*Optica Applicata, Vol. X, No. 3, 1980*

# Evanescent wave structure for total reflection of Gaussian beam at a plane interface

Jerzy Woźnicki
Institute of Electron Technology, Warsaw Technical University, Warsaw, Poland.

By employing a continuous plane-wave spectrum and the continuity of the fields across the boundary, in the case of total reflection of linearly polarized Gaussian beam, the transmitted surface wave has been analysed. The curves of a constant density of energy flux have been computed.

## Introduction

Gaussian light beam plays an essential role in coherent optics [1, 2], providing a model of optical beam being not only more realistic than a plane wave but also than a spherical one. The total internal reflection of Gaussian light beam and the structure of evanescent wave, emerging in optically less dense medium in particular, are of primary importance. This type of surface waves is utilized in optical integrated systems, such as couplers [3] or inhomogeneous wave holography [4]. The total internal reflection has been analysed by a number of authors (see e.g. [4, 5]). A few papers [6, 7] have been also devoted to the case of Gaussian light beam, where the totally reflected wave was treated in two dimensions only.

The present paper discusses a three dimensional structure of the evanescent refracted wave in the case of the total reflection of $TEM_{00}$ Gaussian light beam along an interface separating two isotropic homogeneous lossless media. For the purpose of calculations the incident beam is decomposed into angular plane waves. The reflected beam is obtained by the integration over each individual refracted plane component of the spectrum. The detailed form of reflected surface wave, encountered in the case of the total internal reflection of the incident beam, has been determined in the approximation of energetically fundamental beams (see e.g. [8]), geometrically reflected and geometrically refracted, by using continuity of a tangential field components at the boundary between the two media.

## Angular spectrum of reflected beam

Consider $TEM_{00}$ Gaussian beam with the normal polarization with respect to the plane of incidence falling at an angle $\varphi_1$ at the interface separating two homogeneous lossless media with the refractive indices $n_1$, and $n_2$, respectively ($n_1/n_2 = n_{21}$).

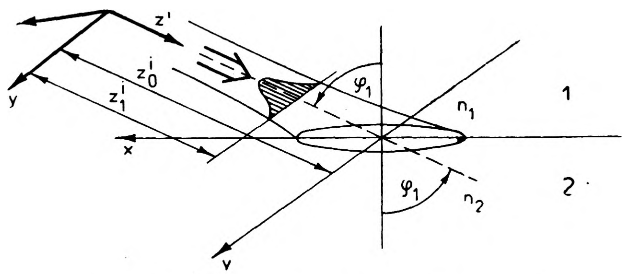


Fig. 1. Geometry of the incident-beam problem

The beam in question, denoted by an index "$i$" standing for incidence, has been represented in Cartesian coordinate system $(x^i, y^i, z^i)$ (fig. 1), where the location of the coordinate system was chosen in such a manner that the beam lays in the plane $y = 0$. For such a configuration $E_x = E_z = 0$, whereas $E_y$ denoted here by $U^i$ is equal to

$$U^i(x^i, y^i, z^i) = A^i \frac{w_0^i}{w^i(z^i)} \exp\left\{-\frac{(\alpha^i x^i)^2+(\beta^i y^i)^2}{[w^i(z^i)]^2}\right\}$$
$$\times \exp\left\{-i\left[k^i z^i - \Phi^i(z^i) + k_i \frac{(\alpha^i x^i)^2+(\beta^i y^i)^2}{2R^i(z^i)}\right]\right\}, \tag{1}$$

where $A^i$ = constant,

$$\begin{aligned} x^i &= x\cos\varphi_1 + z\sin\varphi_1, \\ y^i &= y \\ z^i &= -\ x\sin\varphi_1 + z\cos\varphi_1 + z_0^i,\ z \leqslant 0, \end{aligned} \tag{2}$$

and $z_0^i$ is a distance from the origin of coordinate set $(x, y, z)$ to the beam set. (Here we have to do with a rotation by an angle $\varphi_1$ and a displacement by $z_0$ of the coordinate system such that the origin of the local coordinate system $(x^i, y^i, z^i)$ be positioned at the beam waist). The values of $w^i(z^i)$, $R^i(z^i)$, $\Phi^i(z^i)$, are described by the following relations [2]:

