}{z_o^n}x_o^3 \pm \frac{\mu}{m^3}\frac{\varrho_r^{n-5}}{z_r^n}x_r^3 - \frac{\varrho_{G_{R,V}}^{n-5}}{z_{G_{R,V}}^n}x_{G_{R,V}}^3. \end{aligned} \tag{10}$$

Under conditions $z_c = z_r = \infty$, and $(x_c/z_c) = -(\mu x_r/m z_r)$ the relations (10) for Φ_R are transformed into

$$\begin{aligned} C'_{e_x} &= -\frac{\mu}{m^3}\left(\frac{\varrho_o}{z_o}\right)^{n-3}\frac{x_o}{z_o^3}\left[1-\left(\frac{\mu}{m}\right)^{n-1}\right], \\ C''_{e_{x^3}} &= -\frac{\mu}{m^3}\left(\frac{\varrho_o}{z_o}\right)^{n-5}\frac{x_o^3}{z_o^5}\left[1-\left(\frac{\mu}{m}\right)^{n-1}\right], \end{aligned} \tag{10a}$$

and the elliptical coma disappears at $\mu = m$.
For $z_r = z_o$ and the fulfilled condition $(x_c/z_c) = -(\mu x_r/m z_r)$

$$C'_{e_x} = \frac{\mu}{m^3}\left[\frac{1}{z_o^2} - \left(\frac{\mu}{m}\right)^{n-3} m^2\frac{1}{z_c^2}\right]\left[\left(\frac{\varrho_r}{z_r}\right)^{n-3}\frac{x_r}{z_r} - \left(\frac{\varrho_o}{z_o}\right)^{n-3}\frac{x_o}{z_o}\right].$$

This coefficient disappears for

$$z_c^2 = \left(\frac{\mu}{m}\right)^{n-3} m^2 z_o^2 \quad \text{or} \quad \left(\frac{\varrho_r}{z_r}\right)^{n-3} \frac{x_r}{z_r} = \left(\frac{\varrho_o}{z_o}\right)^{n-3} \frac{x_o}{z_o}.$$

On the other hand,

$$C''_{e_{x^3}} = \frac{\mu}{m^3}\left[\frac{1}{z_o^2} - \left(\frac{\mu}{m}\right)^{n-5} \mu^2 \frac{1}{z_c^2}\right]\left[\left(\frac{\varrho_r}{z_r}\right)^{n-5}\left(\frac{x_r}{z_r}\right)^3 - \left(\frac{\varrho_o}{z_o}\right)^{n-5}\left(\frac{x_o}{z_o}\right)^3\right], \tag{10b}$$

disappears for

$$z_c^2 = \left(\frac{\mu}{m}\right)^{n-5} \mu^2 z_0^2, \quad \text{or} \quad \left(\frac{\varrho_r}{z_r}\right)^{n-5}\left(\frac{x_r}{z_r}\right)^3 = \left(\frac{\varrho_o}{z_o}\right)^{n-5}\left(\frac{x_o}{z_o}\right)^3.$$

For $\mu = m$ the first disappearance conditions for C'_{e_x}, and $C''_{e_{x^3}}$ pass into one common for disappearance of the elliptical coma $z_c = \pm m z_o$. The second kind of conditions assuring disappearance of elliptical coma is $x_r/y_r = x_o/y_o$.

D. Astigmatism

Astigmatism is defined by

$$\begin{aligned}\sum_{n=3}^{\infty}\sum_{q} a_n\left[\binom{n+1}{2} - \frac{1}{2}\binom{n+1}{1}\right]\frac{\varrho_q^{n-3}}{z_q^n}(\boldsymbol{\rho}\cdot\boldsymbol{\rho}_q)^2 \\ = \sum_{n=3}^{\infty} a^n \frac{n^2-1}{2} \varrho^2 (A_{n_x}^{R,V}\cos^2\theta + 2A_{n_{xy}}^{R,V}\cos\theta\sin\theta + A_{n_y}^{R,V}\sin^2\theta),\end{aligned} \tag{11}$$

