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Time and space compression functions in two-dimensional description of the laser pulse propagation in nonlinear medium

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Starting with a nonlinear wave equation written in paraxial approximation, the time and space compression functions have been generalized to the case of a medium with the complex refractive index. The general dependences allowing to analyze the time and space field distribution in the presence of single- and multiphoton resonance interaction and the nonlinear radiation refraction. By taking advantage of the formulae obtained the influence of the nonlinear interaction upon the change in the field distribution has been analyzed for some cases of practical interest.

Introduction

The propagation of the light pulse in a nonlinear dielectric medium is described by the wave equation and the equations for the real and virtual parts of the refractive index of the medium. The analytical solution of the above system of equations, which would be necessary for a complete description of the field change, is impossible (even under considerable simplifications), and the numerical solution is by no means simple either. Under these circumstances any activity allowing to find new simplified methods for the analysis of changes in fundamental parameters of radiation in a medium is of great importance.

In papers [1, 2] starting from the energy transport equations for electromagnetic fields the time and space compression functions have been introduced, which enable the analysis of the time and spatial changes in radiation intensity distribution in the medium with nonlinear virtual part of the refractive index. The next papers [3-8] proved the usefulness of these functions as applied to problems of nonlinear amplification and absorption of strong laser pulses. A number of conclusions following from the analysis of compression function have been verified experimentally [5,8-10].

In the present paper we have started from the nonlinear wave equation written in paraxial approximation, and generalized the time and space compression functions to include the medium with nonlinear real and virtual parts of the refractive index. The effects of both single- and multiphoton resonance interaction as well as the influence of the nonlinear radiation refraction on the changes of the time and spatial field distribution in the medium have been analyzed.

Model of a medium and the propagation equations

The light propagation in a dielectric medium is described by a nonlinear wave equation

$$\text{rot rot } \tilde{E} = -\frac{1}{c^2}\frac{\partial^2}{t^2}(\varepsilon\tilde{E}) - \frac{4\pi p}{c^2}\frac{\partial\tilde{E}}{\partial t}, \tag{1}$$

where: $\tilde{E}$ — electric field vector, ε — dielectric permittivity depending, in general, on $\tilde{E}$, c — vacuum light velocity, $\frac{4\pi p}{c^2}\frac{\partial\tilde{E}}{\partial t}$ — phenomenological term describing the linear nonresonant losses in the medium [11], p — constant term of these losses. Let us choose the coordinate system, in which z-axis direction is consistent with the main direction of light wave propagation, and let us put

$$\tilde{E} = \frac{1}{2}\vec{E}\, e^{i(kz-\omega t)} + \frac{1}{2}\vec{E}^* e^{-i(kz-\omega t)},$$

where: $\vec{E} = \vec{E}(\vec{r}, t)$ — complex slowly varying (as compared with $e^{i(kz-\omega t)}$), field amplitude, ω — central frequency of field vibrations, $k = \frac{\omega}{c}\sqrt{\varepsilon_0}$, ε_0 — real part of dielectric permittivity independent of $\vec{E}$.

Let us assume that

$$\varepsilon = \varepsilon(<\tilde{E}^2>), \quad \Lambda_{\parallel} \gg \Lambda_{\perp}, \quad |\delta_\varepsilon| \ll \varepsilon_0, \quad \tau_p \frac{\Lambda_{\perp}}{\lambda} T', \; \Lambda_{\perp} \gg \lambda, \tag{2}$$

where: $<\tilde{E}^2>$ — value of $\tilde{E}^2$ averaged over the wave period, λ — wavelength, T' — period of light vibrations, $\delta_\varepsilon = \varepsilon - \varepsilon_0$, $\Lambda_{\perp}$ and $\Lambda_{\parallel}$ — characteristic dimension of changes in $\vec{E}$ in directions perpendicular and parallel to z-axis, respectively, τ_p — characteristic dimension for changes in $\vec{E}$ as a function of time. Let us assume also that the characteristic dimension of the time and spatial "nonuniformity" of the medium is great if compared with $\Lambda_{\perp}$, $\Lambda_{\parallel}$ and τ_p, respectively. Under the above assumptions $|\vec{E}_z| \ll |\vec{E}_{\perp}|$, and the equation for the transversal components of $\vec{E}$ vector may be reduced to the form [12]

$$\Delta_{\perp}\vec{E}_{\perp} + 2ik\left(\frac{\partial\vec{E}_{\perp}}{\partial z} + \frac{1}{v}\frac{\partial\vec{E}_{\perp}}{\partial t}\right) + k^2\frac{\delta\varepsilon}{\varepsilon_0}\vec{E}_{\perp} + ik\varrho\vec{E}_{\perp} = 0, \tag{3}$$

where:

$$\vec{E}_{\perp} = (E_x, E_y), \quad \Delta_{\perp} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$v = \frac{c}{\sqrt{\varepsilon_0}}, \qquad \varrho = \frac{4\pi p}{c\sqrt{\varepsilon_0}}.$$

From the fact that in an isotropic medium the equations for E_x and E_y, resulting from (3), are the same it follows that if the field in the $z = z_0$ plane has one definite polarization for all t this polarization does not suffer from any changes in the medium. In particular, the plane-polarized wave preserves its polarization state. The complete description of such a wave in the medium is reduced to the equation

$$\Delta_\perp E + 2ik\left(\frac{\partial E}{\partial z} + \frac{1}{v}\frac{\partial E}{\partial t}\right) + k^2 \frac{\delta\varepsilon}{\varepsilon_0} E + ik\varrho E = 0, \tag{4}$$

where $E \equiv E_x$, and the x-axis is chosen in accordance with the electric vector vibration direction.

Writting $\delta\varepsilon$ in the form

$$\delta\varepsilon = \delta\varepsilon' + i\delta\varepsilon''$$

the equation (4) may be written as follows

$$\Delta_\perp E + 2ik\left(\frac{\partial E}{\partial z} + \frac{1}{v}\frac{\partial E}{\partial t}\right) + k^2 \frac{\delta\varepsilon'}{\varepsilon_0} E = ikK(|E|^2)E, \tag{5}$$

where

$$K(|E|^2) = -\left(k\frac{\delta\varepsilon''}{\varepsilon_0} + \varrho\right). \tag{6}$$

The function $K(|E|^2)$ will be called the amplification function of the medium*.

