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LAMINAR RESISTANCE OF BINGHAM FLUID FLOW IN PIPELINES

The dependence determining the exact value of energy losses for laminar flow of Bingham fluid in pipes has been derived. The solution of the problem has been based on Bingham model, generalized description of laminar flow and the theory of solving the quartic algebraic equations. The equation presented determines explicitly the value of energy losses as the function of plasticity number. It enables to calculate with the same accuracy as before when simplified form of Buckingham-Reiner equation was used.

NOTATIONS

$L.p. = \frac{\tau_0 D}{v\eta_{pl}}$ — plasticity number,

D — pipeline diameter, m,

L — pipeline length, m,

Q — flow rate, $m^3 \cdot s^{-1}$,

$R+$ — set of the real non-negative numbers,

W_1, W_2 — coefficients defined by eqs. (28) and (29),

Δp — loss of pressure, $N \cdot m^{-2}$,

$\overline{\Delta p}$ — auxiliary variable of loss of pressure defined by coordinates translation according the eq. (8), $N \cdot m^{-2}$,

$\breve{\Delta p}$ — approximative value of Δp defined by eq. (7), $N \cdot m^{-2}$,

v_x — local velocity in pipeline, $m \cdot s^{-1}$,

v — mean flow velocity in pipeline, $m \cdot s^{-1}$,

v_{KR} — mean critical velocity of flow in pipeline, $m \cdot s^{-1}$,

a — coefficient defined by eqs. (19), (20) or (21),

$\lambda, \varkappa$ — variables of eqs. (12) and (15), $N^2 \cdot m^{-6}$,

$\delta_\tau = \tau_0 \cdot D^{-1}$, $N \cdot m^{-3}$,

$\delta_{\breve{\Delta p}} = | \Delta p \breve{\Delta p}^{-1} - 1 |$ — approximation error,

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τ — shear stress, $N \cdot m^{-2}$,
τ_s — shear stress on pipeline wall, $N \cdot m^{-2}$,
τ_0 — flow limit, $N \cdot m^{-2}$,
η_{pl} — plastic viscosity, $N \cdot s \cdot m^{-2}$,
$\frac{dv_x}{dr}$ — velocity gradient, s^{-1}.

1. INTRODUCTION

For a rheological characterization of many systems the concept of Bingham body [1] is very useful. This body, subject to tangent stresses exceeding liquid limit, flows like a Newtonian fluid subject to the tangent stress equal to $\tau - \tau_0$

$$\tau - \tau_0 = \eta \frac{dv_x}{dr}. \tag{1}$$

The model of Bingham body was successfully applied in description of the behaviour of a number of viscoelastic fluids. Because of its wide applications there are many publications concerning the regularities governing the flow of Bingham fluids in pipelines and channels [3–5]. They have been reviewed in [6]. The purpose of the present paper is to give a concise and general dependence, determining the values of energy losses for laminar flow of Bingham fluid in pipelines. Despite a number of the related papers, no equation has been derived which would explicitely and precisely determine the magnitude of energy losses for the flow of these fluids in pipelines. This is true also with the equation of BUCKINGHAM [7], whereas in the works by HEDSTRÖM [8] and GOVIER and WINNING [9] the nomograms have been worked out by the method of dimensional analysis.

2. PROBLEM FORMULATION

Bingham's equation is applied to an abstract element of the fluid volume. Its application to the characteristics of laminar flow of viscoelastic fluids in pipelines requires its integration, according to the formula

$$\frac{8Q}{D^3} = \frac{1}{\tau_s} \int_{\tau_0}^{\tau_s} \tau^2 f(\tau)\, d\tau \tag{2}$$

for

$$f(\tau) = \frac{\tau - \tau_0}{\eta_{pl}}$$

which yields the well-known Buckingham-Reiner's equation [10]:

$$\frac{8Q}{\pi D^3}=\frac{1}{\eta_{\text{pl}}}\left[\frac{1}{4}\tau_s-\frac{1}{3}\tau_0+\frac{1}{12}\tau_0^4\cdot\tau_s^{-3}\right]. \tag{3}$$