where the coefficients of astigmatism of n-th order are of the form

$$\begin{aligned} A_{n_x}^{R,V} &= \frac{\varrho_c^{n-3}}{z_c^n} x_c^2 \mp \frac{\mu\varrho_o^{n-3}}{m^2 z_o^n} x_o^2 \pm \frac{\mu\varrho_r^{n-3}}{m^2 z_r^n} x_r^2 - \frac{\varrho_{G_{R,V}}^{n-3}}{z_{G_{R,V}}^n} x_{G_{R,V}}^2, \\ A_{n_{xy}}^{R,V} &= \frac{\varrho_c^{n-3}}{z_c^n} x_c y_c \pm \frac{\mu\varrho_o^{n-3}}{m^3 z_o^n} x_o y_o \pm \frac{\mu\varrho_r^{n-3}}{m^2 z_r^n} x_r y_r - \frac{\varrho_{G_{R,V}}^{n-3}}{z_{G_{R,V}}^n} x_{G_{R,V}} y_{G_{R,V}}. \end{aligned} \tag{12}$$

The dependences (12), for $z_c = z_r = \infty$, and the slopes fulfilling the conditions $(x_c/z_c) = -(\mu x_r/m z_r)$, and $(y_c/z_c) = -(\mu y_r/m z_r)$, take the forms

$$\begin{aligned} A_{n_x}^{R} &= -\frac{\mu}{m^2}\left(\frac{\varrho_o}{z_o}\right)^{n-3}\frac{x_o^2}{z_o^3}\left[1 - \left(\frac{\mu}{m}\right)^{n-1}\right], \\ A_{n_{xy}}^{R} &= -\frac{\mu}{m^2}\left(\frac{\varrho_o}{z_o}\right)^{n-3}\frac{x_o y_o}{z_o^3}\left[1 - \left(\frac{\mu}{m}\right)^{n-1}\right]. \end{aligned} \tag{12a}$$

The astigmatism disappears under an additional condition $\mu = m$.

On the other hand, for $z_r = z_o$, and for such a choice of the reconstructing wave source $(x_c/z_c) = -(\mu x_r/mz_r)$ and $(y_c/z_c) = -(\mu y_r/mz_r)$ the dependences (12) take the form:

$$A^R_{n_x} = -\frac{\mu}{m^2}\left[\frac{1}{z_o}+\left(\frac{\mu}{m}\right)^{n-3}\frac{\mu}{z_c}\right]\left[\left(\frac{\varrho_o}{z_o}\right)^{n-3}\left(\frac{x_o}{z_o}\right)^2-\left(\frac{\varrho_r}{z_r}\right)^{n-3}\left(\frac{x_r}{y_r}\right)^2\right],$$
$$A^R_{n_{xy}} = -\frac{\mu}{m^2}\left[\frac{1}{z_o}+\left(\frac{\mu}{m}\right)^{n-3}\frac{\mu}{z_c}\right]\left[\left(\frac{\varrho_o}{z_o}\right)^{n-3}\frac{x_oy_o}{z_o^2}-\left(\frac{\varrho_r}{z_r}\right)^{n-3}\frac{x_ry_r}{z_r^2}\right]. \tag{12b}$$

The disappearance of the n-th order astigmatism occurs inder the additional condition $z_c = -\left(\frac{\mu}{m}\right)^{n-3}\mu z_o$ or $(x_o/y_o) = (x_r/y_r)$.

