Optica Applicata, Vol. XXV, No. 1, 1995

Application of the Fourier description of discrete phase change method to synthesis of the five-point algorithms

J. R. MŁOKOSIEWICZ, W. PARA, W. ŚMIETAŃSKI

Electromechanical Faculty, Military Academy of Technology, ul. Kaliskiego 2, 00–908 Warszawa, Poland.

In the paper, the methodology of synthesis of five-point algorithms, which are employed in the discrete phase change method, is presented. The algorithms are characterized by small phase measurement errors occurring due to phase shifter errors and nonlinearity of the detector. The analysis of the properties of the algorithms was based on the theoretical apparatus of the Fourier description of the discrete phase change method. The results of simulation of phase measurement method for three new algorithms obtained by using the proposed method are presented.

1. Introduction

The phase measurement of the fringe image by means of the discrete phase change method has been widely applied to precision optical measurements. Many algorithms for the discrete phase change method have been published so far being characterized by the diminishing sensitivity to the error sources occurring in the phase-measuring systems, such as errors introduced by the phase shifter and/or nonlinearity of the detector. Due to a steady demand for an increased accuracy of the optical measurements there exists a need for further development of the method of algorithm synthesis for the discrete phase change method which would assure a further decrease of their sensitivity to the systematic errors of the measuring setups.

One of the newest methods of synthesis of the discrete phase change algorithms has been proposed by LARKIN and OREB [1]. It exploits the Fourier description of the discrete phase change method introduced by FREISCHLAD and KOLIOPOULOS [2]. Larkin and Oreb have obtained a family of symmetric algorithms based on $N+1$ sampling points of the fringe image shifted in phase by $2\pi/N$. For the increasing number of sampling points in the image these algorithms exhibit a diminishing sensitivity to the errors of the phase shifter as well as to the presence of higher harmonics in the sampled image.

In this paper, a method of designing the five-point algorithm based on the Fourier description of the discrete phase change method has been proposed. This method enables us to obtain the algorithms exploiting the samples of an image of an arbitrary known phase shift. The suitable choice of both the weighting coefficients of the image samples and the plane shifts between them rendered it possible to obtain algorithms for which the phase measurement error in the presence of only a a linear error of the phase shifter was several times less than the error of the symmetric algorithms [1] for $N = 5$ and $N = 7$. Also, the algorithm exhibiting no sensitivity to the even harmonics of the image and assuring twice as low sensitivity to the linear errors of the phase shifter has been worked out; the above properties being similar to those of the symmetric algorithm for $N = 5$.

2. General form of the five-point algorithm for the discrete phase change method

The synthesis of the general form of a five-point algorithm will be performed while exploiting the Fourier description of the discrete phase change method proposed by FREISCHLAD and KOLIOPOULOS [2]. The principal results of the last work recalled below will constitute the basis of our analysis.

Let $s(t)$, the intensity distribution in the fringe image at a fixed point (x, y), be a function of the shift parameter t (representing either the space or time shift). Let us consider the correlations $k_1(t)$ and $k_2(t)$ of the signal $s(t)$ with two sampling functions $f_1(t)$ and $f_2(t)$:

$$k_1(t) = \int_{-\infty}^{\infty} s(\tau) f_1(\tau + t) d\tau, \tag{1}$$

$$k_2(t) = \int_{-\infty}^{\infty} s(\tau) f_2(\tau + t) d\tau, \tag{2}$$

It may be shown that

$$k_1 = k_1(t = 0) = 2\mathrm{Re}\left[\int_0^{\infty} S(\nu) F_1^*(\nu) d\nu\right], \tag{3}$$

$$k_2 = k_2(t = 0) = 2\mathrm{Re}\left[\int_0^{\infty} S(\nu) F_2^*(\nu) d\nu\right], \tag{4}$$

where: Re – real part of a complex number,
* – complex conjugate value,
$S(\nu)$, $F_1(\nu)$, $F_2(\nu)$ – Fourier transforms of the signal and the sampling functions, respectively.

