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Structural information in volume holography*

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For the weak diffraction problem in volume holography, the quantization of three-dimensional Fourier space is introduced. The Ewald's construction is generalized over the case of volume gratings, then the structural optical information, for many-exposure case, is discussed.

Introduction

The problem of optical information storage and reconstruction is of particular interest in holography. One of the most important parameters of holographic system resolution is structural information, connected with the number of degrees of freedom for interference field recorded in medium. The structural information, for isoplanatic optical systems, was first introduced by Toraldo di Francia [1] and then developed for plane holography [2–3]. However, since the influence of material thickness on the information capacity is very significant, a more general analysis for three-dimensional materials is required. These developments are discussed in this paper, on the base of weak diffraction approximation. Such an approximation is formally analogous to that in quantum scattering theory and X-ray diffraction analysis.

According to the well-known fact, that an arbitrary distribution of refractive index can be represented as a three-dimensional Fourier spectrum of elementary sinusoidal structures (gratings), it will be shown in the present work, that limitation of grating sizes induce the "uncertainty" of grating vector. This fact implies, moreover, the quantization of the Fourier space. Every quant of this space corresponds to one complex number connected with one of the gratings.

In this paper a new approach to determination of structural information capacity in volume holography is proposed for many-exposure case. For our considerations, the Ewald's construction in a generalized form is used.

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Uncertainty problem in volume holography

The problem of scattering of light on volume diffraction gratings, and their superpositions — volume holograms — is difficult, particulary in the case of high diffraction efficiencies. The most complete analysis of this problem was reported by KOGELNIK [4], who considered the diffraction on one elementary sinusoidal volume grating, under assumption, that the sizes of the grating are restricted in one dimension only, and the modulation of refractive index is small. However, for structural information analysis the Kogelnik method is not sufficient, because the most interesting effects are connected with diffraction on volume holograms the size of which are restricted in all three dimensions. Since only the well-known methods based on the first Born approximation can be applied in such a case, our analysis will be given for small diffraction efficiencies. Moreover, for our purpose the WOLF method [5] concerned to the scattering of light on three-dimensional phase objects is particularly usefull.

When the scattering medium with refractive index $n = n_0+n_1(x, y, z)$, where $n_1 \ll n_0$, is restricted by two planes $z = z^{\pm}$, then, in the case of plane incident wave for two-dimensional Fourier transform of scattering field amplitude U_s in plane $z = z^{\pm}$, respectively, we get the following equation

$$\hat{F}\{U_s(k_x, k_y;\ z = z^{\pm})\} = \frac{-2ik_v^2 n_0}{k_z}\times$$
$$\times\exp(\pm ik_z z)\iiint\limits_{\{z^-\leqslant z\leqslant z^+\}} n(\boldsymbol{r})\exp(i\boldsymbol{k}_0\boldsymbol{r})\exp(-i\boldsymbol{k}\boldsymbol{r})\,d^3\boldsymbol{r}, \tag{1}$$

and $\boldsymbol{k} = (k_x, k_y, k_z)$,

where: $\boldsymbol{k}_0$ and $\boldsymbol{k}$ are wave-vectors of incident and scattering wave, respectively (in medium), and k_v is a wavenumber in vacuum.

Assuming the scattering structure to be a sinusoidal grating, we can write $n = n_0+n_2\cos(\boldsymbol{K}\cdot\boldsymbol{r}+\Phi_0)$, where $\boldsymbol{K}$ is the grating vector, and n_2, Φ_0 are constant values. For rectangular symmetry and for hologram sizes T_x, T_y, T_z, from eq. (1) we recive the following equation for one of the modulating terms:

$$\hat{F}_M\{U_s(k_x, k_y;\ z = z^{\pm})\} = \frac{-ik_v^2 n_0 n_2}{k_z}T_xT_yT_z\times$$
$$\times\frac{\sin[(k_{0x}+K_x-k_x)T_x]}{(k_{0x}+K_x-k_x)T_x}\times\frac{\sin[(k_{0y}+K_y-k_y)T_y]}{(k_{0y}+K_y-k_y)T_y}\times$$
$$\times\frac{\sin[(k_{0z}+K_z-k_z)T_z]}{(k_{0z}+K_z-k_z)T_z}. \tag{2}$$

Therefore, restriction of grating dimensions gives the spread of Bragg condition. However, from the physical point of view, it is equivalent to vector $\boldsymbol{K}$ spread. So, in our considerations, we shall assume that vector

K is spread but Bragg law is fulfilled strictly. Such an approach would appear to be more convenient in our case. Hence, according to formula (2), the spread for K vector components is given by:

$$\Delta K_x \geqslant \frac{2\pi}{T_x}, \quad \Delta K_y \geqslant \frac{2\pi}{T_y}, \quad \Delta K_z \geqslant \frac{2\pi}{T_z}. \tag{3}$$

In agreement with relation (1), both the three-dimensional Fourier space (K_x, K_y, K_z) and configurational space (x, y, z) are canonical conjugate ones. The uncertainty theorem (3) has the fundamental meaning in further analysis.