$$w^i(z^i) = w_0^i \left\{1 + \left[\frac{2z^i}{k^i(w_0^i)^2}\right]^2\right\}^{1/2}, \tag{3}$$

$$R^i(z^i) = z^i \left\{1 + \left[\frac{2z^i}{k^i(w_0^i)^2}\right]^{-2}\right\}, \tag{4}$$

$$\Phi^i(z^i) = \arctan\left[\frac{2z^i}{k^i(w_0^i)^2}\right], \tag{5}$$

while $\alpha^i$, $\beta^i$ are the new constants describing deformation of transverse beam shape, being of primary importance for this paper. The beam can be represented by an angular spectrum of plane waves $V^i$ of normal polarization travelling in the directions specified by directional cosines $p^i, q^i, m^i$

$$U^i(x^i y^i, z^i) = C^i \int\int_{-\infty}^{+\infty} V^i(x^i y^i z^i; p^i, q^i) dp^i dq^i$$

$$= C^i \int\int_{-\infty}^{+\infty} \exp\left\{-a^i(k^i)^2 \left[\left(\frac{p^i}{\alpha^i}\right)^2 + \left(\frac{q^i}{\beta^i}\right)^2\right]\right\}$$

$$+\exp[i\,k^i(p^i x^i + q^i y^i + m^i z^i)]\,dp^i\,dq^i, \tag{6}$$

where

$$C^i = (k^i/2\pi)^2 \frac{4\pi a^i}{\alpha^i \beta^i} U_1^i,$$

$$U_1^i = \frac{A^i w_0^i}{w^i(z_1^i)} \exp\{-i[k^i z_1^i - \Phi^i(z^i)]\}, \tag{7}$$

$$a^i = \frac{1}{4b^i},$$

$$b^i = \frac{1}{[w^i(z_1^i)]^2} + i\frac{k^i}{2R^i(z_1^i)},$$

and $z_1^i$ satisfies the condition $0 < z_1^i < z_0^i$. Further

$$m^i \begin{cases} [1-(p^i)^2-(q^i)^2]^{1/2} & \text{when } (p^i)^2+(q^i)^2 \leqslant 1, \\ i[(p^i)^2+(q^i)^2-1]^{1/2} & \text{when } (p^i)^2+(q^i)^2 > 1. \end{cases} \tag{8}$$

It is convenient to describe both the reflected $V^r$ and refracted $V^g$ elementary plane wave components by using the local coordinate sets $(x^r, y^r, z^r)$ and $(x^g, y^g, z^g)$, respectively. A particularly convenient choice of the local coordinate system is such that $y^i = y^r = y^g$, the directions of $z^r$ and $z^g$ axes being related to the direction of $z^i$ axis by the reflection and refraction laws (fig. 2). By employing the boundary con-

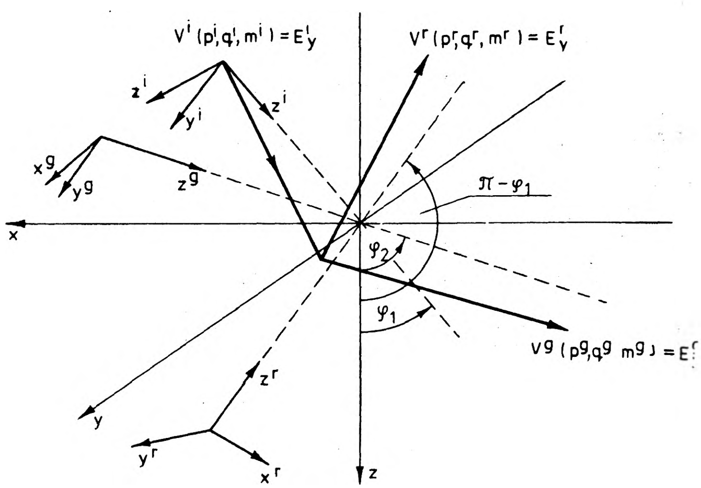


Fig. 2. Coordinate systems

ditions along the plane $z = 0$ and after strightforward manipulations one can get the following expression for the refracted wave

$$V^g(x^g, y^g, z^g; p^g, q^g) = T^g(p^g, q^g) A_0^g(p^g, q^g) \times \exp[ik^g(p^g x^g + q^g y^g + m^g z^g)]. \tag{9}$$