If the image $s(t)$ is a periodic function of the parameter t, *i.e.*, $s(t + T_s) = s(t)$, where T_s – period of the image, $S(\nu)$ can be written as

$$S(\nu) = \sum_{n=0}^{\infty} s_n \delta(\nu - n\nu_s) \tag{5}$$

where: $s_n = |s_n| e^{j\varphi_n}$, and φ_n is the phase of the n-th harmonic of the fundamental frequency $\nu_s = 1/T_s$.

In order to determine the phase φ_n of the image $s(t)$ consider the ratio of the correlations (3) and (4)

$$r = k_1/k_2. \tag{6}$$

For a periodical image spectrum (5) we have

$$r = \frac{s_0 F_1(0) + 2\mathrm{Re}\left[\sum\limits_{n=1}^{\infty} s_n F_1^*(n\nu_s)\right]}{s_0 F_2(0) + 2\mathrm{Re}\left[\sum\limits_{n=1}^{\infty} s_n F_2^*(n\nu_s)\right]}. \tag{7}$$

If the spectra $F_1(\nu)$, $F_2(\nu)$ and $S(\nu)$ have only one common frequency component $m\nu_s$ (image $s(t)$ has only the component ν_s or the spectra of the sampling functions are non-zero only for component $m\nu_s$) and when additionally

$$F_1(m\nu_s) = jF_2(m\nu_s), \tag{8}$$

then the relation (7) simplifies to

$$r = \tan(\varphi_m). \tag{9}$$

Consider the general form of a five-point algorithm for the discrete phase change method described by the sampling functions

$$f_1(t) = C\left\{C_1\left[\delta\left(t+T_f\frac{2\alpha}{2\pi}\right)-\delta\left(t-T_f\frac{2\alpha}{2\pi}\right)\right]\right.$$
$$\left.+C_2\left[\delta\left(t+T_f\frac{\alpha}{2\pi}\right)-\delta\left(t-T_f\frac{\alpha}{2\pi}\right)\right]\right\}, \tag{10}$$

$$f_2(t) = C_0\delta(t)-C_3\left[\delta\left(t+T_f\frac{2\alpha}{2\pi}\right)+\delta\left(t-T_f\frac{2\alpha}{2\pi}\right)\right]$$
$$-C_4\left[\delta\left(t+T_f\frac{\alpha}{2\pi}\right)+\delta\left(t-T_f\frac{\alpha}{2\pi}\right)\right]\Bigg\} \tag{11}$$

(where: $T_f = 1/\nu_f$ – period of the sampling functions, α – phase shift between the consecutive samples in the image, C_0, C_1, C_2, C_3, C_4 – weighting coefficients of the discrete samples of the image $s(t)$, C – constant), the spectra of which are described by the relations:

$$F_1(\nu) = C\left[2jC_1\sin\left(2\alpha\frac{\nu}{\nu_f}\right)+2jC_2\sin\left(\alpha\frac{\nu}{\nu_f}\right)\right] = j|F_1(\nu)|, \tag{12}$$

$$F_1(\nu) = C_0-2C_3\cos\left(2\alpha\frac{\nu}{\nu_f}\right)-2C_4\cos\left(\alpha\frac{\nu}{\nu_f}\right) = |F_2(\nu)|. \tag{13}$$

Determination of the phase of fundamental frequency ($m = 1$) from relation (7) requires the fulfilment of the following conditions:

$$F_1(0) = 0, \tag{14a}$$
$$F_2(0) = 0, \tag{14b}$$
$$F_1(\nu_f) = jF_2(\nu_f). \tag{14c}$$

Additionally, it is required that

$$F_1(\nu_s) = jF_2(\nu_s), \tag{15}$$

which is obtained when the phase shifter is error-free ($\nu_f = \nu_s$).