Generalized Ewald's sphere

The conventional Ewald's construction was introduced in connection with the problem of X-ray scattering on periodic structures (see e.g. [6]), but its meaning was rather didactic for graphical determination of "reflected" beam directions. Since, the volume holograms can be regarded as a superposition of periodic structures, thus, the Ewald's construction can be adapted also to this problem. Moreover, in our case, there appears a very interesting fact, namely, that the characteristic dimensions of holograms are comparable with period of the gratings $d = 2\pi/K$, [7]. Therefore, using the new geometrical construction based on Ewald's sphere with addition of uncertainty theorem (3), we can get some new results; for example, structural information capacity which would be difficult to calculate without application of this construction.

In order to designe the generalized Ewald's construction we must introduce the Fourier space (K_x, K_y, K_z) of the grating vectors K. Then, for each vector k_0 of the restoring plane wave, the sphere with the radius k_0 is constructed, as in fig. 1. This is so-called Ewald's sphere. In consequence of Bragg law, this sphere determines the loci of the ends of grating vectors, which are registrated during one exposure by the same vectors k_0, as the restoring one.

In fig. 1 the both cases of transmission (B) and reflection one (C) holograms are presented. The spherical bowl B (or C) represents the geometrical locus of vector ends of the angular spectrum plane waves of object beam. It is usefull to introduce the second coordinate system (k_x, k_y, k_z) — translated with respect to the first one — for the spherical bowl B_{xy}, the projection on the (k_x, k_y) plane determines the so-called Fourier area of an object beam (the hologram plane fulfils the equation $z =$ constant). Moreover, the conjugate Ewald's sphere corresponding to the conjugate image is plotted.

Generalized Ewald's sphere connected with uncertainty relation (3) is presented in fig. 2 (we assume that $T_x, T_y \gg T_z$, and $T_z = T$). This figure shows, in (K_y, K_z) plane, the illustration of two-exposure recording,

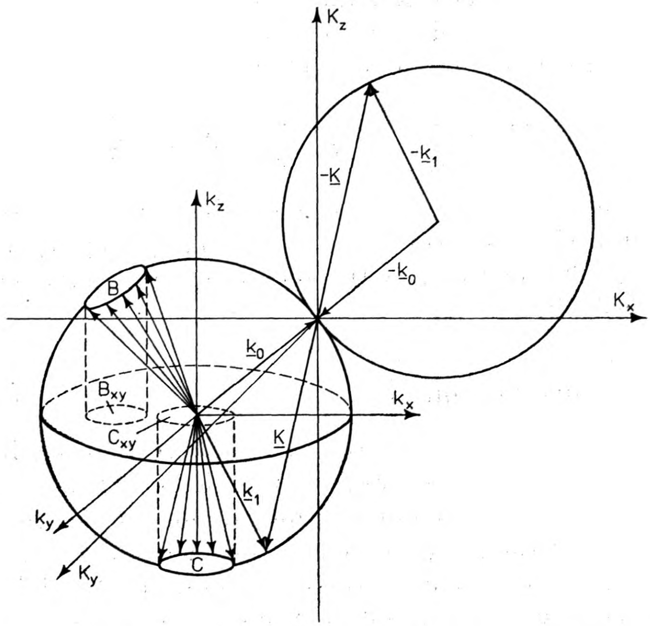

Fig. 1. Ewald's construction adapted to volume holography

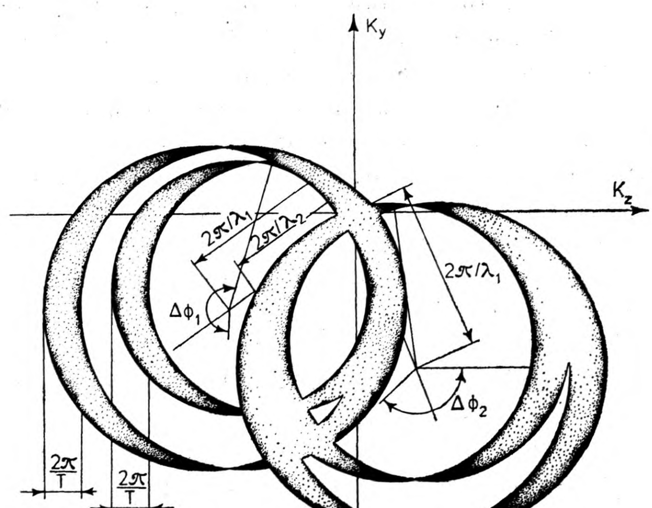

Fig. 2. Application of generalized Ewald's construction to colour holography
$(T_x, T_y \gg T_z = T)$

using two wavelengths λ_1 and λ_2 simultaneously (this analysis is usefull, for example, in colour holography). We can see that the uncertainty relation effects the spread of Ewald's spheres, so, it may be the reason of the information disturbation. The angles $\Delta\Phi_1$ and $\Delta\Phi_2$ determine the intervals of angular spectrum of plane waves for which the information may be recorded without disturbance.