As a consequence of both the Snellius law $\sin\varphi_2/\sin\varphi_1 = n_{21}$ and what was stated above we obtain the following relations:

$$A_0^g(p^g, q^g) = \exp\left\{-a^g(k^g)^2\left[\left(\frac{p^g}{\alpha^g}\right)^2 + \left(\frac{q^g}{\beta^g}\right)^2\right]\right\},$$

$$k^g = \frac{1}{n_{21}} k^i, \quad a^g = \left(\frac{1}{n_{21}}\right)^2 a^i, \quad z_0^g = n_{21} z_0^i, \tag{10}$$

$$\frac{p^g}{\alpha^g} = \frac{p^i}{\alpha^i}, \quad \frac{q^g}{\beta^g} = \frac{q^i}{\beta^i}.$$

The Fresnel transmission coefficient in the above notation has a form

$$T^g(p^g, q^g) = \frac{\{1 - 1/n_{21}^2[1 - (p^g \sin\varphi_2 + m^g \cos\varphi_2)^2]\}^{2/1}}{\{1 - 1/n_{21}^2[1 - (p^g \sin\varphi_2 + m^g \cos\varphi_2)^2]\}^{1/2} + p^g \sin\varphi_2 + m^g \cos\varphi_2}. \tag{11}$$

Unfortunately, rigorous integration of the refracted bem

$$U^g(x^g, y^g, z^g) = C^g \int_{-\infty}^{+\infty}\int V^g(x^g, y^g, z^g; p^g, q^g)\, dp^g\, dq^g \tag{12}$$

cannot be done. Notice, however, that the Gaussian beam is paraxial. It means that directional cosines of energetically important components of its angular spectrum satisfy the condition $(p^g)^2 + (q^g)^2 \ll 1$. The Fresnel coefficient $T^g(p^g, q^g)$ is nearly constant: in the range of variation of $p^g$ and $q^g$

$$T^g(p^g, q^g) = T^g(0, 0) \tag{13}$$

(zero value of directional cosines is an attribute of the strongest energetically spectrum component). Under the approximation (13) the beam (12) is a formally geometrically refracted Gaussian beam. The validity of the assumption (13) has been verified on a computer [9]. From the computation performed for $|p^g| + |q^g| < 10^{-5}$ (when $|p^g| + |q^g| > 10^{-5}$ the amplitude of refracted plane wave $A_0^g < 10^{-4}$) we get

$$\Delta[T^g(p^g, q^g)/T^g(0, 0)]_{\max} < 10^{-5} \tag{14}$$

in the case of nontotal reflection ($\varphi_1 < \varphi_{cr}$), and

$$\Delta \mathrm{Re}[T^g(p^g, q^g)/T^g(0, 0)]_{\max} < 10^{-4},$$

$$\Delta \mathrm{Im}[T^g(p^g, q^g)/T^g(0, 0)]_{\max} < 10^{-4}, \tag{15}$$

when a total reflection occurs. All the calculations have been performed for the following data: $n_1 = 1.7$, $n_2 = 1.6$, $\varphi_1 = 60°$ or $\varphi_1 = 71°$, $\varphi_{cr} = 70.25°$. The similar arguments hold for reflected wave. The assumption of a constant reflection coefficient $T^r(p^r, q^r) \cong T^r(0, 0)$ results, in such case, in a Gaussian geometrically reflected beam described in [6, 7].

## Evanescent wave in optically less dense medium

Within the framework of geometrical approximation, the equations (1)–(5) determine the incident, reflected and refracted beam, where for the reflected and refracted beam the index "$i$" has to be replaced by an index "$r$" or "$g$", respectively. The continuity of a tangential electrical field components along the interface separating two media is sufficient to obtain the set of equations:

$$\begin{gathered}\frac{2z^i}{k^i(w_0^i)^2} = \frac{2z^r}{k^r(w_0^r)^2} = \frac{2z^g}{k^g(w_0^g)^2},\\ k^i z^i = k^r z^r = k^g z^g,\\ \frac{(\alpha^i x^i)^2 k^i}{2z^i} = \frac{(\alpha^r x^r)^2 k^r}{2z^r} = \frac{(\alpha^g x^g)^2 k^g}{2z^g},\\ \frac{(\beta^i y^i)^2 k^i}{2z^i} = \frac{(\beta^r y^r)^2 k^r}{2z^r} = \frac{(\beta^g y^g)^2 k^g}{2z^g},\\ \frac{\alpha^i x^i}{w^i} = \frac{\alpha^r x^r}{w_0^r} = \frac{\alpha^g x^g}{w_0^g},\\ \frac{\beta^i y^i}{w_0^i} = \frac{\beta^r y^r}{w_0^r} = \frac{\beta^g y^g}{w_0^g}.\end{gathered} \tag{16}$$