The sampling functions (10) and (11) satisfy the conditions (14) for:

$$C_0 = 2(C_3+C_4),$$

$$C = \frac{C_3+C_4-C_3\cos 2\alpha - C_4\cos\alpha}{C_1\sin 2\alpha + C_2\sin\alpha}. \tag{16}$$

Denoting:

$$I_1 = s\left(-2\alpha\frac{T_f}{2\pi}\right),\quad I_2 = s\left(-\alpha\frac{T_f}{2\pi}\right),\quad I_3 = s(0),$$

$$I_4 = s\left(\alpha\frac{T_f}{2\pi}\right),\quad I_5 = s\left(2\alpha\frac{T_f}{2\pi}\right) \tag{17}$$

a five-point algorithm for the discrete phase change method is obtained which allows us to determine the phase φ for arbitrary phase shifts between the image samples and arbitrary weighting coefficients of the image samples from the relation

$$\varphi = \arctan\left[C\frac{C_1(I_1-I_5)+C_2(I_2-I_4)}{2(C_3+C_4)I_3-C_3(I_1+I_5)-C_4(I_2+I_4)}\right]. \tag{18}$$

The accuracy of the phase measurement in the presence of linear errors of the phase shifter ($\nu_s \neq \nu_f$) and higher harmonics of the image $s(t)$ is defined by the choice of both the phase shift and the weighting coefficients of the image samples.

3. Algorithms of low sensitivity to the linear errors of the phase shifter

In the real measuring systems, condition (15) is frequently not fulfilled which is most often caused by the existence of the linear error of the phase shifter ($\nu_f \neq \nu_s$). By exploiting the analysis presented in [1] for the case of cosine distribution of intensity $s(t)$ (lack of higher harmonics), we can write

$$\tan(\varphi') = \frac{|F_1(\nu_s)|}{|F_2(\nu_s)|}\tan(\varphi) \tag{19}$$

where: $\varphi' = \varphi - \Delta\varphi$ – measured phase, $\Delta\varphi$ – phase error.

The minimization of the phase errors following from the shifter errors requires that the changes of $|F_1(\nu)|$ resulting from the change of ν_f would correspond to the changes $|F_2(\nu)|$. Since $F_1(\nu_s)$ and $F_2(\nu_s)$ may be approximated by the Taylor series around the point ν_f

$$F_1(\nu_s) \simeq F_1(\nu_f)+\frac{dF_1(\nu_f)}{d\nu}(\Delta\nu)+\frac{1}{2}\frac{d^2F_1(\nu_f)}{d\nu^2}(\Delta\nu)^2, \tag{20}$$

$$F_2(\nu_s) \simeq F_2(\nu_f) + \frac{dF_2(\nu_f)}{d\nu}(\Delta\nu) + \frac{1}{2}\frac{d^2F_2(\nu_f)}{d\nu^2}(\Delta\nu)^2, \tag{21}$$

(where: $\Delta\nu = \nu_s - \nu_f$), the changes of $|F_1(\nu)|$ correspond to the changes of $|F_2(\nu)|$ when:

$$\frac{dF_1(\nu_f)}{d\nu} = j\frac{dF_2(\nu_f)}{d\nu}, \tag{22}$$

$$\frac{d^2F_1(\nu_f)}{d\nu^2} = j\frac{d^2F_2(\nu_f)}{d\nu^2}. \tag{23}$$

The choice of the suitable phase shift and the weighting coefficients of the image samples of the algorithm (18) makes it possible to satisfy the conditions (22) and (23) to obtain small phase measurement errors in the presence of the linear errors of the phase shifter.