By substituting

$$E(t, z, r) = A(t, z, r)e^{ik\,\Psi(t, z, r)}, \tag{7}$$

where r — radius-vector in the plane perpendicular to the z-axis for the axially-symmetrical system, or $r = x$ for the system symmetric with respect to y-, z-plane, the equation (5) may be written for the real amplitude A and the eikonal Ψ, separately, in an axially-symmetric or a symmetric system with respect to the x-, y-plane. From (5) and (7) we obtain

$$2\left(\frac{\partial\Psi}{\partial z} + \frac{1}{v}\frac{\partial\tau}{\partial t}\right) + \left(\frac{\partial\Psi}{\partial r}\right)^2 = \frac{\delta\varepsilon'}{\varepsilon_0} + \frac{1}{k^2 A}\Delta_\perp A, \tag{8}$$

$$\frac{\partial A}{\partial z} + \frac{1}{v}\frac{\partial A}{\partial t} + \frac{\partial A}{\partial r}\frac{\partial\Psi}{\partial r} + \frac{1}{2}A\Delta_\perp\Psi = \frac{1}{2}K(A^2)A, \tag{9}$$

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* In the case when $\delta\varepsilon'' > 0$ for all $\vec{r}$, t the function K will be called the absorption function of the medium.

where:

$$\Delta_\perp = \frac{\partial^2}{\partial r^2} + \frac{a-1}{r}\frac{\partial}{\partial r},$$

and $a = 2$ for the axially-symmetric system or $a = 1$ for the system symmetric with respect to y-, z-plane.

It is convenient to reduce the equations (8) and (9) to those for eikonal Ψ and the light intensity ($I = \frac{c}{8\pi} A^2$) for the case when

$$\frac{\delta\varepsilon'}{\varepsilon_0} \gg \frac{\Delta_\perp A}{k^2 A} \sim \frac{1}{4\pi^2}\frac{\lambda^2}{\Lambda_\perp^2}. \tag{10}$$

These equations have the form

$$\frac{\partial\psi}{\partial z} + \frac{1}{v}\frac{\partial\psi}{\partial t} + \frac{1}{2}\left(\frac{\partial\Psi}{\partial r}\right)^2 = \frac{\delta n'}{n_0}, \tag{11}$$

$$\frac{\partial I}{\partial z} + \frac{1}{v}\frac{\partial I}{\partial t} + \frac{\partial I}{\partial r}\frac{\partial\psi}{\partial r} + I_\perp\Delta\psi = K(I)I. \tag{12}$$

In these equations the electric permittivity has been replaced by the refractive index in accordance with the relationship: $\varepsilon = n^2$, $\varepsilon_0 = n_0^2$, $\delta\varepsilon' = 2n_0\delta n'$.

For the paraxial rays, i.e. those satisfying $r \ll \Lambda_\perp$, the equation for the light intensity (and analogically the equation (9)) takes the form:

$$I\frac{\partial I}{\partial z} + \frac{1}{v}\frac{\partial I}{\partial t} + I\Delta_\perp\Psi = K(I)I. \tag{13}$$

In the case of radiation with plane, cylindric or spherical wave front we obtain from (13)*:

$$\frac{\partial I}{\partial z} + \frac{1}{v}\frac{\partial I}{\partial t} + \frac{a}{z}I = K(I)I, \tag{14}$$

where $a = 0, 1, 2$ for plane, cylindric and spherical wave, respectively.

The field equations formulated above may be conveniently analyzed when expressed in variables r, z, τ, where $\tau = t - \frac{z}{v}$. In these variables the operator $\frac{\partial}{\partial z} + \frac{1}{v}\frac{\partial}{\partial t}$ is transformed into $\frac{\partial}{\partial z}$, and consequently the equations (11) and (12) take the form

$$\frac{\partial\Psi(r, z, \tau)}{\partial z} = \frac{\delta n'}{n_0} - \frac{1}{2}\left(\frac{\partial\Psi}{\partial r}\right)^2, \tag{15}$$

$$\frac{\partial I(r, z, \tau)}{\partial z} = K(I)I - \frac{\partial I}{\partial r}\frac{\partial\Psi}{\partial r} - I\Delta_\perp\Psi. \tag{16}$$

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* For the cylindric or spherical wave $\Psi = \frac{r^2}{2z}$, where the point $z = 0$ is the convergence point for rays.

From (16) it is possible to get a clear interpretation of K. This is namely a relative rate of changes in light intensity (field amplitude) at an arbitrary point of the beam cross-section, caused by both the resonance (Raman, etc.) interaction of the radiation with a medium and the linear losses occurring in this medium: $K = \frac{1}{I}\frac{\partial}{\partial z}\Big|_{\Psi=\text{const}}$ . .

If the eikonal Ψ does not depend upon the time (both explicitly and implicitly), then, by introducing the function

$$\mathscr{E}(r, z, \tau) = \int\limits_{\tau_0}^{\tau} J(r, z, \tau')d\tau',$$

(where $\mathscr{E}(r, z, \tau)$ — the radiation energy density (in I/cm^2)), which passed through the given point of the medium at the time $\tau-\tau_0$) and integrating (16) with respect to τ, we obtain

$$\frac{\partial \mathscr{E}(r, z, \tau)}{\partial z} = K_\varepsilon \mathscr{E} = \frac{\partial \mathscr{E}}{\partial r}\frac{\partial \Psi}{\partial r} = \mathscr{E}\Delta_\perp \Psi, \tag{17}$$

where

$$K_\varepsilon = \frac{1}{\mathscr{E}}\int\limits_{\tau_0}^{\tau} KId\tau, \; \mathscr{E}(\tau < \tau_0) = 0.$$

The function K_ε will be called the energy amplification function of the medium. The equation (17) may be employed to describe the changes in the energy radiation density for the case when $K_\varepsilon = K_\varepsilon(\mathscr{E}, r)$.