The example of colour selectivity analysis, based on the generalized Ewald's sphere, is presented in fig. 3. We assume that the reconstruction

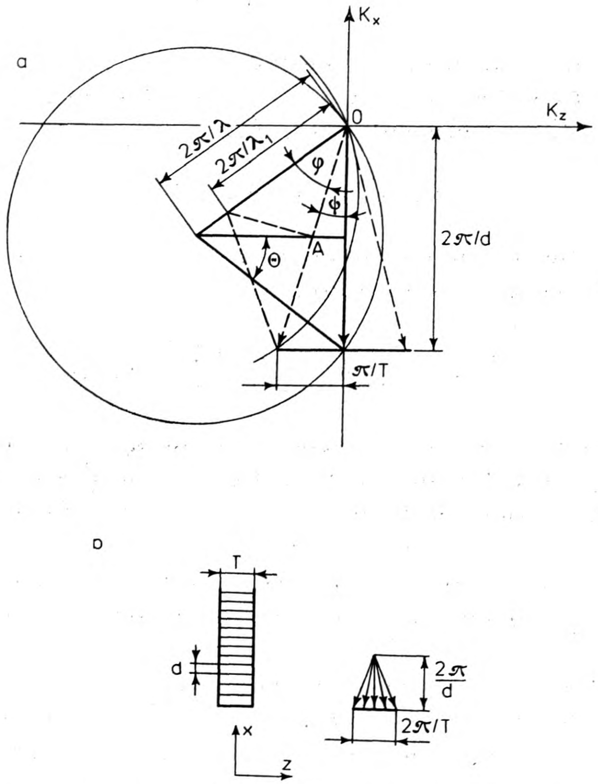

Fig. 3. Application of generalized Ewald's construction method to colour selectivity calculation

a) illustrating geometrical considerations b) illustrating the grating vector spread

may also occur for the wavelength λ_1. In our accounts we also assume that $T_x, T_y \gg T_z = T$, and that $d \ll T$. Then $\Psi \ll 1$ and $A0 \approx \pi/d$; hence $\lambda_1 \approx 2d\ \cos\varphi$, and $\lambda \approx 2d\sin\theta$. Therefore, in that approximation:

$$\Delta\lambda = \lambda\left(\frac{\lambda_1}{\lambda} - 1\right) = \frac{\lambda_v^2}{2Tn_0\sin\theta\tan\theta}. \tag{4}$$

This expression agrees, in its order of magnitude, with the relation recived on the basis of Kogelnik theory (see e.g. [8] p. 282).

Structural information capacity

The structural information capacity is determined, in our case, by the number of degrees of freedom which can be recorded in material. Its maximum value depends on sizes of permissible domain of Fourier space (K_x, K_y, K_z). Using the generalized Ewald's construction it is easy to show that this quantity equals:

$$\Omega_{\max} = \frac{4}{3}\pi\left(\frac{4\pi}{\lambda}\right)^3,$$

where λ — average wavelength in medium.

According to uncertainty theorem (3), the spread of grating vector $\boldsymbol{K}$ determines the sizes of elementary cell Ω_0:

$$\Omega_0 = (2\pi)^3 (T_x T_y T_z)^{-1}. \tag{5}$$

Therefore, the maximum number of degrees of freedom, for rectangular sizes of hologram, is given by:

$$N_{\max} = \frac{\Omega_{\max}}{\Omega_0} = \frac{4}{3}\pi\left(\frac{2}{\lambda}\right)^3 T_x T_y T_z.$$

Unfortunately, after many-exposure recording (using different values of $\boldsymbol{k}_0$ vector), not a full information stored in deep even ideal material can be restored without disturbance, because of the following restrictive conditions:

1. For each exposure there are two (not one) Ewald's spheres (see fig. 1). Hence, the number of independent degrees of freedom decreases two times.

2. Information from one diffraction grating may be reconstructed by the set of the wave-vectors $\boldsymbol{k}_0$ situated on the cone surface, which is obtained by rotating the vector $\boldsymbol{k}_0$ about the grating vector $\boldsymbol{K}$. Due to this fact, in order to get the independent degrees of freedom we must take into account only these vectors $\boldsymbol{k}_0$ which lie on one plane perpendicular to the hologram plane [8]. Finally this restrictive condition reduces the Fourier space domain to the torus with identical radii $r = 2\pi/\lambda$.