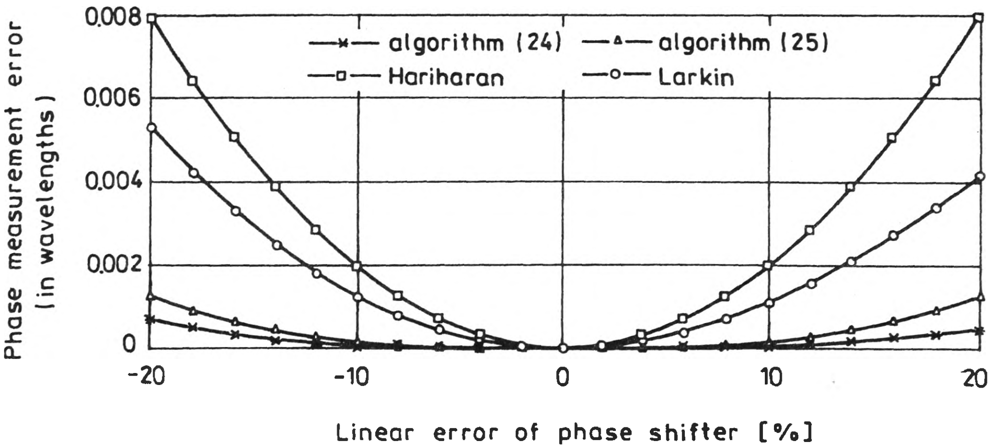

Fig. 1. Dependence of the phase measurement error on the linear phase shifter error

Examples of algorithms (18) satisfying the conditions (22) and (23) are:
– for $\alpha = 60°$

$$\varphi = \arctan\left[\frac{\sqrt{3}}{3}\frac{2(I_1-I_5)-5(I_2-I_4)}{-6I_3+3(I_2+I_4)}\right], \tag{24}$$

– for $\alpha = 90°$

$$\varphi = \arctan\left[\frac{(I_5-I_1)+4(I_2-I_4)}{6I_3-(I_1+I_5)-2(I_2+I_4)}\right]. \tag{25}$$

When applying the model of linear error of the phase shifter

$$\alpha' = \alpha(1+\varepsilon), \tag{26}$$

(in which α' — real phase shift, ε — normalized error of the phase shifter), the dependence of the measurement error on the linear error of the phase shifter for algorithms (24) and (25) as well as for Hariharan algorithm [3] and the seven-point algorithm proposed by LARKIN and OREB [1] has been presented in Fig. 1.

4. Algorithms insensitive to the second harmonic of the fringe image

The dependence (7) simplifies to the form (9) if the condition (8) is satisfied and when the spectra $F_1(\nu)$, $F_2(\nu)$ and $S(\nu)$ have only one common frequency component. This happens if the signal $s(t)$ is of cosine character (lack of higher harmonics) or when:

$$\begin{aligned} &F_1(m\nu_f) = 0 \quad \text{for } m \geqslant 2, \\ &F_2(m\nu_f) = 0 \quad \text{for } m \geqslant 2, \\ &\nu_f = \nu_s. \end{aligned} \tag{27}$$

If the components $m\nu_s$ $(m = 2, 3, \ldots)$ appear in the signal $s(t)$ due to nonlinearity of the measuring system (nonlinearity of detector, for instance), the phase measurement is charged with an error. A reduction of the phase measurement error caused by the nonlinearity of the measuring system is possible by applying the algorithm of the discrete phase change method, while the sampling functions have non-zero spectra only for frequency ν_s.

The lack of resistance to the phase measurement errors caused by appearance of the second harmonic of the signal $s(t)$ is a shortcoming of the algorithms (24) and (25). This follows from the fact that the spectra of the sampling functions of these algorithms are non-zero for the frequency $2\nu_s$, which is in contrast to the Hariharan algorithm. This means that when applying, for instance, a detector exhibiting the nonlinearity of the second order, the phase measurement errors of these algorithms will be greater than those for the Hariharan algorithm.

Let us consider the possibility of synthesis of an algorithm fulfilling, similarly to that of Hariharan, the conditions:

$$F_1(2\nu_s) = 0, \tag{28a}$$

$$F_2(2\nu_s) = 0, \tag{28b}$$

$$\frac{dF_1(\nu_s)}{d\nu} = j\frac{dF_2(\nu_s)}{d\nu} \tag{28c}$$

on the basis of the general form of the five-point algorithm of the discrete phase change method, *i.e.*, an algorithm of similar resistance to the linear phase shifter errors and the measuring system nonlinearity. The analysis of the spectra of the sampling functions (12) and (13) allows us to show that the conditions (28) are fulfilled by the five-point algorithm for which $\alpha = 90°$, $C_1 = 0$ and $C_4 = 0$, *i.e.*, by the Hariharan algorithm.