Now, we want to discuss the model of the medium and the equations describing the interaction of the radiation and the matter.

Consider the matter of refractive index $n' = n_0'+\delta n'$, containing L classes of non-interacting multilevel active centres*. Two energy levels, between which there occurs a resonant m-photon transition stimulated by the laser radiation of ω frequency, will be denoted by a and b. Among the L classes of active centres the following sub-classes will be distinguished: L_1 classes, for which the change of populations of a, b levels occurs due to a single-photon process ($\Omega_1^{ab} \approx \omega$, where Ω_1^{ab} is the transition frequency at the midpoint of the active centre line from the L_1 class), L_2 classes, in which this change occurs due to a two-photon process ($\Omega_2^{ab} \approx 2\omega$), and — generally — L_m classes, in which this change occurs due to an m-photon process ($\Omega_m^{ab} \approx m\omega$, where $m = 1, 2, \ldots, M$, and $L_1+L_2+ \ldots +L_M = L$). Let us assume that

$$\Delta\omega, \Gamma_{lm}^{ab} \ll \Omega_{lm}^{ab}, \; \tau_p \gg \tau_{lm}^{ab}, \quad W_{lm}^{ab} \ll \frac{\omega}{2\pi}, \quad \Lambda_\parallel, \Lambda_\perp \gg \lambda,$$

$$|m\omega - \Omega_{lm}^{ab}| \ll \Gamma_{lm}^{ab}, \quad \Delta\omega,$$

where: the index "lm" numbers the l-th class included in L_m group, $\Delta\omega$ is spectral

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* i.e. the atoms, ions, and the molecules introduced to the host material of refractive index n'. In general, it may be a set of media of refractive indices n_k'.

width of the laser pulse, Γ_{lm}^{ab} — width of the uniform transition line, W_{lm}^{ab} — probability of m-photon stimulated transition averaged over the period, τ_{lm}^{ab} — cross-relaxation time of the dipol transition. Under the above assumptions the resonant interaction of the pulse with the medium is of incoherent nature [16], and the equation for difference in population density of the a and b states may be represented in the form [1, 2]:

$$\frac{\partial N_{lm}}{\partial t}+\frac{N_{lm}-N_{lm}^{e}}{T_{lm}}+s_{lm}\sigma_{lm}N_{lm}I^{m}=0, \tag{18}$$

$$m=1,2,\ldots,M,\ l=1,2,\ldots,L_m,$$

under the two-level approximation; where: N_{lm} — effective difference of the population density of a and b states, N_{lm}^{e} — difference in population density at the thermodynamic equilibrium state for $I=0$, T_{lm} — effective relaxation time, s_{lm} — parameters depending upon the laser action scheme (in the two-level scheme $s_{lm}\hbar\omega=2$), $\sigma_{lm}I^{m-1}$ — cross-section for the m-photon transition.

The amplification function K for the medium may be defined basing on the energy conservation law. In the model considered this function is given by the expression [1, 2]:

$$K=\sum_{m=1}^{M}\sum_{l=1}^{L_m}m\sigma_{lm}N_{lm}I^{m-1}-\varrho, \tag{19}$$

where $N_{lm}>0$ for the inverse population of a and b states. In this expression the term attributed to the spontaneous transitions is neglected (as being small).

The explicit form of the energy amplification function K_ε may be determined for the case of single-photon nonstationary interaction at $\tau_p \ll T_{lm}$ [2]:

$$K_\varepsilon=\sum_{l=1}^{L_1}a_l\frac{\mathscr{E}_l^{s}}{\mathscr{E}}\left[1-\exp\left(-\frac{\mathscr{E}}{\mathscr{E}_l^{s}}\right)\right]-\varrho, \tag{20}$$

where

$$a_l=\sigma_{l1}N_{l1}^{e},\ \mathscr{E}_l^{s}=(s_{l1}\sigma_{l1})^{-1}.$$

The complete description of the pulse propagation in the considered medium requires additionally the formulation of an equation describing the dependence of nonlinear part of the refractive index $\delta n'$ on the light intensity. This equation has different form depending upon the physical phenomena (like the molecule orientation, electrostriction, molecule libration, polarization of the electron shell, and so on [13], which are the effect of the dependence $n'(I)$). For short radiation pulses (of nano- and subnanosecond duration) the dominant effect in the majority of the media used in the laser technique is the orientational or electron Kerr effect [12]. In the case of the orientational Kerr effect the equation for $\delta n'$ has the form [13]:

$$\frac{\partial(\delta n')}{\partial t}+\frac{\delta n'}{T_{\mathrm{Kerr}}}=\frac{1}{T_{\mathrm{Kerr}}}n_2 I, \tag{21}$$

where: T_{Kerr} — time of Kerr effect stabilization in the medium, n_2 — constant characteristic of given material. For the electron Kerr effect, in the face of the fact that $T_{\text{Kerr}} = 10^{-15} s$ [12, 13], the equation for $\delta n'$ has the simple form [13]:

$$\delta n' = n_2 I. \tag{22}$$

The equation for $\delta n'$ has the form (22) also for orientational Kerr effect, if $\tau_p \gg T_{\text{Kerr}}$.

Change in the time distribution of the field in the medium. Time compression function

We shall write the equation for the light intensity (16) in the general form

$$\frac{\partial I(r, z, \tau)}{\partial z} = IF[I(r, z, \tau), r, z, \tau], \tag{23}$$

where

$$F = K - \frac{1}{I}\frac{\partial I}{\partial r}\frac{\partial \Psi}{\partial r} - \Delta_{\perp}\Psi. \tag{24}$$

The effective length τ_p of the radiation pulse is defined by the relations

$$\tau_p(r, z) = \tau_2(r, z) - \tau_1(r, z),$$

$$I[r, z, \tau_1(r, z)] = I[r, z, \tau_2(r, z)] = \frac{1}{b} I_h(r, z), \tag{25}$$

where

$$I_h(r, z) = I(r, z, \tau_h),\ b > 1, \text{ and } \tau_h, \tau_1, \tau_2$$

are the respective points at the time maximum and the front and back pulse sides.

