Determination of the optimal reference sphere in the case of coherent illumination

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In an earlier paper, the author proposed a new image quality measure and a new definition of the centre of the point spread. Determination of the value of this measure is equivalent to simultaneous optimization of the reference sphere. This measure resolves itself into a mean square of the wave aberration in the case of small aberration. In the present paper, the parameters of the optimal reference sphere are calculated for the circular, annular and rectangular exit pupils.

1. Introduction

As it was shown in [1], a mean square of the wave aberration may be treated as a coherent measure of the image quality in the case of small aberration. Therefore, two local image quality measures were introduced:

$$Q_A(x_0, y_0) = \min_{\Delta x, \Delta y, z} I(\Delta x, \Delta y, z), \tag{1}$$

$$Q_B(x_0, y_0) = \min_{\Delta x, \Delta y, z, M} I(\Delta x, \Delta y, z, M) \tag{2}$$

where

$$I(\Delta x, \Delta y, z, M) = \frac{k^2}{S} \iint W_1^2[x_0, y_0, \xi', \eta'] d\xi' d\eta' \tag{3}$$

and $S = \iint_\sigma d\xi' d\eta'$ is the pupil area, $k = 2\pi/\lambda$, λ is the light wavelength.

Aberration $W_1(x_0, y_0, \xi', \eta')$ can be expressed as

$$W_1(x_0, y_0, \xi', \eta') = W(x_0, y_0, \xi', \eta') + H(\xi'^2 + \eta'^2) + K\xi' + L\eta' + M \tag{4}$$

while the following notation was used:

$$H = -\frac{a^2 z}{2R^2}, \quad K = -\frac{\Delta x a}{R}, \quad L = -\frac{\Delta y a}{R}.$$

Here $W(x_0, y_0, \xi', \eta')$ is calculated in relation to the Gaussian reference sphere, ξ and η are the Cartesian coordinates in the pupil plane (Fig. 1), Δx, Δy denote the transverse aberration on the image plane, z denotes the longitudinal aberration
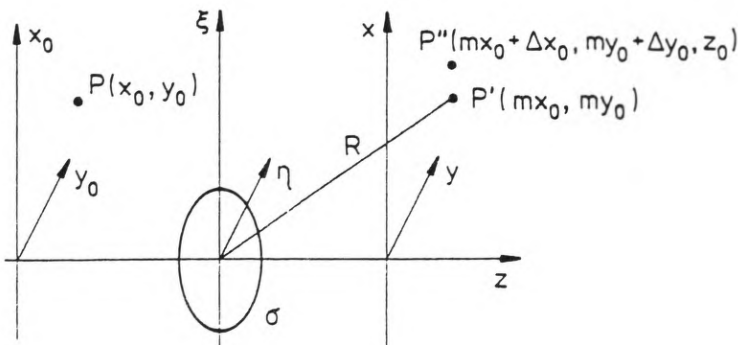
Fig. 1. Schematic diagram of the optical system. The illuminating point $P(x_0, y_0)$ in the object plane is imaged into $P'(mx_0, my_0)$ in the Gaussian image plane. $P''(mx_0 + \Delta x_0, my_0 + \Delta y_0, z_0)$ — is the point of the best image, R — reference sphere radius, σ — exit pupil

on the image plane, R denotes the reference radius, a is the characteristic dimension of pupil (*e.g.*, the radius of circular pupil).

2. Determination of the optimal reference sphere

An analytical form of the parameters H_{opt}, K_{opt}, L_{opt}, M_{opt}, for the optimal reference sphere cannot be determined from the exact formula [1]. Therefore, some calculations were carried out using the approximate formula (3). Thus, it is necessary to solve the following set of equations:

i) Optimization with respect to the measure Q_A

$$M = -\left(\frac{R}{a}\right)^2 H,$$

$$\frac{\delta\langle W_1^2\rangle}{\delta} = \frac{\delta\langle W_1^2\rangle}{\delta^2 K} = \frac{\delta\langle W_1^2\rangle}{\delta L} = 0. \tag{5a}$$

ii) Optimization with respect to the measure Q_B (M does not depend on H)

$$\frac{\delta\langle W_1^2\rangle}{\delta H} = \frac{\delta\langle W_1^2\rangle}{\delta K} = \frac{\delta\langle W_1^2\rangle}{\delta L} = \frac{\delta\langle W_1^2\rangle}{\delta M} = 0 \tag{5b}$$

where $\langle\ldots\rangle$ denotes a mean value with respect to the pupil area divided by S. Further detailed calculations of $H - M$ parameters have been performed for a circular pupil with a radius a. In such a case, it is more convenient to introduce polar coordinates ρ and Θ, related to the Cartesian

$$\zeta' = \rho\sin\Theta, \quad 0 \leqslant \rho \leqslant 1,$$
$$\eta' = \eta\sin\Theta, \quad 0 \leqslant \Theta \leqslant 2\pi.$$