We synthesize the algorithms fulfilling the conditions (28a) and (28b), *i.e.*, the algorithm showing lack of sensitivity to the presence of the second harmonic of the signal $s(t)$, but sensitive to the linear errors of the phase shifter due to failure of satisfying the condition (28c).

For the sampling functions (10) and (11), we have the spectra:

$$F_1(2\nu_s) = 2jC[C_1\sin(4\alpha)+C_2\sin(2\alpha)], \tag{29}$$
$$F_2(2\nu_s) = 2\{C_3[1-\cos(4\alpha)]+C_4[1-\cos(2\alpha)]\} \tag{30}$$

and assuming $C \neq 0$, we can write the condition (28a) as follows:

$$[2C_1\cos(2\alpha)+C_2]\sin(2\alpha) = 0. \tag{31}$$

The above condition is fulfilled for arbitrary C_1 and C_2 different from zero and $\alpha = 90°$. For such nominal phase shift we have

$$F_2(2\nu_s) = 4C_4, \tag{32}$$

i.e., the condition (28b) is fulfilled for arbitrary $C_3 \neq 0$ and $C_4 = 0$.

Let us introduce two five-point algorithms fulfilling the conditions (28a) and (28b), (in general form it suffices that $\alpha = 90°$ and $C_4 = 0$). Denoting the weights of the image samples by C_i and D_i ($i = 1 \ldots 5$), respectively, we have:

$$\varphi = \arctan C\frac{C_1(I_1-I_5)+C_2(I_2-I_4)}{2C_3I_3-C_3(I_1+I_5)}, \tag{33}$$

$$\varphi = \arctan D\frac{D_1(I_1-I_5)+D_2(I_2-I_4)}{2D_3I_3-D_3(I_1+I_5)}. \tag{34}$$

Algorithms (33) and (34) assure a zero order of the phase measurement in the presence of the second harmonic of the image $s(t)$. Their shortcomings are relatively great measurement errors caused by the linear errors of the phase shifter. A reduced sensitivity to the linear errors of the phase shifter may be obtained if the phase measurement errors for algorithms (33) and (34) are in anti-phase and the sought phase is determined from the dependence

$$\varphi = \frac{1}{2}\arctan C\frac{C_1(I_1-I_5)+C_2(I_2-I_4)}{2C_3I_3-C_3(I_1+I_5)}+\frac{1}{2}\arctan D\frac{D_1(I_1-I_5)+D_2(I_2-I_4)}{2D_3I_3-D_3(I_1+I_5)}. \tag{35}$$

Denoting by P_1, P_2, Q_1 and Q_2 the sampling functions for the algorithms (33) and (34), respectively, and taking advantage of (19), the phase measurement error from the dependence (35) may be written as

$$\Delta\varphi = \arctan(\tan\varphi)-\frac{1}{2}\left\{\arctan\left[\frac{|P_1(\nu_s)|}{|P_2(\nu_s)|}\tan\varphi\right]+\arctan\left[\frac{|Q_1(\nu_s)|}{|Q_2(\nu_s)|}\tan\varphi\right]\right\}, \tag{36}$$

hence after some transformations we obtain

$$\Delta\varphi = \frac{1}{2}\arctan\frac{\left[2-\frac{|P_1(\nu_s)|}{|P_2(\nu_s)|}-\frac{|Q_1(\nu_s)|}{|Q_2(\nu_s)|}\right]\tan\varphi+\left[\frac{|P_1(\nu_s)|}{|P_2(\nu_s)|}+\frac{|Q_1(\nu_s)|}{|Q_2(\nu_s)|}-2\frac{|P_1(\nu_s)||Q_1(\nu_s)|}{|P_2(\nu_s)||Q_2(\nu_s)|}\right]\tan^3\varphi}{1+\left[2\frac{|P_1(\nu_s)|}{|P_2(\nu_s)|}+2\frac{|Q_1(\nu_s)|}{|Q_2(\nu_s)|}-\frac{|P_1(\nu_s)||Q_1(\nu_s)|}{|P_2(\nu_s)||Q_2(\nu_s)|}-1\right]\tan^2\varphi+\frac{|P_1(\nu_s)||Q_1(\nu_s)|}{|P_2(\nu_s)||Q_2(\nu_s)|}\tan^4\varphi}. \tag{37}$$