This equation describes the relationship between kinematic and dynamic parameters of Bingham fluids. By introducing into the eq. (3) the value of τ_s determined from the equilibrium of forces acting on the fluid flowing in pipelines

$$\tau_s=\frac{\Delta_p\cdot D}{4L} \tag{4}$$

and multiplying its both sides by $\Delta p/L$ we get

$$\frac{\pi D^4}{128\eta_{pl}}\left(\frac{\Delta p}{L}\right)^4-\left(Q+\frac{\pi D^3\tau_0}{24\eta_{\text{pl}}}\right)\left(\frac{\Delta p}{L}\right)^3+\frac{2\pi\tau_0^4}{3\eta_{pl}}=0. \tag{5}$$

This equation, after suitable transformation, represents an incomplete polynomial of 4th order

$$\left(\frac{\Delta p}{L}\right)^4-32\left(\frac{v\cdot\eta_{\text{pl}}}{D^2}+\frac{\tau_0}{6D}\right)\left(\frac{\Delta p}{L}\right)^3+\frac{256}{3}\left(\frac{\tau_0}{D}\right)^4=0. \tag{6}$$

Its simplified solution, under the assumption that $(256/3)(\tau_0/D)^4=0$, has the form

$$\frac{\Delta p}{L}\approx\frac{\widetilde{\Delta p}}{L}=32\left(\frac{v\cdot\eta_{\text{pl}}}{D^2}+\frac{\tau_0}{6D}\right). \tag{7}$$

According to Reiner [11], in the case of $\tau_0\cdot\tau_s^{-1}<0.5$ the last component of the formula (6) may be neglected which, after Parzonka [3], takes place in most cases occurring in practice.

Equation (6) is an implicit function of two variables

$$F(v,\frac{\Delta p}{L}(v),$$

in which to each $v\in(0,v_{\text{KR}})$ there corresponds precisely one number

$$\frac{\Delta p}{L}\in\text{R}^+$$

such that

$$\bigwedge_{v\in(0.v_{KR})}F\left(v,\frac{\Delta p}{L}(v)\right)=0.$$

The usability of the Buckingham-Reiner 's equation in the engineering designing depends on the explicitness of the function

$$F\left(v, \frac{\Delta p}{L}\right)$$

defined in the set of real positive numbers. In order to apply the simplified formula (7) as a function approximating a precise solution of the formula (6), the magnitude of the error must be known. This type of error is called truncation error and to determine it the precise value of $\Delta p/L$ and approximate one of $\widetilde{\Delta p}/L$ should be known.

In order to derive a formula determining explicitely the form of the function $\Delta p/L = f(v, D, \eta_{pl}, \tau_0)$, it is necessary to solve the eq. (6). This solution is an effective solution of the problem.

3. GENERAL SOLUTION

A suitable transformation of the eq. (6) by means of

$$\frac{\Delta p}{L} = \frac{\overline{\Delta p}}{L} + \frac{1}{4}\left(\frac{\widetilde{\Delta p}}{L}\right) \tag{8}$$

yields the eq. (9) without the term containing $(\overline{\Delta p}/L)^3$:

$$\left(\frac{\overline{\Delta p}}{L}\right)^4 - \frac{3}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^2\left(\frac{\overline{\Delta p}}{L}\right)^2 - \frac{1}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^3\left(\frac{\overline{\Delta p}}{L}\right) + \frac{3}{256}\left(\frac{\widetilde{\Delta p}}{L}\right)^4 + \frac{256}{3}\,\delta_\tau^4 = 0 \tag{9}$$

where $\delta_\tau = \tau_0 \cdot D^{-1}$.

The left-hand side of eq. (9) will be presented in the form of the difference of squares by completing the term $(\overline{\Delta p}/L)^4$ according to the formula (10)

$$\left[\left(\frac{\overline{\Delta p}}{L}\right)^2 + \frac{1}{2}\lambda\right]^2 = \left(\frac{\overline{\Delta p}}{L}\right)^4 + \lambda\left(\frac{\overline{\Delta p}}{L}\right) + \frac{1}{4}\lambda^4. \tag{10}$$

Hence, we get the eq. (11) which is identical with (9)

$$\left[\left(\frac{\overline{\Delta p}}{L}\right)^2 + \frac{1}{2}\lambda\right]^2 - \left\{\left[\lambda + \frac{3}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^2\right]\left(\frac{\overline{\Delta p}}{L}\right)^2 + \frac{1}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)\left(\frac{\overline{\Delta p}}{L}\right) + \left[\frac{1}{4}\lambda^2 + \frac{3}{256}\left(\frac{\widetilde{\Delta p}}{L}\right)^4 - \frac{256}{3}\delta_\tau{}^4\right]\right\} = 0. \tag{11}$$