Let us introduce the function

$$T \equiv -\frac{1}{\tau_p}\frac{d\tau_p}{dz}, \tag{26}$$

and put

$$\left.\frac{\partial I(r, z, \tau)}{\partial \tau}\right|_{\tau_1} = \frac{1}{\delta_1(r, z)}\frac{I_h(r, z)}{\tau_p(r, z)},$$

$$\left.\frac{\partial I(r, z, \tau)}{\partial \tau}\right|_{\tau_2} = -\frac{1}{\delta_2(r, z)}\frac{I_h(r, z)}{\tau_p(r, z)}, \tag{27}$$

where $\delta_1, \delta_2 > 0$. By differentiating (25) with respect to z and taking advantage of (23), (26), (27) we obtain

$$T = \frac{\delta_1}{b}\left[F(I_h, \tau_h, r, z) - F\left(\frac{1}{b} I_h, \tau_1, r, z\right)\right] + \frac{\delta_2}{b}\left[F(I_h, \tau_h, r, z) - F\left(\frac{1}{b} I_h, \tau_2, r, z\right)\right]. \tag{28}$$

The function T will be called the time compression function. This function defines the relative rate of changes in the pulse duration during the propagation process across the medium. In accordance with the definition (26) a shortening (compression) of the pulse occurs for $T > 0$, its elongation (decompression) being stated for $T < 0$. The coefficients δ_1 and δ_2 defined in (27) will be called the slopes of the front and back pulse sides, respectively. For a monotonic function K they are usually slow-varying functions if compared with $I_h(z)$ and $\tau_p(z)$, respectively.

From the expression (28) it follows that the change in time distribution of the field in the medium is, in general, different for different points of the transversal distribution. For the sake of simplicity we shall neglect, hereafter, the dependence upon r, z, assuming that this dependence is considered at a definite point of the (r, z)-system.

By substituting (24) into (28) we obtain

$$T = T_K + T_\Psi, \tag{29}$$

where:

$$T_K = \frac{\delta_1}{b}\left[K(I_h, \tau_h) - K\left(\frac{1}{b} I_h, \tau_1\right)\right] + \frac{\delta_2}{b}\left[K(I_h, \tau_h) - K\left(\frac{1}{b} I_h, \tau_2\right)\right], \tag{30}$$

$$T_\Psi = \frac{\delta_1}{b}\left\{\theta\left[\frac{\partial\Psi(I_h, \tau_h}{\partial r} - \frac{\partial\Psi\left(\frac{1}{b} I_h, \tau_1\right)}{\partial r}\right] + \Delta_\perp\Psi\left(\frac{1}{b} I_h, \tau_1\right) - \Delta_\perp\Psi(I_h, \tau_h)\right\} +$$

$$\frac{\delta_2}{b}\left\{\theta\left[\frac{\partial\Psi(I_n, \tau_h)}{\partial r} - \frac{\partial\Psi\left(\frac{1}{b} I_h, \tau_2\right)}{\partial r}\right] + \Delta_\perp\Psi\left(\frac{1}{b} I_h, \tau_2\right) - \Delta_\perp\Psi(I_h, \tau_h)\right\}, \tag{31}$$

$$\theta = -\frac{1}{I_h}\frac{\partial I_h}{\partial r}.$$

The function T_K describes the change in the time distribution of the field due to nonlinear resonant (Raman-type, and others) interaction with the matter, while the function T_Ψ describes the change of this distribution due to the nonlinear refraction and diffraction of the radiation. We want to discuss some general conclusions following from the time compression function. We shall assume further that $b = 2$, i.e. we shall analyse the changes in the pulse duration measured at its half-height points.

Change in the time field distribution due to the resonant interaction

In the case of nonstationarity interaction, as may be seen from (30), the direction and rate of change in pulse duration strongly depend on both the slope of the pulse sides and the pulse symmetry (ratio δ_1/δ_2). The sufficient condition for the pulse compression to occur ($T > 0$) in an amplifying medium ($K > 0$) is $\delta_1 \ll \delta_2$ (the front side of the pulse being much steeper than its back side). This condition is not a necessary one.

For the nonstationary interaction with the absorbing medium the condition $\delta_1 \ll \delta_2$ is sufficient for the pulse elongation. The symmetric pulse takes in the medium an asymmetric shape.

If the function K does not depend explicitly on time (the case of quasi-stationary interaction or low singal) the function T_K takes the form

$$T_K = \frac{\delta_1+\delta_2}{2}\,[K(I_h)-K(\tfrac{1}{2}I_h)]. \tag{32}$$

The sufficient conditions for pulse compression and for elongation are $\partial K/\partial I > 0$, and $\partial K/\partial I < 0$, respectively. The direction of τ_p changes does not depend upon the pulse symmetry. By taking advantage of the definitions of coefficients δ_1, δ_2 it may be pointed out that also the rate of τ_p changes depends weakly upon the pulse symmetry. From (32) it follows, moreover, that the symmetric pulse remains symmetric in the medium. In the uniform medium, when the dependence $K(\tau)$ is not explicit and the radiation divergence is low, the relation between the pulse length and the top light intensity I_h is determined by the formula

$$\tau_p = \tau_p^0 \exp\left[-\int\limits_{I_h^0}^{I_h} \frac{T_K(I_h)}{I_h K(I_h)}\,dI_h\right], \tag{33}$$

where $\tau_p^0 = \tau_p(z = z_0)$, or in accordance with (31)

$$\tau_p = \tau_p^0 \exp\left[-\delta\int\limits_{I_h^0}^{I_h} \frac{K(I_h)-K(\frac{1}{2}I_h)}{I_h K(I_h)}\,dI_h\right], \tag{34}$$

where δ — average value of the function $\frac{1}{2}[\delta_1(I_h)+\delta_2(I_h)]$ within the considered interval of I_h*.