E. Astigmatism of the second kind

It is defined by the sum

$$\sum_{n=5}^{\infty}\sum_{q} a_n\left[\binom{n+1}{2}-\frac{1}{2}\binom{n+1}{1}\right]\frac{\varrho^{n-3}}{z_q^n}(\rho\cdot\rho_q)^2$$
$$= \sum_{n=5}^{\infty} a_n\frac{n^2-1}{2}\varrho^{n-1}[\mathscr{A}^{R,V}_{n_x}\cos^2 0+2\mathscr{A}^{R,V}_{n_{xy}}\cos\theta\sin\theta+\mathscr{A}^{R,V}_{n_y}\sin^2\theta], \tag{13}$$

where the coefficients of the n-th order aberrations are of the form

$$\mathscr{A}^{R,V}_{n_x} = \frac{x_c^2}{z_c^n}\mp\frac{\mu}{m^{n-1}}\frac{x_o^2}{z_o^n}\pm\frac{\mu}{m^{n-1}}\frac{x_r^2}{z_r^n}-\frac{x^2_{G_{R,V}}}{z^n_{G_{R,V}}},$$
$$\mathscr{A}^{R;V}_{n_{xy}} = \frac{x_cy_c}{z_c^n}\mp\frac{\mu}{m^{n-1}}\frac{x_oy_o}{z_o^n}\pm\frac{\mu}{m^{n-1}}\frac{x_ry_r}{z_r^n}-\frac{x_{G_{R,V}}y_{G_{R,V}}}{z^n_{G_{R,V}}}. \tag{14}$$

For $z_c = z_r = \infty$, the slopes fulfilling the conditions $(x_c/z_c) = -(\mu x_r/mz_r)$, and $(y_c/z_c) = -(\mu y_t/mz_r)$

$$\mathscr{A}^R_{n_x} = -\frac{\mu}{m^{n-1}}\frac{x_o^2}{z_o^n}\left[1-\left(\frac{\mu}{m}\right)^{n-1}\right],\quad \mathscr{A}^R_{n_{xy}} = -\frac{\mu}{m^{n-1}}\frac{x_oy_o}{z_o^n}\left[1-\left(\frac{\mu}{m}\right)^{n-1}\right]. \tag{14a}$$

Under these conditions this aberration disappears if $\mu = m$. For $z_r = z_o$, and $(x_c/z_c) = -(\mu x_r/mz_r)$, and $(y_c/z_c) = -(\mu y_r/mz_r)$ the relations (14) take the forms

$$\mathscr{A}^R_{n_x} = -\frac{\mu}{m^2}\left[\frac{\mu}{z_c^{n-2}}+\frac{1}{m^{n-3}}\frac{1}{z_o^{n-2}}\right]\left[\left(\frac{x_o}{z_o}\right)^2-\left(\frac{x_r}{z_r}\right)^2\right],$$
$$\mathscr{A}^R_{n_{xy}} = -\frac{\mu}{m^2}\left[\frac{\mu}{z_c^{n-2}}+\frac{1}{m^{n-3}}\frac{1}{z_o^{n-2}}\right]\left[\frac{x_oy_o}{z_o^2}-\frac{x_ry_r}{z_r^2}\right]. \tag{14b}$$

This aberration disappears at $z_c^{n-2} = -\mu m^{n-3} z_o^{n-2}$, or at $(x_o/y_o) = (x_r/y_r)$.

F. Field curvature

It is defined by the sum

$$\sum_{n=3}^{\infty}\sum_{q} a_n \frac{1}{2}\binom{n+1}{1}\frac{\varrho^2}{z_q^n}\varrho_q^{n-1} = \sum_{n=3}^{\infty} a_n \frac{n+1}{2}\varrho^2 F_n^{R,V}, \tag{15}$$

where the coefficient of the field curvature of n-th order is

$$F_n^{R,V} = \frac{\varrho_c^{n-1}}{z_c^n} \mp \frac{\mu\varrho_o^{n-1}}{m^2 z_o^n} \pm \frac{\mu}{m^2}\frac{\varrho_r^{n-1}}{z_r^n} - \frac{\varrho_{G_{R,V}}^{n-1}}{z_{G_{R,V}}^n}. \tag{16}$$

For $z_c = z_r = \infty$, and the slope fulfilling the conditions $(x_c/z_c) = -(\mu x_r/mz_r)$, and $(y_c/z_c) = -(\mu y_r/mz_r)$

$$F_n^R = -\frac{\mu}{m^2}\frac{1}{z_o}\left(\frac{\varrho_o}{z_o}\right)^{n-1}\left[1-\left(\frac{\mu}{m}\right)^{n-1}\right]. \tag{16a}$$

The field curvature disappears at $\mu = m$.