The expression of the eq. (11) given in brackets is a square of binomial in the case when the discriminant $\Delta(\lambda) = 0$, from which we get the relation (12):

$$\lambda^3 + \frac{3}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^2 \lambda^2 + \left[\frac{3}{64}\left(\frac{\widetilde{\Delta p}}{L}\right)^4 - \frac{1024}{3}\delta_\tau^{\,4}\right]\lambda + \left[\frac{1}{512}\left(\frac{\widetilde{\Delta p}}{L}\right)^4 - 128\,\delta_\tau^4\right]\left(\frac{\widetilde{\Delta p}}{L}\right)^2 = 0. \tag{12}$$

The idea of the above transformation was to reduce the solution of the equation of fourth order (6) to the solution of cubic eq. (12) and quadratic eqs. (13.1), (13.2):

$$\left(\frac{\overline{\Delta p}}{L}\right)^2 - \sqrt{\lambda - \frac{3}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^2} \cdot \frac{\overline{\Delta p}}{L} + \frac{1}{2}\lambda - \frac{\left(\frac{\widetilde{\Delta p}}{L}\right)^3}{16\sqrt{\lambda - \frac{3}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^2}} = 0, \tag{13.1}$$

$$\left(\frac{\overline{\Delta p}}{L}\right)^2 + \sqrt{\lambda - \frac{3}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^2} \cdot \frac{\overline{\Delta p}}{L} + \frac{1}{2}\lambda + \frac{\left(\frac{\widetilde{\Delta p}}{L}\right)^3}{16\sqrt{\lambda - \frac{3}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^2}} = 0. \tag{13.2}$$

Simplification of the form of eq. (12) enables the substitution (14)

$$\lambda = \varkappa - \frac{1}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^2 \tag{14}$$

which leads to the eq. (15):

$$\varkappa^3 - \frac{1024}{3}\,\delta_\tau^4 \cdot \varkappa - \frac{256}{3}\,\delta_\tau^4\left(\frac{\widetilde{\Delta p}}{L}\right)^2 = 0. \tag{15}$$

The discriminant of eq. (15) has a positive value, which is presented by the formula (16):

$$\Delta(\varkappa) = \frac{1}{4}\left[\frac{256}{3}\,\delta_\tau^4\left(\frac{\widetilde{\Delta p}}{L}\right)^2\right]^2 - \frac{1}{27}\left(\frac{1024}{3}\,\delta_\tau^4\right)^3. \tag{16}$$

After substituting the value of $\Delta p/L$ according to the eq. (7) and applying the necessary transformation, it is presented also by the relation (17):

$$\Delta(\varkappa) = \frac{256^4}{2916}\delta_\tau^{12}\left(1+\frac{\nu\cdot\eta_{pl}}{6D^2\delta_\tau^4}\right) - \frac{256^4}{5184}\delta_\tau^{12} > 0. \tag{17}$$

Thus, the eq. (15) possesses one real root defined by the eq. (18):

$$\varkappa = \sqrt[3]{\frac{256}{6}\left(\frac{\widetilde{\Delta p}}{L}\right)^2\cdot\delta_\tau^4}\left(\sqrt[3]{1+\sqrt{1-\frac{256}{27}\frac{\frac{256}{3}\delta_\tau^4}{(\widetilde{\Delta p}/L)^4}}} + \sqrt[3]{1-\sqrt{1-\frac{256}{27}\frac{\frac{256}{3}\delta_\tau^4}{(\widetilde{\Delta p}/L)^4}}}\right) \tag{18}$$

and two complex coupled roots, which are beyond the scope of the problem considered. Based on the eqs. (14) and (18) we have determined the sought value of the root of the eq. (12) by the relation (14.1):

$$\lambda = \sqrt[3]{\frac{256}{6}\left(\frac{\widetilde{\Delta p}}{L}\right)^2\delta_\tau^4}\cdot\alpha - \frac{1}{8}\left(\frac{\widetilde{\Delta p}}{L}\right)^2 \tag{14.1}$$

where

$$\alpha = \sqrt[3]{1+\sqrt{1-\left(\frac{16}{3}\frac{\delta_\tau\cdot L}{\widetilde{\Delta p}}\right)^4}} + \sqrt[3]{1-\sqrt{1-\left(\frac{16}{3}\frac{\delta_\tau\cdot L}{\widetilde{\Delta p}}\right)^4}} \tag{19}$$