By substituting (19) into (30) and assuming $\varrho \neq \varrho(t)$, we obtain a general expression for the function T_K in the multicomponent medium

$$T_K = \frac{\delta_1}{2}\sum_{m=1}^{M} I_h^{m-1}\sum_{l=1}^{L_m}\Big[\beta_{lm}(I_h, \tau_h)-2^{1-m}\beta_{lm}(\tfrac{1}{2}I_h, \tau_1)+$$
$$+\frac{\delta_2}{2}\sum_{m=1}^{M} I_h^{m-1}\sum_{l=1}^{L_m}[\beta_{lm}(I_h, \tau_h)-2^{1-m}\beta(\tfrac{1}{2}I_h, \tau_2)], \tag{35}$$

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* Precisely speaking, the formula (34) defines the relation $\tau_p(I_h)$ within the interval in which the function $K(I_h)$ is monotonic and does not change the sign. In general case

$$\tau_p = \tau_p^0 \exp\left[-\sum_{n=0}^{N}\delta_n \int\limits_{I_h^{(n)}}^{I_h^{(n+1)}} \frac{K(I_h)-K1/2I_h)}{I_h K(I_h)}\,dI_h\right] \tag{34'}$$

holds, where: $I_h^{(n)}$, $I_h^{(n+1)}$ — limits of the interval in which the function K is monotonic and of constant sign. Additionally, if $K < 0$, then $I_h^{(n+1)} < I_h^{(n)}$.

where

$$\beta_{lm} = m\sigma_{lm}N_{lm}.$$

Particular cases of the relations (35) and (34) have been exploited in papers [3-8] to analyze time distribution of the field in single-, two- and three-component media for both single- and two-photon transitions. The analysis of these dependences, together with the analysis of the medium amplification function allowed to obtain the fundamental qualitative information concerning the changes in the time distribution of the field in the cases discussed, as well as some important quantitative relations. The results obtained in this way are consistent with the results of both the numerical solutions of the respective propagation equations and the experimental examinations [5, 8, 10].

As an example we shall discuss the change of pulse duration due to m-photon interaction in a single-component medium for the low signal case. From (35) we obtain

$$T_K = (1-2^{1-m})\frac{\delta_1+\delta_2}{2}\beta^e_{1m}I_h^{m-1}, \tag{36}$$

where $\beta^e_{1m} = m\sigma_{1m}N^e_{1m}$. In this case the change in the pulse duration occurs only for $m > 1$, its relative rate being proportional to the m-th power of the peak light intensity. The pulse compression occurs in the medium with the population inversion.

By substituting the amplification function for a low signal $K = \beta^e_{1m}I_h^{m-1}\text{-}\varrho$ into the formula (34) we obtain the dependence of the pulse duration upon the light intensity

$$\tau_p = \tau_p^0\left|\frac{\beta^e_{1m}(I_h^0)^{m-1}-\varrho}{\beta^e_{1m}I_h^{m-1}-\varrho}\right|^{\frac{1-2^{1-m}}{m-1}}. \tag{37}$$

If, in turn, we insert to this formula the expression for $I_h(z)$, which may be easily obtained from the formula (14) for β^e_{1m}, $\varrho = \text{const.}$ [2], and $a = 0$, we obtain the following dependence for the pulse duration upon the path travelled in the medium

$$\tau_p = \tau_p^0\left|1-\frac{\beta^e_{1m}}{\varrho}(I_h^0)^{m-1}[1-e^{\varrho(1-m)(z-z_0)}]\right|^{\frac{1-2^{1-m}}{m-1}\delta}. \tag{38}$$

In the case of $\beta^e_{1m}(I_h^0)^{m-1} > \varrho$ (amplifying medium) for $z \to z_K = z_0+\frac{1}{\varrho}\ln \times$ $\times \frac{\beta^e_{1m}(I_h^0)^{m-1}}{\beta^e_{1m}(I_h^0)^{m-1}-\varrho}$, $\tau_p \to 0$. Further, in the case of $\beta^e_{1m} < 0$ (absorbing medium) or $0 < \beta^e_{1m}(I_h^0)^{m-1} < \varrho$ (medium with population inversion but with losses higher than amplification) we have for $z \to \infty$:

$$\left|\frac{\beta^e_{1m}(I_h^0)^{m-1}-\varrho}{\varrho}\right|^{\frac{1-2^{1-m}}{m-1}\delta}.$$

This relation may be also obtained immediately from (37).

We shall shortly discuss one more conclusion following from (34). Let us put $L_m = 1$ for $m = p, n$, and $L_m = 0$ for $m = p, n$. The function T_K for low signal takes the form

$$T_K = \frac{\delta_1+\delta_2}{2} [(1-2^{1-p})\beta^e_{1p} I_h^{p-1} + (1-2^{1-n})\beta^e_{1n} I_h^{n-1}]. \tag{39}$$

If $\beta^e_{1p} \beta^e_{1n} < 0$, the change in pulse duration results from two opposite processes: compression and recompression. The expression (39) allows to simply distinguish which of these processes dominates. For instance, if $p > n$, the process conected with p-photon interaction dominates when

$$I_h > \left[\frac{(1-2^{1-n})\beta^e_{1n}}{(1-2^{1-p})\beta^e_{1p}}\right].$$

Change in time distribution of the field as a result of nonlinear refraction

The influence of nonlinear refraction upon the change of time distribution of the field will be illustrated by an example of a medium with $\delta n' = n_2 I$. In this case from (31) we obtain

$$T_{\Psi} = \frac{\delta_1+\delta_2}{2} \left\{\theta\left[\frac{\partial \Psi(I_h)}{\partial r} - \frac{\partial \Psi(\frac{1}{2} I_h)}{\partial r}\right] + \Delta_\perp \Psi(\tfrac{1}{2} I_h) - \Delta_\perp \Psi(I_h)\right. . \tag{40}$$

This expression describes the change in pulse duration at an arbitrary point r of the cross-sectional distribution in the beam. We restrict ourselves to determining the changes in τ_p on the beam axis (at the point $r = 0$), where the function T_Ψ takes the form

$$T_\Psi = \frac{\delta_1+\delta_2}{2} [\Delta_\perp \Psi(\tfrac{1}{2} I_h) - \Delta_\perp \Psi(I_h)]. \tag{41}$$