It may be shown that the error amplitude (37) will be small

$$\frac{|P_1(\nu_s)|}{|P_2(\nu_s)|} = \frac{|Q_2(\nu_s)|}{|Q_1(\nu_s)|}. \tag{38}$$

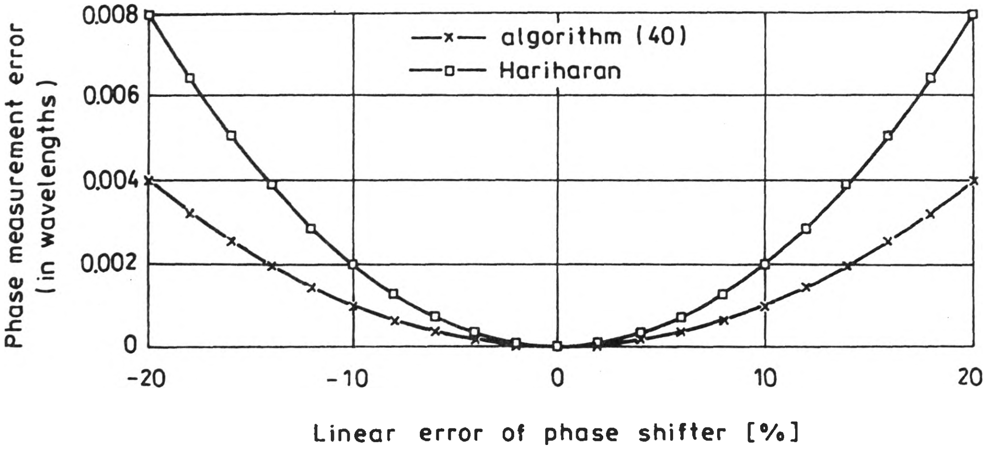

Fig. 2. Dependence of the phase measurement error on the linear phase shifter error

The sampling algorithms (33) and (34) satisfy the condition (38) for:

$$\frac{C_1}{C_2} = \frac{1}{2}, \quad \frac{D_1}{D_2} = -\frac{1}{2},$$
$$C_3 \neq 0, \quad D_3 \neq 0. \tag{39}$$

Taking account of the conditions (28a), (28b) and (39), the five-point algorithm for the discrete phase change method can be obtained in the form:

$$\varphi = \frac{1}{2}\left[\arctan\frac{I_1 - I_5 + 2I_2 - 2I_4}{2I_3 - I_1 - I_5} + \arctan\frac{I_5 - I_1 + 2I_2 - 2I_4}{2I_3 - I_1 - I_5}\right] \tag{40}$$

assuring small errors of the phase measurement in the measuring systems of second order nonlinearity and linear errors of the phase shifter.

The dependence of the phase measurement error on the linear errors of the phase shifter for algorithm (40) and Hariharan algorithm, respectively, is shown in Fig. 2. Since both the algorithm (40) and the Hariharan algorithm fulfil the conditions (28a) and (28b), the phase measurement error evoked by existence of the second harmonic of the $s(t)$ image will equal zero for these algorithms in the absence of linear errors of the phase shifter. For non-zero errors of the phase shifter, the measurement error increases with nonlinearity of the measuring system.