After some algebraic rearrangements we obtain the following relations or the refracted wave of interest:

$$\begin{gathered}k^g = (1/n_{21})k^i, \quad w_0^g = n_{21} w_0^i, \quad z_0^g = n_{21} z_0^i,\\ \alpha^g = n_{21}\frac{\cos\varphi_1}{\cos\varphi_2}\alpha^i, \quad \beta^g = n_{21}\beta^i.\end{gathered} \tag{17}$$

It can be seen that the refracted beam is astigmatically distorted. The cross-section of the beam has a form of an ellipse described by the equation

$$\left(\frac{x^g}{\cos\varphi_2}\right)^2 + \left(\frac{y^g}{\cos\varphi_1}\right)^2 = \left(\frac{w^g}{n_{21}\cos\varphi_1}\right)^2. \tag{18}$$

If $n_1 > n_2$ and $\varphi_1 > \varphi_{cr} = \arcsin(n_2/n_1)$ (total reflection), then $\cos\varphi_2 = n_{21}[1/n_{21}^2 - \sin^2\varphi_1]^{1/2} = -in_{21}\cdot v$, where $v = (\sin^2\varphi_1 - 1/n_{21}^2)^{1/2}$. Further

$$\alpha^g x^g = (\alpha^i n_{21}\cos\varphi_1)x + i(\sin 2\varphi_1/2v)z,$$

$$\beta^g y^g = \beta^i n_{21} y,$$

$$z^g = n_{21}[(z_0^i - \sin\varphi_1\cdot x) - ivz] \cong n_{21}(z_i^0 - \sin\varphi_1\cdot x), \tag{19}$$

where the imaginary part has been neglected, since the values of variable of interest $z$ are small (of order of $\lambda_0^i = 2\pi n_1/k^i$) and $v \ll 1$. If $U_c^g$ describes a wave in the medium 2 in the case of total reflection, then after substituting (19) into (1) (but with index "$g$") and taking into account the equations (17) relating the parameters of refracted beam to the parameters of incident beam, we obtain ($A_c^g$ is the complex amplitude and index "$i$" for simplicity has been omitted):

$$U_c^g(x, y, z) = A_c^g \frac{n_{21}w_0}{w} \exp\left\{-\left[\left(\frac{an_{21}\cos\varphi_1}{w}\right)^2 x^2 - \left(\frac{a^2 n_{21} k\sin\varphi_1\cos^2\varphi_1}{vR}\right)xz - \left(\frac{an_{21}\sin 2\varphi_1}{2vw}\right)^2 z^2 + (kv)z + \left(\frac{\beta n_{21}}{w}\right)y^2\right]\right\}$$
$$\times \exp\left\{-i\left[kz_0 - \Phi + \left(\frac{a^2 n_{21} k\cos^2\varphi_1}{2R}\right)x^2 + \left(\frac{2a^2 n_{21}^2 \sin\varphi_1\cos^2\varphi_1}{vw^2}\right)xz - \left(\frac{a^2 n_{21} k\sin^2 2\varphi_1}{8v^2 R}\right)z^2 - (k\sin\varphi_1)x - \left(\frac{\beta^2 n_{21} k}{2R}\right)y^2\right]\right\}. \tag{20}$$

Notice that the quantities $w$, $R$ and $\Phi$ are here the functions of variable $x$ (see eqs. (3)–(5), but with "$g$" indices, and eq. (19)). The family of curves of a constant density of energy flux can be obtained from the equation

$$|U_c^g| = |A_c^g| \frac{n_{21}w_0}{w} \exp(-m^2), \tag{21}$$

where $m$ is a parameter. It can be seen that in the medium 2, where the wave is fast decying, for all the values of $z$ which are of interest in this case, the relation $|A_c^g| n_{21} w_0/ /w \cong$ const holds. From (20) and (21) we get

$$x^2 - \left(\frac{kw^2\sin\varphi_1}{vRn_{21}}\right)xz - \left(\frac{\sin\varphi_1}{v}\right)^2 z^2 + \left(\frac{kvw^2}{a^2 n_{21}^2\cos^2\varphi_1}\right)z + \left(\frac{\beta}{a\cos\varphi_1}\right)^2 y^2 = \left(\frac{mw}{an_{21}\cos\varphi_1}\right)^2. \tag{22}$$