One then obtains:
— For the case i)

$$H_{\text{opt}} = C\langle W\rho^2\rangle - 2C\left(\frac{R}{a}\right)^2\langle W\rangle,$$

$$K_{\text{opt}} = -4\langle W\rho\sin\Theta\rangle,$$

$$L_{\text{opt}} = -4\langle W\rho\cos\Theta\rangle,$$

$$M_{\text{opt}} = -2\left(\frac{R}{a}\right)^2 H_{\text{opt}}, \tag{6a}$$

where:

$$C = \left[4\left(\frac{R}{a}\right)^4 - 2\left(\frac{R}{a}\right)^2 + \frac{1}{3}\right]^{-1}.$$

— For the case ii)

$$H_{\text{opt}} = -12\langle W\rho^2\rangle + 6\langle W\rangle,$$

$$K_{\text{opt}} = -4\langle W\rho\sin\Theta\rangle,$$

$$L_{\text{opt}} = -4\langle W\rho\cos\Theta\rangle,$$

$$M_{\text{opt}} = 6\langle W\rho^2\rangle - 4\langle W\rangle. \tag{6b}$$

Calculation of all the second derivatives of a mean square $\langle W_1^2\rangle$ reveals that a square form of the second derivatives is positive at the point $(H_{\text{opt}}, K_{\text{opt}}, L_{\text{opt}})$ for the case i), and at the point $(H_{\text{opt}}, K_{\text{opt}}, L_{\text{opt}}, M_{\text{opt}})$ for the case ii). Thus, it has been proved that the parameters (6a) and (6b) denote a minimum of the function $I(\quad)$, Eq. (3). It appears [2] that the optimal parameters (6b) simultaneously optimize the variance of wave aberration which is equivalent to the Strehl number maximization within the range of small aberrations. It indirectly proves the hypothesis that this measure can be applied simultaneously both to coherent and non-coherent systems.

3. Conclusions

1. The coordinates of P'', which is the "best image" of $P(x_0, y_0)$, are as follows (Fig. 1):

$$mx_0 + \Delta x_0 = mx_0 - \frac{R}{a}K_{\text{opt}},$$

$$mx_0 + \Delta y_0 = my_0 - \frac{R}{a}L_{\text{opt}}, \tag{7}$$

$$z_0 = -2\left(\frac{R}{a}\right)^2 H_{\text{opt}}.$$

The point P'' is the centre of the optimal reference sphere whose radius is of the value [3]

$$R' = R + z_0 + M_{\text{opt}}. \tag{8}$$

2. The quantities Δx_0, Δy_0 denote the transverse aberrations on the image plane, and z_0 denotes the longitudinal aberrations.

3. In the case of an isoplanatic system, *i.e.*, when $W(x_0, y_0, \xi, \eta)$ does not depend on x_0 and y_0, the best image plane can be determined and it will be shifted with respect to the Gaussian plane by a vector $(\Delta x_0, \Delta y_0, z_0)$. Then, the optimal transfer function can be determined

$$\mathcal{H}_{\text{opt}}(f_x, f_y) = \text{rect}\left[\frac{\lambda R}{2a}(f_x + f_y)^{1/2}\right]\exp[ik\Phi(f_x, f_x)]$$

where

$$\Phi(f_x, f_y) = W(\lambda R f_x, \lambda R f_y) + \left(\frac{\lambda R}{a}\right)^2 H_{\text{opt}}(f_x^2 + f_y^2) + \frac{\lambda R}{a}(K_{\text{opt}} f_x + L_{\text{opt}} f_y) + M_{\text{opt}}, \qquad (9)$$

which is a transfer function of the system imaging the object plane into the best image plane. The autocorrelation function $\mathcal{H}_{\text{opt}}(f_x, f_y)$ may be an optimal transfer function of the incoherent system under the condition that optimal parameters $H_{\text{opt}} - M_{\text{opt}}$ of Eq. (9) are now determined according to Eqs. (6b). Therefore, the best image plane will differ from that previously found.