However, in the case when $z_r = z_o$ and $(x_c/z_c) = -(\mu x_r/mz_r)$, and $(y_c/z_c) = -(\mu y_r/mz_r)$

$$F_n^R = -\frac{\mu}{m^2}\left[\frac{1}{z_o}+\left(\frac{\mu}{m}\right)^{n-2}\frac{m}{z_c}\right]\left[\left(\frac{\varrho_o}{z_o}\right)^{n-1}-\left(\frac{\varrho_r}{z_r}\right)^{n-1}\right]. \tag{16b}$$

The conditions $z_c = -\left(\frac{\mu}{m}\right)^{n-2} m z_o$ or $\varrho_o = \pm\varrho_r$ assure the disappearance of this aberration.

G. Field curvature of the second kind

It is defined by

$$\sum_{n=5}^{\infty}\sum_{q} \bar{a}_n \frac{1}{2}\binom{n+1}{1}\frac{\varrho^{n-1}}{z_q^n}\varrho_q^2 = \sum_{n=5}^{\infty} a_n \frac{n+1}{2}\varrho^{n-1}\mathscr{F}_n^{R,V}, \tag{17}$$

where

$$\mathscr{F}_n^{R,V} = \frac{\varrho_c^2}{z_c^n} \pm \frac{\mu}{m^{n-1}}\frac{\varrho_o^2}{z_o^n} \pm \frac{\mu}{m^{n-1}}\frac{\varrho_r^2}{z_r^n} - \frac{\varrho_{G_{R,V}}^2}{z_{G_{R,V}}^n} \tag{18}$$

is the coefficient of n-th order of this aberration.

For $z_c = z_r = \infty$, and the slope of the reconstructing beam fulfilling the conditions $(x_c/z_c) = -(\mu z_r/mz_r)$, and $(y_c/z_c) = -(\mu y_r/mz_r)$

$$\mathscr{F}_n^R = -\frac{\mu}{m^2}\frac{\varrho_o}{z_o^3}\left[1-\left(\frac{\mu}{m}\right)^{n-1}\right]. \tag{18a}$$

The condition of the aberration elimination is $\mu = m$.

In the case when $z_r = z_o$, and the position of the source of the wave φ_c fulfills the conditions $(x_c/z_c) = -(\mu x_r/mz_r)$ and $(y_c/z_c) = -(\mu y_r/mz_r)$

$$\mathscr{F}_n^R = -\frac{\mu}{m^2}\left[\frac{1}{m^{n-3}}\frac{1}{z_o^{n-2}} + \frac{\mu}{z_c^{n-2}}\right]\left[\left(\frac{\varrho_o}{z_o}\right)^2 - \left(\frac{\varrho_r}{z_r}\right)^2\right]. \tag{18b}$$

The aberration disappears for $z_c^{n-2} = -\mu m^{n-3} z_o^{n-2}$ or $\varrho_o = \pm\varrho_r$. For $n = 3$ $\mathscr{F}_n^R = F_n^R$.