or

$$\alpha = \sqrt[3]{1+\sqrt{1-\left(\frac{\tau_0/6D}{\nu\eta_{pl}/D^2+\tau_0/6D}\right)^4}} + \sqrt[3]{1-\sqrt{1-\left(\frac{\tau_0/6D}{\nu\eta_{pl}/D^2+\tau_0/6D}\right)^4}}, \tag{20}$$

$$\alpha = \sqrt[3]{1+\sqrt{1-\left(\frac{1}{1+6\,\nu\cdot\eta_{pl}/D\cdot\tau_0}\right)^4}} + \sqrt[3]{1-\sqrt{1-\left(\frac{\tau_0/6D}{\nu\eta_{pl}/D^2+\tau_0/6D}\right)^4}}. \tag{21}$$

Using the relation (18) we determine the roots of eqs. (13.1) and (13.2), which have the form:

$$\left(\frac{\overline{\Delta p}}{L}\right)_{1,2} = \frac{\widetilde{\Delta p}}{L}\left[\sqrt{\left(\frac{2}{3}\frac{\delta_\tau^4\cdot L^4}{\widetilde{\Delta p}^4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}\right.$$

$$\pm\sqrt{\frac{1}{8}-\left(\frac{2}{3}\frac{\delta_\tau^4\cdot L^4}{\tilde{\Delta p}^4}\right)^{1/3}\cdot\alpha+\frac{1}{32\sqrt{\left(\frac{2}{3}\frac{\delta_\tau^4\cdot L^4}{\tilde{\Delta p}^4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}}}\Bigg], \tag{22.1}$$

$$\left(\frac{\Delta p}{L}\right)_{3,4}=\frac{\tilde{\Delta p}}{L}\Bigg[-\sqrt{\left(\frac{2}{3}\frac{\delta_\tau^4\cdot L^4}{\tilde{\Delta p}^4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}$$

$$\pm\sqrt{\frac{1}{8}-\left(\frac{2}{3}\frac{\delta_\tau^4\cdot L^4}{\tilde{\Delta p}^4}\right)^{1/3}\cdot\alpha-\frac{1}{32\sqrt{\left(\frac{2}{3}\frac{\delta_\tau^4\cdot L^4}{\tilde{\Delta p}^4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}}}\Bigg]. \tag{22.2}$$

Substituting the formulae (22.1) and (22.2) to the orthocartesian transformation (8) we get the solution of the eq. (6) and we were in search of:

$$\left(\frac{\Delta p}{L}\right)_1=\frac{\tilde{\Delta p}}{L}\Bigg[\frac{1}{4}+\sqrt{\left(\frac{2}{3}\delta_\tau^4\left(\frac{\tilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}+\frac{1}{16}}$$

$$+\sqrt{\frac{1}{8}-\left(\frac{2}{3}\delta_\tau^4\left(\frac{\tilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\cdot\alpha+\frac{1}{32\sqrt{\left(\frac{2}{3}\delta_\tau^4\left(\frac{\tilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\alpha+\frac{1}{16}}}}\Bigg], \tag{23.1}$$

$$\left(\frac{\Delta p}{L}\right)_2=\frac{\tilde{\Delta p}}{L}\Bigg[\frac{1}{4}+\sqrt{\left(\frac{2}{3}\delta_\tau^4\left(\frac{\tilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}$$

$$-\sqrt{\frac{1}{8}-\left(\frac{2}{3}\delta_\tau^4\left(\frac{\tilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\cdot\alpha+\frac{1}{32\sqrt{\left(\frac{2}{3}\delta_\tau^4\left(\frac{\tilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}}}\Bigg], \tag{23.2}$$

$$\left(\frac{\Delta p}{L}\right)_3=\frac{\tilde{\Delta p}}{L}\Bigg[\frac{1}{4}-\sqrt{\left(\frac{2}{3}\delta_\tau^4\left(\frac{\Delta p}{L}\right)^{-4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}$$

$$+\sqrt{\frac{1}{8}-\left(\frac{2}{3}\delta_\tau^4\left(\frac{\widetilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\alpha-\frac{1}{32\sqrt{\left(\frac{2}{3}\delta_\tau^4\left(\frac{\widetilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}}}\Bigg], \tag{23.3}$$