As a result the equation describing the curve composed of two branches (see fig. 3) has been obtained. The lower branch, depicted by a dashed line, has no physical interpretation. The equation describing the family of curves of a constant phase can be obtained in the same manner. Taking into account that $\Phi \cong$ const we get:

$$z^2-\left(\frac{4n_{21}vR}{k\sin\varphi_1 w^2}\right)zx-\left(\frac{v}{\sin\varphi_1}\right)^2x^2$$
$$+\left(\frac{2v^2R}{a^2n_{21}\sin\varphi_1\cos^2\varphi_1}\right)x-\left(\frac{2\beta v}{a\sin 2\varphi_1}\right)^2y^2 = \text{const.} \tag{23}$$

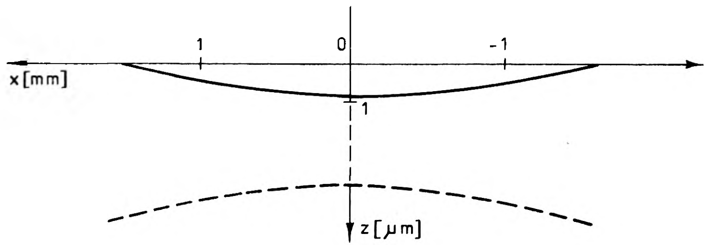


Fig. 3. Calculated curve of a constant density energy flux

The numerically computed curves of a constant density of energy flux in the plane $y = 0$ are plotted in fig. 4 abc, for different values of $m$ and three discrete values of $\varphi_1$. Figure 4d illustrates a transverse structure of the wave in the plane $x = 0$. The direction of energy flow is indicated by the arrows, while the dashed lines show the approximate shape of constant phase curves.

## Conclusions

It has been shown (see inequalities (14), (15)) that in the case of reflection and refraction only the geometrically reflected and refracted beam are of practical importance.

In the case of total reflection of a Gaussian beam, certain surface wave, fast decaying in the direction of $z$-axis, penetrates an optically less dense medium. The energy flow into optically less dense medium is followed by a surface wave penetration and its return into original medium 1. The extent of the penetration connected with Goos–Hänchen effect [5–7] is limited by a transverse dimensions of the incident beam along the interface. The penetration depth of the surface wave changes accross the beam reaching the maximum value of about one wavelength (see fig. 4). An increase of the angle of incidence results in a decrease of energetic penetration depth along the whole penetration area by the same ammount. Quasi-hyperboloidal surfaces of a constant density of energy flux are heavily flatten. If a small variations of parameters $w$, $R$, $\Phi$ are neglected then the cross-sections of these surfaces along the plane $y =$ const as well as $x =$ const. are hyperbolas, whereas the cross-sections along the planes $z =$ const. produce certain ellipses.

The solution obtained has a singularity at $\varphi_1 = \varphi_{cr}$. That is why the theory presented is invalid in the immediate vicinity of this point. The limits of its applicability are determined by the conditions that all the energetically important components