H. Distortion

Distortion is defined by

$$-\sum_{n=3}^{\infty}\sum_q a_n\binom{n+1}{1}\frac{\varrho_q^{n-1}}{z_q^n}(\rho\cdot\rho_q) = -\sum_{n=3}^{\infty} a_n(n+1)\varrho$$
$$\times[D_{n_x}^{R,V}\cos\theta + D_{n_y}^{R,V}\sin\theta], \tag{19}$$

where $D_{n_x}^{R,V}$, $D_{n_y}^{R,V}$ ... are the coefficients of the distortion of the n-th order

$$D_{n_x}^{R,V} = \frac{\varrho_c^{n-1}}{z_c^n}x_c \mp \frac{\mu}{m}\frac{\varrho_o^{n-1}}{z_o^n}x_o \pm \frac{\mu}{m}\frac{\varrho_r^{n-1}}{z_r^n}x_r - \frac{\varrho_{G_{R,V}}^{n-1}}{z_{G_{R,V}}^n}x_{G_{R,V}}. \tag{20}$$

Independently of z_o, z_r, z_c, for fulfilled conditions $(x_c/z_c) = -(\mu x_r/mz_r)$, and $(y_c/z_c) = -(\mu y_r/mz_r)$

$$D_{n_x}^R = \frac{\mu}{m}\left[1-\left(\frac{\mu}{m}\right)^{n-1}\right]\left[\left(\frac{\varrho_r}{z_r}\right)^{n-1}\frac{x_r}{z_r} - \left(\frac{\varrho_o}{z_o}\right)^{n-1}\frac{x_o}{z_o}\right]. \tag{20a}$$

The coefficients $D_{n_x}^R$, $D_{n_y}^R$ become simultaneously equal to zero for $\mu = m$, or $(x_r/y_r) = (x_o/y_o)$.

As it may be seen from the above analysis the simultaneous elimination of all the aberrations occur if $\mu = m$ under condition that the plane waves of properly chosen slope angles are used for the recording and the reconstruction of hologram. If $z_r = z_o$ and the proper choice of the reconstructing wave source position is made (i.e. $(x_c/z_c) = -(\mu x_r/mz_r)$, and $(y_c/z_c) = -(\mu y_r/mz_r)$) this condition $\mu = m$ is also required. Owing to the assumption of $\mu = m$ all the additional conditions for disappearance of coma, astigmatism, field curvature, etc. are reduced to a single condition. The aberrations of all orders for Φ_R disappear if additionally $z_c = -\mu z_o$. This means that for sources of the wavefronts φ_o and φ_r lying in one plane, the source of the wavefront φ_c should be positioned at the points $x_c = -\mu x_r$, $y_c = -\mu y_r$, $z_c = -\mu z_r$. The other set of conditions for the aberration elimination, valid for the phase Φ_R at $z_r = z_o$, requires that the conditions $x_r = x_o$ and $y_r = y_o$ be fulfilled which means that the sources of wavefronts φ_r and φ_o are located at the same point. The condition $\rho_o = \rho_r$ does not satisfy the physical condition imposed on the problem. This means that in the lensless Fourier hologram only the spherical aberration becomes equal to zero [2].

Far region aberrations. Acceleration of convergence

In practical applications it is often necessary to determine the far region aberrations. At the limit of hologram resolution the aperture angles of order of one radian may be accounted. For large apertures the series describing the particular aberrations obtained from the expansion (3) converges very slowly and for $|\xi| > 1$ the series (3) becomes divergent. Following the method proposed in [3] allowing to extend the determinability of the spherical aberration outside the range of the classical binomial expansion, we shall do the same with the other aberrations. There are no physical reasons, for which the aberrations should be cut out at $\xi = 1$. This difficulty is eliminated by chosing such k for each ξ that $((\xi-k)/(1+k)) < 1$, then

$$\sqrt{1+\xi} = \sqrt{1+k+\xi-k} = \sqrt{1+k}\sqrt{1+\frac{\xi-k}{1+k}}, \tag{21}$$

$\sqrt{1+\frac{\xi-k}{1+k}}$ may be developed into convergent series of the form (3). Depending on the values $\left(\frac{\rho-\rho_q}{z_q}\right)^2$, and $\left(\frac{\rho_q}{z_q}\right)^2$ each of the roots in the expression (2) for the phase φ_q may be developed into series according to (3) or (20).