$$\left(\frac{\Delta p}{L}\right)_4=\frac{\widetilde{\Delta p}}{L}\Bigg[\frac{1}{4}-\sqrt{\left(\frac{2}{3}\delta_\tau^4\left(\frac{\widetilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}$$

$$-\sqrt{\frac{1}{8}-\left(\frac{2}{3}\delta_\tau^4\left(\frac{\widetilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\alpha-\frac{1}{32\sqrt{\left(\frac{2}{3}\delta_\tau^4\left(\frac{\widetilde{\Delta p}}{L}\right)^{-4}\right)^{1/3}\cdot\alpha+\frac{1}{16}}}}\Bigg]. \tag{23.4}$$

5. DISCUSSION OF RESULTS

Further analysis of the above problem has been based on the eq. (23.1), which — contrary to the remaining solutions — satisfies the condition

$$\frac{\Delta p}{L}=\left(\frac{\widetilde{\Delta p}}{L}\right) \quad \text{for} \quad \frac{256}{3}\delta_\tau^4=0, \tag{24}$$

thus may be useful in determining the error due to the application of the approximate function (7).

In analogical case the eqs. (23.2)–(23.4) will take the following form:

$$\frac{\Delta p}{L}=0 \quad \text{for} \quad \frac{256}{3}\delta_\tau^4=0. \tag{25}$$

Let $\delta_{\widetilde{\Delta p}}$ denote the relative error of the approximation $\widetilde{\Delta p}/L$, determined by the relationship (26)

$$\delta_{\widetilde{\Delta p}}=\left|\frac{\Delta p}{\widetilde{\Delta p}}-1\right|. \tag{26}$$

The change of value of the ratio $\Delta p/\widetilde{\Delta p}$ depends on that of dimensionless coefficient

$$\Delta p(\widetilde{\Delta p})^{-1}=f\left(\frac{\delta_\tau L}{\widetilde{\Delta p}}\right). \tag{27}$$

Denotation f is a function determined by the expression in brackets of the eq. (23.1), in which dimensionless coefficient occurs in form of exponential expressions

$$W_1 = \left(\frac{2}{3} \frac{\delta_\tau^4 \cdot L^4}{\widetilde{\Delta p}^4}\right)^{1/3}, \tag{28}$$

$$W_2 = \left(\frac{16}{3} \frac{\delta_\tau \cdot L}{\widetilde{\Delta p}}\right)^4. \tag{29}$$

After substituting

$$\frac{\widetilde{\Delta p}}{L} = 32\left(\frac{v\eta_{pl}}{D^2} + \frac{\tau_0}{6D}\right)$$

and

$$\delta_\tau = \tau_0 \cdot D^{-1}$$

these formulae are reduced to (30) and (31), respectively:

$$W_1 = \frac{3}{32}\left(\frac{1}{1+6(L.p.)^{-1}}\right)^{4/3}, \tag{30}$$

$$W_2 = \left(\frac{1}{1+6(L.p.)^{-1}}\right)^4 \tag{31}$$

where the number $L.p.$ introduced by GOVIER and WINNING [9] is determined by the relationship

$$L.p. = \frac{\tau_0 \cdot D}{v \cdot \eta_{pl}}. \tag{32}$$

If we introduce the expression

$$\tilde{\tau}_s = \frac{\widetilde{\Delta p} \cdot D}{4L} \tag{33}$$

and $\delta_\tau = \tau_0 \cdot D^{-1}$ into the formulae (28) and (29), then they can be transformed into the forms (34) and (35):

$$W_1 = \frac{1}{4\sqrt[3]{6}}\left(\frac{\tau_0}{\tilde{\tau}_s}\right)^{4/3}, \tag{34}$$

$$W_2 = \left(\frac{4}{3} \frac{\tau_0}{\tilde{\tau}_s}\right)^4. \tag{35}$$

Assuming the criterion of plasticity as an argument it allows us to determine upper and lower limits of the functions $W_1(L.p.)$, $W_2(L.p.)$, $\alpha(L.p.)$ and $\Delta p/\widetilde{\Delta p}$ $(L.p.)$for $L.p. \in \langle 0; \infty \rangle$:

$$\sup W_1(L.p.) = \frac{3}{32}, \quad \inf W_1(L.p.) = 0, \tag{36}$$

$$\sup W_2(L.p.) = 1, \quad \inf W_2(L.p.) = 0, \tag{37}$$

$$\sup \alpha(L.p.) = 2, \quad \inf \alpha(L.p.) = \sqrt[3]{2}, \tag{38}$$

$$\sup \frac{\Delta p}{\widetilde{\Delta p}}(\alpha) = 1, \quad \inf \frac{\Delta p}{\widetilde{\Delta p}}(\alpha) = \frac{3}{4}. \tag{39}$$

The assumption that $L.p. = 0$ in physical interpretation represents a Newtonian fluid, whereas $L.p. = \infty$ — Saint-Vantan's body.