Applying the model of detector nonlinearity of the form

$$I' = I + \zeta I^2 \tag{41}$$

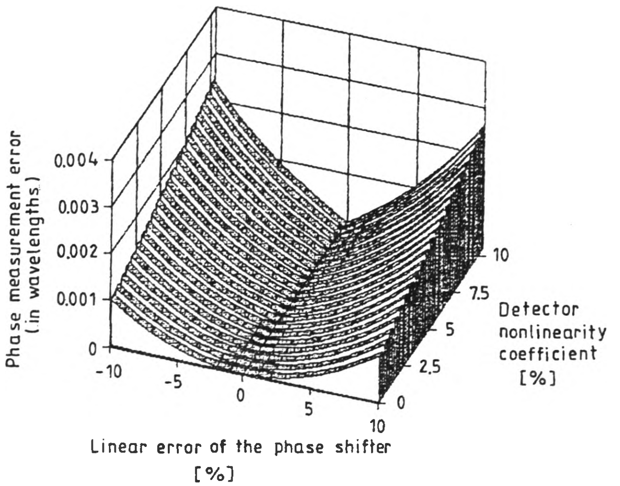

Fig. 3. Dependence of the phase measurement error on the linear phase shifter error and the detector nonlinearity for the algorithm (40)

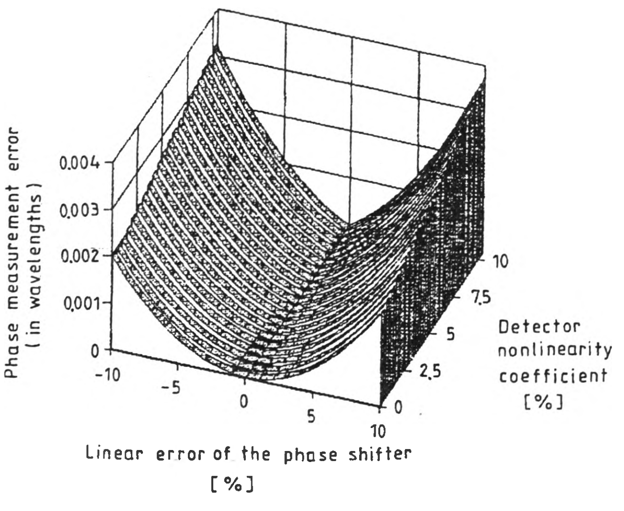

Fig. 4. Dependence of the phase measurement error on the linear phase shifter error and the detector nonlinearity for the Hariharan algorithm

(where: I' — output signal of the detector, ζ — coefficient of detector nonlinearity) the dependence of the phase measurement errors on both the linear errors of the phase shifter and the detector nonlinearity has been presented in Figs. 3 and 4.

5. Conclusions

In this paper, a general form of the five-point algorithm for a discrete phase change method has been proposed. The algorithms derived from the general form render the phase measurement of the fringe image possible for arbitrary, equal phase shifts between the image samples. The selection of the weighting coefficients of image samples and the phase shifts between the samples allows us to shape the sensitivity of the algorithms to the sources of the measurement errors occurring in the measuring system.

The authors pointed out that the exploitation of the Fourier description of the discrete phase change method renders it possible to reduce the sensitivity of the five-point algorithms obtained from the suggested general form to two typical sources of errors: linear error of the phase shifter and the second order nonlinearity of the detector. The results of simulation show that, if only linear errors of the phase shifter occur, the two proposed algorithms enable the phase measurement errors to be diminished by the factor of 10^{-1} as compared to the popular Hariharan algorithm. The algorithm worked out shows, similarly to that of Hariharan, lack of sensitivity to the second order nonlinearity of the detector and assures two times smaller phase measurement errors in the presence of only linear phase shifter errors, as well as their essential reduction if the phase shifter errors and the detector nonlinearity occur simultaneously.

In order to further improve the proposed method of synthesis of discrete phase change algorithms the authors intend to verify experimentally the algorithms presented in this paper and to examine the possibility of reducing their sensitivity to other sources of errors (for instance, the third order detector nonlinearity) occurring in the measuring systems.

References

[1] Larkin K. G., Oreb B. F., J. Opt. Soc. Am. A **9** (1992), 1740.
[2] Freischlad K., Koliopoulos C. L., J. Opt. Soc. Am. A **7** (1990), 542.
[3] Hariharan P., Oreb B. F., Eiju T., Appl. Opt. **26** (1987), 2504.

Received September 9, 1994