Eg. with great $\frac{\rho-\rho_q}{z_q}$ and $\frac{\rho_q}{z_q}$ and the values of z_o, z_r, and z_c close to each other, as it is the case at the limit of resolution, the expression for the phase φ_q becomes the following

$$\varphi_q = \frac{2\pi}{\lambda_q} z_q\sqrt{1+k}\left(\sqrt{1+\frac{\left(\frac{\rho-\rho_q}{z_q}\right)^2-k}{1+k}} - \sqrt{1+\frac{\left(\frac{\rho_q}{z_q}\right)^2-k}{1+k}}\right) \approx \frac{2\pi}{\lambda_q} z_q\sqrt{1+k}$$

$$\times\Bigg\{\left[\binom{1/2}{1}\frac{1}{1+k} - \binom{1/2}{2}\frac{2k}{(1+k)^2} + \binom{1/2}{3}\frac{3k^2}{(1+k)^3} - \binom{1/2}{4}\frac{4k^3}{(1+k)^4}\ldots\right]\left[\left(\frac{\rho-\rho_q}{z_q}\right)^2 - \left(\frac{\rho_q}{z_q}\right)^2\right]$$

$$+\left[\binom{1/2}{2}\binom{2}{0}\frac{1}{(1+k)^2} - \binom{1/2}{3}\binom{3}{1}\frac{k}{(1+k)^3} + \binom{1/2}{4}\binom{4}{2}\frac{k^2}{(1+k)^4}\ldots\right]\left[\left(\frac{\rho-\rho_q}{z_q}\right)^4 - \left(\frac{\rho_q}{z_q}\right)^4\right]$$

$$+\left[\binom{1/2}{3}\binom{3}{0}\frac{1}{(1+k)^3} - \binom{1/2}{4}\binom{4}{1}\frac{k}{(1+k)^4}\ldots\right]\left[\left(\frac{\rho-\rho_q}{z_q}\right)^6 - \left(\frac{\rho_q}{z_q}\right)^6\right]$$

$$+\left[\binom{1/2}{4}\binom{4}{0}\frac{1}{(1+k)^4}\ldots\right]\left[\left(\frac{\rho-\rho_q}{z_q}\right)^8 - \left(\frac{\rho_q}{z_q}\right)^8\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Bigg\} \tag{22}$$

The phase difference $\Phi_{R,V} - \Phi_{G_{V,R}}$ written down by using (22) gives the Gaussian reference sphere in the first row, while in the second and next rows we find the aberrations of third, fifth and further orders taken with the corresponding coefficients. Denoting by w_2, w_4, w_6, and so on, the polynomials appearing in the expansion and keeping in mind the structure of the aberration represented in the form (4) it is possible to represent the far region aberrations by using several aberration coefficient taken with the proper weighting factors.

For instance, the spherical aberration is described by the sum

$$\frac{2\pi}{\lambda}\sqrt{1+k}\,(w_4 S_3^{R,V}\varrho^4 + w_6 S_5^{R,V}\varrho^6 + w_8 S_7^{R,V}\varrho^8 + w_{10} S_9^{R,V}\varrho^{10} + \ldots), \tag{23}$$

while the coma by

$$\frac{2\pi}{\lambda}\sqrt{1+k}[4w_4(C_{3_x}^{R,V}\cos\theta + C_{3_y}^{R,V}\sin\theta) + 6w_6(C_{5_x}^{R,V}\cos\theta + C_{5_y}^{R,V}\sin\theta)$$

$$8\,w_8(C_{7_x}^{R,V}\cos\theta + C_{7_y}^{R,V}\sin\theta)\varrho^7 + 10\,w_{10}(C_{9_x}^{R,V}\cos\theta + C_{9_y}^{R,V}\sin\theta)\varrho^9 \ldots]. \quad (24$$

The other aberrations may be expressed similarly. For this purpose it is necessary to take the $\binom{n+1}{r}w_{n+1}$ factor, proper for the given aberration, according to the scheme in fig. 2, and the relations (4), and next multiply it by the aberration expression of the suitable order. The advantage of this procedure is obvious. In the example for spherical aberrations given in [4] it was necessary to take more than 20 orders of aberration to calculate the complete aberration in the vicinity of $(\varrho/z_q) \leqslant 1$ (much slower convergence may be expected in the case of other aberrations). The same result was obtained by calculating for $k = 1$ only 5 initial coefficients of aberration and summing up the aberrations of particular order with the respective weighting factors according to (22).