6. CONCLUSIONS

The Buckingham-Reiner's equation has a complex form, not convenient for designing purposes. Neglecting the last component

$$\frac{256}{3}(\tau_0 \cdot D^{-1})^4$$

of this equation leads to its simplified form, commonly used in experiments and designing. A precise calculation of energy losses during laminar motion of Bingham fluid in a pipeline is possible when the following equation, derived in the paper, is applied:

$$\frac{\Delta p}{L} = \frac{\widetilde{\Delta p}}{L}\left[\frac{1}{4} + \sqrt{\frac{3}{32}\left(\frac{1}{1+6(L.p.)^{-1}}\right)^{4/3} \cdot \alpha + \frac{1}{16}} + \sqrt{\frac{1}{8} - \frac{3}{32}\left(\frac{1}{1+6(L.p.)^{-1}}\right)^{4/3} \cdot \alpha + \frac{1}{32\sqrt{\frac{3}{32}\left(\frac{1}{1+6(L.p.)^{-1}}\right)^{4/3} \cdot \alpha + \frac{1}{16}}}}\right]$$

where:

$$\alpha = \sqrt[3]{1 + \sqrt{1 - \left(\frac{1}{1+6(L.p.)^{-1}}\right)^4}} + \sqrt[3]{1 - \sqrt{1 - \left(\frac{1}{1+6(L.p.)^{-1}}\right)^4}}.$$

Relative error of the reduced form of Buckingham-Reiner's equation may be determined by means of the function:

$$\Lambda \delta_{\widetilde{\Delta p}} \quad (L.p.),$$

$$L.p. \in (0, \infty).$$

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OPORY PRZEPŁYWU LAMINARNEGO CIECZY BINGHAMA W RURACH

Wyprowadzono w postaci jawnej zależność określającą dokładną wartość strat energetyznych dla ruchu laminarnego cieczy binghamowskiej w rurach. Rozwiązanie zagadnienia oparto na modelu ciała Binghama, uogólnionej charakterystyce ruchu laminarnego oraz teorii rozwiązania równań algebraicznych czwartego stopnia. Podane równanie określa jednoznacznie wielkość strat energetycznych w funkcji liczby plastyczności. Umożliwia ono obliczenie dokładności stosowanej dotychczas uproszczonej postaci równania Buckinghama-Reinera.

HYDRAULISCHE WIDERSTÄNDE IN ROHREN WÄHREND DER LAMINARSTRÖMUNG EINER BINGHAM'SCHEN FLÜSSIGKEIT

In offensichtlicher Form wurden genaue Werte der Energieverluste während der Laminarströmung einer Bingham'schen Flüssigkeit in Rohren ausgerechnet. Die Lösung des Problems stützte man auf dem Modell des Bingham'schen Körpers, der allgemeinen Charakteristik der Laminarströmung sowie der Lösungstheorie von algebraischen Gleichungen vierten Grades. Die aufgestellte Gleichung beschreibt eindeutig die Energieverluste als Funktion der Plastizitätszahl. Das wiederum ermöglicht die Berechnung der Genauigkeit der bisher nur in annähernder Form verwendeten Gleichung von Buckingham und Reiner.

СОПРОТИВЛЕНИЕ ЛАМИНАРНОМУ ТЕЧЕНИЮ ЖИДКОСТИ БИНГАМА В ТРУБАХ

Выведена в явном виде зависимость, определяющая точное значение энергетических потерь для ламинарного движения бингамовской жидкости в трубах. Решение вопроса основано на модели тела Бингама, обобщённой характеристике ламинарного движения, а также теории решения алгебраических уравнений четвёртой степени. Приведённое уравнение однозначно определяет величину энергетических потерь в функции числа пластичности. Оно даёт возможность точного вычисления применяемого до настоящего времени упрощённого вида уравнения.