In accordance with [14], in the presence of the following boundary conditions

$$(r, \Psi, z = 0) = 0,$$

$$I(r, \tau, z = 0) = I_m^0(\tau)\left(1 - \frac{r^2}{r_0^2}\right)$$

the eikonal Ψ is determined by the expression

$$\Psi = -\frac{n_2 I_m^0(\tau)}{2n_0 r_0^2}\left[1 - \frac{n_2 I_m^0(\tau)}{n_0 r_0^2}\right]^{-1} zr^2 + \varphi(z), \tag{42}$$

where $\varphi(0) = 0$. By substituting (42) into (41) we obtain

$$T_\Psi = (\delta_1+\delta_2)\frac{\dfrac{z}{z_{fh}^2}}{\left[1-\left(\dfrac{z}{z_{fh}}\right)^2\right]\left[2-\left(\dfrac{z}{z_{fh}}\right)^2\right]}, \tag{43}$$

where

$$z_{fh} = \frac{r_0\sqrt{n_0}}{\sqrt{n_2 I_{hm}^0}}$$

is the length of the selffocussing path for the top (in time) light intensity, $I_{hm}^0 = I$ $(r = 0,\ \tau = \tau_h,\ z = 0)$. Since from the definition of the time compression function it follows that

$$\tau_p = \tau_p^0 \exp\left[-\int_0^z T_\Psi dz\right], \tag{44}$$

where $\tau_p^0 = \tau_p(z = 0)$, then by introducing to this formula the expression (43) we obtain the dependence of the pulse duration on the beam axis upon the path in the medium

$$\tau_p = \tau_p^0 \left|\frac{2\left[1-\left(\dfrac{z}{z_{fh}}\right)^2\right]}{2-\left(\dfrac{z}{z_{fh}}\right)^2}\right|^\delta. \tag{45}$$

It may be seen that with the increase of z the pulse duration on the beam axis decreases monotonically for $z < z_{fh}$, and for $z \to z_{fh}$, $\tau_p \to 0$*. The result obtained is due to "nonlinear aberration" of the radiation, i.e. due to the focussing of the radiation corresponding to different moments τ at different points on the z-axis. It is worth noting that for $r > 0$ the dependence $\tau_p(z)$ will be, in general, other than that on the beam axis. By taking advantage of the complete function T_Ψ, defined by formula (40), it may be shown that time compression of the field distribution will occur for these points of the transversal distribution, for which $r < \frac{1}{\sqrt{2}} r_p(z)$, where $2r_p(z)$ is the beam aperture at the point z**. For $r > \frac{1}{\sqrt{2}} r_p(z)$ the time distribution of the field will be broadened.

From the formula (45) it follows that for quasi-stationary nonlinear refraction the time distribution of the field is subject to essential changes for the values of z close to z_{fh}. For example, if $z = \frac{1}{2} z_{fh}$, then $\tau_p = \left(\frac{6}{7}\right)^\delta \tau_p^0$, and since for typical time distributions of the field (like Gaussian, Lorentzian, exponential, and so on)

* The singularity at the point $z = z_{fh}$ results from the assumption that $\delta n' \gg \frac{\Delta_\perp A}{k^2 A}$, i.e. that the diffraction is neglected.

** The dependence $r_p(z)$ will be defined in the next section.

$\delta \sim 1$, then $\tau_p \sim \frac{6}{7}\tau_p^0$. For $z < \frac{1}{2} z_{fh}$ the changes in the time distribution of the field due to quasi-stationary nonlinear refraction are usually low when compared with the changes caused by resonant interaction.

Change i n the spatial distribution of the field in the medium. Space compression function

The general relations describing the changes in spatial (transversal) field distribution in the medium may be found in a way analogical to that for the case of time compression. In the axially-symmetrical systems or systems symmetrical with respect to the plane passing through the z-axis the effective width of the space distribution r_p may be determined by the relation

$$I[\tau, z, r_p(\tau, z)] = \frac{1}{b} I_m(\tau, z), \tag{46}$$

where $I_m(\tau, z) = I(\tau, z, r = 0)$ — light intensity at the maximum of the spatial distribution, $b > 1$. Let us introduce the function

$$S \equiv -\frac{1}{r_p} \frac{dr_p}{dz}, \tag{47}$$

and assume

$$\left.\frac{\partial I(\tau, z, r)}{\partial r}\right|_{r_p} = -\frac{1}{\gamma(\tau, z)} \frac{I_m(\tau, z)}{r_p(\tau, z)}, \tag{48}$$

where $\gamma > 0$ — slope of the distribution. By manipulations analogical to those used in the previous chapter we obtain the expression

$$S = \frac{\gamma}{b}\left[F(I_m, r = 0, \tau, z) - F\left(\frac{1}{b} I_m, r_p, \tau, z\right)\right]. \tag{49}$$

The function S will be called the space compression function. It defines the relative rate of changes in spatial distribution width during the radiation propagation through the medium. In accordance with (47) for $S > 0$ the distribution compression occurs, while for $S < 0$ we observe some broadening (recompression) of the distribution.

From the expression (49) it follows that, in general, the change in the spatial distribution is different at different moments. In the sequel we shall omit the dependence upon τ, z, keeping simultaneously in mind the above conclusion.

By substituting (24) into (49) we obtain

$$S = \frac{\gamma}{b}\left[K(I_m, r = 0) - K\left(\frac{1}{b} I_m, r_p\right)\right] + \frac{\gamma}{b}\left(\Delta_\perp \Psi\Big|_{r_p} - \Delta_\perp \Psi|_{r=0}\right) - \frac{1}{r_p}\left.\frac{\partial \Psi}{\partial r}\right|_{r_p}. \tag{50}$$

The function

$$S_K = \frac{\gamma}{b}\left[K(I_m, 0) - K\left(\frac{1}{b} I_m, r_p\right)\right] \tag{51}$$

describes the changes in the transversal field distribution due to resonant interaction with the matter. These changes, as it may be seen, result from the nonlinearity of the interaction and nonuniformity of the active centre distribution in the plane perpendicular to the z-axis. The function

$$S_{\Psi} = \frac{\gamma}{b}(\Delta_{\perp}\Psi|_{r_p} - \Delta_{\perp}\Psi|_{r=0}) - \frac{1}{r_p}\frac{\partial \Psi}{\partial r}\Big|_{r_p} \tag{52}$$

describes the changes in the transversal field distribution caused by nonlinear refraction and the radiation diffraction.