For small ρ_q/z_q it is convenient to use the expansion of the expression

$$\varphi_q = \frac{2\pi}{\lambda_q} z_q\sqrt{1+k}\left\{\sqrt{1+\frac{\left(\dfrac{\rho-\rho_q}{z_q}\right)^2 - k}{1+k}} - \frac{1}{\sqrt{1+k}}\sqrt{1+\left(\frac{\rho_q}{z_q}\right)^2}\right\}. \tag{25}$$

Two new terms will appear in the aberrational expressions, i.e.

$$\frac{2\pi}{\lambda_q}\sqrt{1+k}\left[\binom{1/2}{0} - \binom{1/2}{1}\frac{k}{1+k} + \binom{1/2}{2}\left(\frac{k}{1+k}\right)^2 - \binom{1/2}{3}\left(\frac{k}{1+k}\right)^3 + \ldots - \frac{1}{\sqrt{1+k}}\right]z_q = \frac{2\pi}{\lambda_q}\sqrt{1+k}\left[w_0 - \frac{1}{\sqrt{1+k}}z_q\right], \tag{26}$$

and

$$\frac{2\pi}{\lambda_q}\left[\left(\sqrt{1+k}\,w_2-\binom{1/2}{1}\right)\frac{\varrho_q^2}{z_q}+\left(\sqrt{1+k}\,w_4-\binom{1/2}{2}\right)\frac{\varrho_q^4}{z_q^3}\right.$$
$$\left.+\left(\sqrt{1+k}\,w_6-\binom{1/2}{3}\right)\frac{\varrho_q^6}{z_q^5}+\left(\sqrt{1+k}\,w_8-\binom{1/2}{4}\right)\frac{\varrho_q^8}{z_q^7}+\ldots\right]. \tag{27}$$

The coefficient associated with z_q tends to zero, when the number of terms in the expansion tends to ∞. Since we are forced to take into account several terms of the expansion it is necessary to preserve this term for numerical calculations. The structure of the expression (27) reminds that of spherical aberration, the difference being that (27) is referred to ϱ and not to ϱ.

Conclusions

For the analysis of the aberration influence on the imaging quality, it is necessary to know the form of the higher order aberrations, and, consequently, the complete aberrations of the given kind. This knowledge is also necessary when aiming at diminishing the given aberration. The suggested method of accelerating the convergence, allowing also to determine the aberrations outside the classical binomial expansion supported by the proposed form of aberration structure facilitate this task. By referring to the local properties of the functions accounted the proper choice of k allows to determine the full aberrations with the needed accuracy.

The expressions given in section *Far region aberrations. Acceleration of convergence* do not represent all the possibilities. Generally speaking, for z_o, z_r, z_c, and ϱ_o, ϱ_r, ϱ_c differing considerably from each other, the expansion of each of the phases φ_q with the same accuracy will require a different choice of k. Under these conditions it seems that the concept of a modified coefficient of aberrations may be introduced, to assure that each of phases φ_q be represented in the respective expansions with the same accuracy. This problem will be the subject of the next publication.

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Аберрации высших порядков в голограммах

Работа содержит анализ структуры аберрации высших порядков для точечных и сточников волновых фронтов, участвующих в отображении. Приводятся выражения для аберрации высших порядков, а также описываются условия их исчезновения. Предлагается простой метод, позволяющий ускорить сходимость аберрационных выражвний в пределах двучленного разложения, дающего возможность также определить полные аберрации вне пределов этого же разложения.