4. The position of the reference sphere depends on the shape of the pupil. For example, the parameters $H - M$ for annular pupil with an outher radius a and inner radius εa ($0 \leqslant \varepsilon \leqslant 1$), Fig. 2, have the forms:

— For the case i)

$$H_{\text{opt}} = \left(-\langle W\rho^2\rangle + 2\frac{R}{a}\langle W\rangle\right)\left\{(1 - \varepsilon^2)\left[4\left(\frac{R}{a}\right)^4(1 - \varepsilon^2)\right.\right.$$
$$\left.\left. - 2\left(\frac{R}{a}\right)(1 - \varepsilon^4) + \frac{1}{3}(1 - \varepsilon^6)\right]\right\},$$

$$K_{\text{opt}} = -\frac{4}{1 + \varepsilon^2}\langle W\rho\sin\Theta\rangle,$$

$$L_{\text{opt}} = -\frac{4}{1 + \varepsilon^2}\langle W\rho\cos\Theta\rangle,$$

$$M_{\text{opt}} = -2\left(\frac{R}{a}\right)H_{\text{opt}}. \qquad (10a)$$

— For the case ii)

$$H_{\text{opt}} = [-12(1 - \varepsilon^2)\langle W\rho^2\rangle + 6(1 - \varepsilon^4)\langle W\rangle]\{4(1 - \varepsilon^6) - 3(1 + \varepsilon^2)(1 - \varepsilon^4)\}^{-1},$$

$$K_{\text{opt}} = -\frac{4}{1 - \varepsilon^2}\langle W\rho\sin\Theta\rangle,$$

$$L_{\text{opt}} = -\frac{4}{1 + \varepsilon^2}\langle W\rho\cos\Theta\rangle,$$

$$M_{opt} = [6(1-\varepsilon^4)\langle W\rho^2\rangle - 4(1-\varepsilon^6)\langle W\rangle]\{4(1-\varepsilon^6)-3(1+\varepsilon^2)(1-\varepsilon^4)\}^{-1}. \tag{10b}$$

Inserting $\varepsilon = 0$ into (10a) and (10b), Eqs. (6a), (6b) are obtained again. The above expressions may be applied when analysing the apodizing systems with annular aperture. On the other hand, the expressions (6a) and (6b) take different forms for a rectangular pupil of dimensions $2a \times 2b$ (Fig. 2):
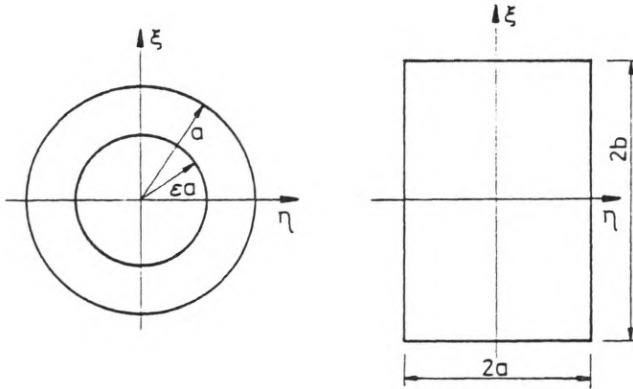

Fig. 2. Exit pupils considered in the paper: annular pupil with an outer radius a and inner radius εa $(0 \leqslant \varepsilon \leqslant 1)$, rectangular pupil of dimensions of $2a \times 2b$

— For the case i)

$$H_{opt} = -\langle W(\xi'^2 + \eta'^2)\rangle + 2\left(\frac{R}{a}\right)\langle W\rangle$$

$$\times \left[4\left(\frac{R}{a}\right)^4 - 4\left(\frac{R}{a}\right)^2\left(\frac{1}{3}+\frac{1}{3}\gamma^2\right)+\frac{1}{5}\gamma^4+\frac{2}{9}\gamma^2+\frac{1}{5}\right]^{-1},$$

$$K_{opt} = -3\langle W\xi'\rangle,$$

$$L_{opt} = -\frac{3}{\gamma^2}\langle W\eta'\rangle,$$

$$M_{opt} = -2\left(\frac{R}{a}\right)^2 H_{opt}. \tag{11a}$$

— For the case ii)

$$H_{opt} = [-45\langle W(\xi'^2 + \eta'^2)\rangle + 15(1+\gamma^2)\langle W\rangle \times [4(1+\gamma^2)]^{-1},$$

$$K_{opt} = -3\langle W\xi'\rangle,$$

$$L_{opt} = -\frac{3}{\gamma^2}\langle W\eta'\rangle,$$

$$M_{opt} = [15(1+\gamma^2)\langle W(\xi'^2 + \eta'^2)\rangle - (9+10\gamma^2+9\gamma^4)\langle W\rangle \times [4(1+\gamma^4)]^{-1} \tag{11b}$$

where ζ' and η' — dimensionless rectangular coordinates $\zeta' = \zeta/a$, $\eta' = \eta/a$, while $-1 \leqslant \zeta' \leqslant 1$, $-\gamma \leqslant \eta' \leqslant \gamma$, $\gamma = b/a$.

5. All the optimal parameters $H - M$ are linear functionals of aberration. Consequently, if the aberration W is a sum of all the component aberrations, the reference sphere optimization may be done separately with respect to each aberration component and next all the obtained component parameters $H - M$ summed up. This feature can be helpful when designing an optical system.

References

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[3] BORN M., WOLF E., *Principles of Optics*, 6th edition, Pergamon Press, New York 1980.