Now, we shall discuss some conclusions following from the functions S_K and S_{Ψ}. Similarly as it was in the case in the previous section we will assume that $b = 2$.

Change in the spatial distribution of the field as a result of resonant interaction

Let us assume that the medium is uniform and the function does not depend explicitly upon the time. From (51) we obtain the expression

$$S_K = \frac{\gamma}{2}\left[K(I_m) - K\left(\frac{1}{2}I_m\right)\right] \tag{53}$$

analogical to the expression (32) for the function of time compression T_K. Hence, it follows that, in the case considered, the changes in spatial field distribution occur analogically to those for time distribution. All the conclusions and dependences obtained in the previous section for the case of quasi-stationary interaction and low signals may be used to describe the spatial changes in distribution provided that I_h, τ_p, T_K and $\delta_1+\delta_2$ in these relations will be changed to I_m, r_p, S_K and γ, respectively. In particular, the following general formulae are valid

$$r_p = r_p^0 \exp\left[-\int_{I_m^0}^{I_m} \frac{S_K(I_m)}{I_m K(I_m)}\, dI_m\right], \tag{54}$$

$$r_p = r_p^0 \exp\left[-\frac{\bar{\gamma}}{2}\int_{I_m^0}^{I_m} \frac{K(I_m) - K\left(\frac{1}{2}I_m\right)}{I_m K(I_m)}\, dI_m\right], \tag{55}$$

where $r_p^0 = r_p(z = z_0)$, $\bar{\gamma}$ — average value of $\gamma(I_m)$. Such a symmetry of time and spatial distributions does not appear in the case of nonstationary interaction or of a nonuniform medium.

The general expression for the function S_K in the multicomponent medium has the form

$$S_K = \frac{\gamma}{2}\sum_{n=1}^{M} I_m^{n-1}\sum_{l=1}^{L_m}\left[\beta_{ln}(I_m, r=0) - 2^{1-n}\beta_{ln}\left(\frac{1}{2}I_m, r_p\right)\right]. \tag{56}$$

In the quasi-stationary case of n-photon interaction, for the single-component medium with nonuniform distribution of active centres we obtain

$$S_K = \frac{\gamma}{2}P_m^{n-1}\left[\frac{\beta_{sn}(r=0)}{1+P_m^n} - \frac{2\beta_{sn}(r_p)}{2^n+P_m^n}\right], \tag{57}$$

where: $P_m = \frac{I_m}{I^s_{1n}}$, $\beta_{sn}(r) = n\sigma_{1n}N^e_{1n}(r)(I^s_{1n})^{n-1}$, $\quad I^s_{1n} = (s_{1n}\sigma_{1n}T_{1n})^{-1/n}$.

For the low signal we have, in turn,

$$S_K = \frac{\gamma}{2}\,[\beta^e_{1n}(r=0) - 2^{1-n}\beta^e_{1n}(r_p)]\,I^{n-1}_m. \tag{58}$$

From (57) and (58) it follows that the nonuniform distribution of the active centres with the maximum on the z-axis (i.e. if $|N^e_{1n}(r=0)| > |N^e_{1n}(r_p)|$) is the agent leading to field distribution compression in the case of population inversion. In contrast to this the same nonuniform distribution for normal population acts as a factor causing the broadening of the field distribution. Based on the formula (58) the detailed discussion of the changes in spatial field distribution in a two-photon medium with parabolic distribution of the active centres has been given in [3].

In the limiting case $\tau_p \ll T_{lm}$ the function S_k in a single-component uniform medium takes the form

$$S_K = \frac{\gamma}{2} f(\tau, r=0)\beta^e_{1n}I^{n-1}_{hm}\{\exp[-\chi_n(\tau, r=0)I^n_{hm}] - \\ -2^{1-n}\exp[-2^{-n}\chi_n(\tau, r=r_p)I_{hm}]\}, \tag{59}$$

where I_{hm} is a light intensity at time-spaced of the field and the functions f and χ_n define the relations

$$f(r, z, \tau) = \frac{I(r, z, \tau)}{I_h(r, z)},$$

$$\chi_n(r, z, \tau) = s_{1n}\sigma_{1n}\int\limits_{-\infty}^{\tau} f^n(r, z, \tau')\,d\tau'.$$

From (59) it follows that the direction of changes in r_p at a given moment τ depends upon the sing of β^e_{1n}, the value of I_{hm}, and the time shape of the pulse, being independent of the shape of the spatial distribution. The latter effects only the rate of changes r_p (via γ). The formula (59) shows the coupling between the time and space field distributions under nonstationary interaction conditions. This allows to obtain in concrete cases some important qualitative information concerning the process of spatial distribution changes in the medium.

Change of the spatial distribution of the field due to diffraction and nonlinear refraction

We shall analyse the changes in spatial distribution of the field in the medium, for the case when particular solutions for the eikonal may be written in the form

$$\Psi = \frac{r^2}{2u(z, \tau)} + \varphi(z, \tau). \tag{60}$$

This case includes, in particular, the propagation of both the Gaussian beam in a linear uniform medium [15] and the spherical wave in a nonlinear medium with $\delta n' = n_2 I$ [14] which will be discussed below.

By substituting (60) into (52) we obtain

$$S_\psi = -\frac{1}{u(z,\tau)}. \tag{61}$$

The function S_ψ determines the curvature of the wavefront at the point z and the moment τ. Taking account of the fact that from the definition of the function S

$$r_p = r_p^0 \exp\left[-\int_{z_0}^{z} S_\psi(z')dz'\right] \tag{62}$$

we obtain

$$r_p = r_p^0 \exp\left[\int_{z_0}^{z} \frac{dz'}{u(z',\tau)}\right]. \tag{63}$$

The formula (63) determines the relation between the width of spatial distribution and the wavefront curvature $1/u$ for the radiation.