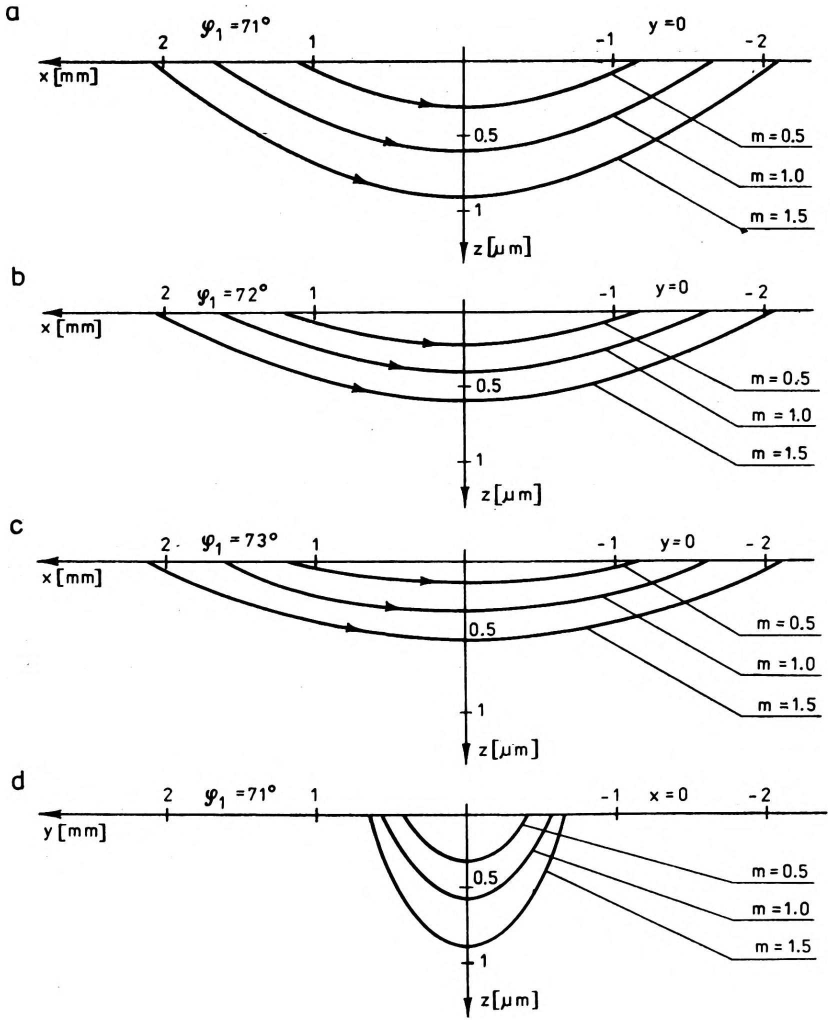


Fig. 4. Evanescent wave structure in a less dense medium for the following numerical data: $\lambda_0^i = 0.6328 \cdot 10^{-6}$ m; $w_0^i = 0.5 \cdot 10^{-3}$ m; $z_0^i = 2$ m; $\alpha^i = \beta^i = 1$; $n_1 = 1.7$; $n_2 = 1.6$

of the angular spectrum are incident at the interface between two media at the angle greater that the critical one. The singularity of this kind is typical and often encountered in papers on similar subject [5–7]. It is basically connected with the limited transverse extent of the incident beam and has noting to do with the specific Gaussian structure of the wave.

The paper discusses the normal polarization case. Accounting for the parallel polarization is strightforward and yields quite similar results.

*Acknowledgement* — I am very much obligated to Prof. Bohdan Karczewski for critical discussion and to Dr. Tomasz Jannson for valuable hints which helped me in the final shaping of the text.

## References

[1] MARCUSE D., *Light Transmission Optics*, Van Nostrand Reinhold Company, New York 1972.
[2] KOGELNIK H., LI T., Appl. Opt. **5** (1966), 1550.
[3] ULRICH R., J. Opt. Soc. Am. **61**, (1971), 1467.
[4] BRYNGDAHL O., *Evanescent Waves in Optical Imaging*, Progress in Optics, Vol. XI, ed. E. Wolf, North Holland Publ. Co., Amsterdam 1973, pp. 169–221.
[5] LOTSCH H. K. V., J. Opt. Soc. Am. **58**, (1968), 551.
[6] HOROWITZ B. R., TAMIR T., J. Opt. Soc. Am. **62**, (1971), 586.
[7] ANTAR Y. M., BOERNER W. M., Can. J. Phys. **58**, (1974), 962.
[8] BREKHOVSKIKH L. M., *Waves in Layered Media*, Academic Press, New York 1960.
[9] WOŹNICKI J., Doctor's Thesis, Warsaw Technical University, Warsaw 1978 (in Polish).

*Received, March 5, 1979*

## Структура затухающей волны при полном отражении гауссова пучка на плоской границе двух сред

При использовании условий непрерывности электромагнитниго поля на границе сред анализируется случай полного внутреннего отражения линейно поляризованного гауссова пучка света, представленного угловным спектром плоских волн. Обсуждается структура поверхностной волны, переломленной на плоской границе двух однородных сред. Приведены вычисленные с помощью цифровой вычислительной машины кривые постоянной плотности потока энергии этой волны.