3. Since the object beam can be represented as a combination of plane waves components, the latter produce the additional interference terms in recording process. Those terms can be restored only in many-exposure case. To avoid such disturbing effects of second order, we must limit Fourier spectrum of each object beam. From the generalized Ewald's construction it may be shown that in this case the angles between $\boldsymbol{K}$ and $\boldsymbol{k}_0$ vectors must range within 120–240°. Finally, this condition permits to reduce the volume of torus to a part restricted by the sphere with radius $\sqrt{3}\, r$, as in fig. 4.

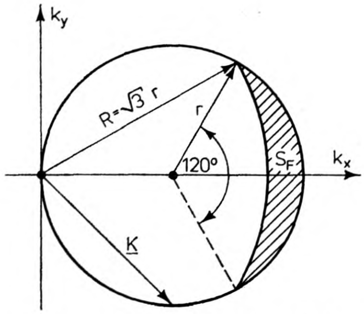

Fig. 4. Limitation of Fourier space domain. Intersection of torus is restricted to S_F surface

The resulting permissible volume of Fourier space is then given by

$$\Omega = (1/2)2\pi r \cdot S_F, \tag{6}$$

where S_F is result of intersection, as in fig. 4.

Finally, the maximum number of independent degrees of freedom which can be recorded in many-exposure case is given by

$$N_0 = \frac{3.74}{\lambda^3} T_x T_y T_z. \tag{7}$$

For optical frequencies we get approximately $N_0 = 10^{10}/\text{mm}^3$.

The above result has been derived under the assumption that MTF of material has no influence. It is equivalent, in our case, to cut-off frequency of material $f \geqslant 4750$ lines/mm, for $\lambda_v = 0.63$ μm (He–Ne laser radiation) and $n = 1.5$. On the other hand, the optimum capacity for many-exposure case, considered above, requires $f > 4050$ l/mm. When $4050 \leqslant f \leqslant 4750$ l/mm, the MTF induces decreasing of N. For example, for $f = 4500$ l/mm, we get $N = 0.41\,N_0$.

The next problem, which may be important, consists in calculation of the number of exposures M in optimum capacity case. If the number of elementary cells which are intersected by one Ewald's sphere is denoted by N_H, then the quantity M equals the ratio of torus volume divided by Ω_0 (see eq. (5)), to N_H, because the number of exposure is independent of the third restriction considered above. Then, after simple calculations

$$M = \frac{\pi}{\lambda}\frac{V}{S}, \tag{8}$$

where V and S are the volume and the total surface of hologram, respectively.

It is clear, that for the hologram with cubic shape (with side T) we get the maximum value of $M \approx T/2\lambda$. For example, for $\lambda_v = 0.63$ μm $n = 1.5$; $T = 15$ μm we get $M = 18$, but for $T = 1$ mm we have $M = 1240$.

Additionally, the ratio N_0/M would be equal to the average number of independent degrees of freedom for one hologram. It should be noted, however, that for each particular exposure, except for cubic case, different values of independent degrees of freedom are obtained. In fact, considering the third restrictive condition, we get different numbers of elementary cells crossed by permissible Ewald's sphere, the maximum number is obtained when the angle between $\boldsymbol{k}_0$ and hologram plane required is 90° (for $T_x, T_y > T_z$).

In practice the value of M can be limited by additional effects (nonlinearity of material characteristic curve, influence of geometry of optical system), our analysis, however, permits to determine the most uniform (therefore, the most economic) distribution of energy for several degrees of freedom, thus it may be particularly useful for holographic memory devices.

Conclusions

Application of generalized Ewald's construction facilitates immensely the analysis of structural information in volume holography, for it leads the entire problem to purely geometrical considerations.

The graphical method presented in this paper allows (in many-exposure case) to determine the number of independent degrees of freedom as a function of the following parameters: size and shape of object beam Fourier spectrum, size and MTF of material. Only some of them were discussed in details. In particular, the most optimal object beam spectrum for the case of many-exposure recording case was considered in detail. It was shown that the second order disturbances (see eq. (7)) cause a drop in of information capacity to 37 %. For this reason, MTF must be sufficiently large ($f > 4050$ l/mm, for $\lambda_v = 0.63$ μm, $n = 1.5$).

It seems that our results hold even if the diffraction efficiencies are large as they have been derived under the assumption of only Bragg law and uncertainty relation (3); these assumptions being not discrepant with the coupling wave theory [4], at least, for phase materials (see eq. (4) and [8], p. 282).

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Структурная информация в объемной голографии

Введено квантование трехмерного пространства Фурье для слабой дифракционной задачи. Обобщена сфера Эвальда для случая объемных решеток и обсужден оптимум структурной информации для случая мультиэкспозиционного освещения.