For the case of Gaussian beam in vacuum or in a linear uniform medium, at $z_0 = 0$, we have [15]

$$u(z) = z\left[1+\left(\frac{ka_0^2}{2z}\right)^2\right],$$

where $2a_0$ — aperture diameter (at the height I_m/e^2) at the point $z = 0$. By the substitution of this expression to (63) we obtain the known formula describing the change in the transversal distribution width of the Gaussian beam during propagation

$$r_p = r_p^0 \sqrt{1+\left(\frac{2z}{ka_0^2}\right)^2}.$$

In a nonlinear medium, with $\delta n' = n_2 I$ for a beam of radiation fulfilling the boundary conditions

$$u(z=0) = \infty,\ \varphi(z=0) = 0,\ I(r, z=0) = I_m^0\left(1-\frac{r^2}{r_0^2}\right),$$

we have the following form [14]

$$u(z) = -\frac{z_f^2}{z}\left[1-\left(\frac{z}{z_f}\right)^2\right],$$

where

$$z_f = \frac{r_0\sqrt{n_0}}{\sqrt{n_2 I_m^0}}.$$

The substitution of this expression to the formula (63) yields

$$r_p = r_p^0 \sqrt{1-\left(\frac{z}{z_f}\right)^2}. \tag{64}$$

Due to the fact that, in general, I_m^0 (and by the same means z_f) is a function of τ, different points τ of the time distribution of radiation are focussed at different points on the z-axis. From (64) it follows also that for values of z several times less than z_f the changes in spatial field distribution due to nonlinear refraction are very small; they become significant first in the close vicinity of z_f. By using the functions S_K and S_Ψ it may be also pointed out that the changes of r_p caused by diffraction in the region of strong nonlinearity are usually small in comparison with the changes connected with resonant interaction for the case of quasi-Gaussian beam of divergence $\theta_p \lesssim 10^{-3}$ rad and of the $\beta_{lm}^e I^{m-1}$ values interesting for praxis. Under such conditions the type of changes in spatial distribution is determined by the dependence $K(I, r)$, the right choice of which creates the possibility of suitable shaping of this distribution.

The space compression function, introduced above, describes the changes in spatial field distribution at the definite point τ on the axis of local time. The changes integrated over time, i.e. the changes of energy distribution in the radiation beam are of interest in a number of cases. In these cases it is usefull to introduce the "energetic" function of space compression:

$$S_\varepsilon = -\frac{1}{r_\varepsilon}\frac{dr_\varepsilon}{dz}, \tag{65}$$

where r_ε is an effective width of the space distribution of energy defined by the relation

$$\mathscr{E}[z, r_\varepsilon(z)] = \frac{1}{b}\mathscr{E}_m(z), \tag{66}$$

in which $\mathscr{E}_m(z) = \mathscr{E}(z, r=0)$ — energy density at the maximum distribution.

If the change in radiation energy density in the medium is described by the equations (17) and $K_\varepsilon = K_\varepsilon(\mathscr{E}, r)$, then the function S_ε is given by the expression, which may be obtained from (50) by changing K, I_m, r_p, γ, to K_ε, $\mathscr{E}_m$, r_ε, γ_ε, respectively, where

$$\gamma_\varepsilon = -\frac{\mathscr{E}_m(z)}{r_\varepsilon(z)}\left[\frac{\partial\mathscr{E}(z,t)}{\partial r}\bigg|_{r_\varepsilon}\right]^{-1}.$$

In the multicomponent medium with the energetic amplification function given by (20), at $\Psi = \text{const.}$ we obtain

$$S_\varepsilon = -\frac{\gamma_\varepsilon}{2}\sum_{l=1}^{L_1} a_1 X_l^{-1}\left[1+\exp(-X_l)-2\exp\left(-\frac{1}{2}X_l\right)\right], \tag{67}$$

where $X_l = \dfrac{\mathscr{E}_m}{\mathscr{E}_l^s}$. From the expression (67), it may be seen that in the amplifying medium ($\alpha_1 > 0$) the single-photon nonstationary saturation leads to a broadening of energy distribution, while in the absorbing medium it causes some compression of this distribution. There exists an optimal value of energy density at the distribution maximum $\mathscr{E}_m \approx 2.5\ \mathscr{E}_l^s$, at which the change in distribution occurs at the highest rate. In the limiting case of infinitely great and infinitesimally small values the radiation propagates without any change of energy distribution.

Concluding remarks

In the present paper the time and space compression functions have been generalized to include the case when the light propagation in a medium with nonlinear complex refractive index is described by the wave equation with paraxial approximation. The general relations formulated in the paper allow to perform the analysis of time and space distributions of the field due to resonant single- and multiphoton interaction and to nonlinear radiation refraction as well. By taking advantage of the obtained relations, the influence of nonlinear interaction upon the change in the field distribution has been analysed for some practically interesting cases.

The relations given in this paper allow to obtain the fundamental qualitative information about the process of field changes, as well as to get a number of important quantitative relations from simple algebraic formulae without necessity of solving the respective differential equations. Thus, by simple means, they enable to obtain the picture of the phenomenon and to carry out the proper programming of both the experiments and numerical calculations. It seems that the compression functions introduced in this paper can be used to describe also other (nonelectromagnetic) kind of fields, provided that their changes are describable by equations analogical to (23).

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Функции временного и пространственного уплотнения в двумерном описании распространения импульса лазера в нелинейной среде

Исходя из нелинейного волнового уравнения в параксьяльной аппроксимации, были обобещены функции временного и пространственного уплотнения в случае среды с нелинейным комплексным коэффициентом преломления. Сформулированы общие зависимости, дающие возможность анализа изменений временного и пространственного распределения поля в результате резонансного, одно- и многофотонного взаимодействия, а также нелинейной рефракции излучения. Пользуясь полученными зависимостями проанализировано влияние нелинейного воздействия на изменение распределения поля в некоторых практически интересных случаях.