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Influence of optical system parameters on the light distribution at output

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Optical systems can be analyzed in terms of geometrical or diffractive optics. The diffraction integral of wave propagation depends on elements of transfer ray matrix that describe an optical system. This article discusses relationship presenting a diffraction integral for some holographic system written in terms of transfer ray matrix.

1. Introduction

Holographic optical element can be used as a conventional optical element in the development of coherent optical processor systems. The capabilities of this holographic element to perform more than one function at a time can be utilized profitably in the development of multifunction signal processor systems. This paper illustrates that each holographic system can be analyzed by use of either diffraction or geometrical optics, and the choice of the approach depends on the particular situation. A diffractional integral is derived when one relates the optical fields on the input plane of an optical system to those on its output plane. It is written in terms of the parameters that describe the holographic system under consideration. Thus, the kernel of the diffraction integral determines a connection between diffraction and the geometrical optics limited to the paraxial approximation. Geometrical optics is couched in terms of ray tracing matrix, where refraction (or diffraction) and translation matrices are multiplied together to form an optical system matrix [1], [2].

2. Diffraction integral and eikonal function

Holographic optical element is just a hologram that contains the full information of the recorded wave fronts. Apart from being more compact than its conventional optical element, it can provide simultaneous channels for carrying out different kinds of signal processing operations such as spectral analysis, filtering, pattern recognition, *etc.* Consider a description for recording and reconstructing a hologram. A typical configuration is shown in Fig. 1. Let $u_O(x_O, y_O)$ be the object distribution field inserted in the input plane (x_O, y_O) at z_O on the axis. If beside amplitude information the phase

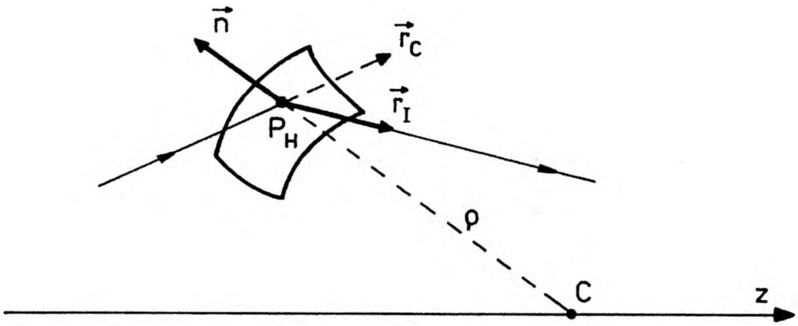

Fig. 1. Holographic system with an object transparency in front of the lens for studying the transform operations.

information is required, then diffraction theory is necessary in analyzing the quality of the optical system. The relationship between output and input planes of the holographic system ray matrix [2] is given by the equation

$$\begin{bmatrix} x_I \\ \xi_I \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_O \\ \xi_O \end{bmatrix}, \tag{1}$$

whereas the expression for optical field distribution described by Huygens–Fresnel principle [3] between these two planes is as follows:

$$U_I(x_I, y_I) = \int_{-\infty}^{\infty}\!\!\int U_O(x_O, y_O) h(x_O, y_O; x_I, y_I) \mathrm{d}x_O \mathrm{d}y_O, \tag{2}$$

where the integral kernel $h(x_O, y_O; x_I, y_I)$ is a transmission function (impulse response) that determines the field amplitude at point (x_I, y_I) of output plane produced by a point source of unit strength and zero phase at the point $(x_O = x'_O, y_O = y'_O)$, *i.e.*, when $U_O(x_O, y_O) = \delta(x_O - x'_O)\delta(y_O - y'_O)$.

As we know, geometrical optics laws follow from Maxwell's equations at limit $\lambda \to 0$ (for large wave number $2\pi/\lambda$), and the basic equation of geometrical optics is then the eikonal equation

$$(\mathrm{grad} S)^2 = n^2 \tag{3}$$

or

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = n^2(x, y, z),$$

where $n(x, y, z)$ is the refraction index of ray propagation region. The eikonal function $S(x, y, z)$ is the optical path of ray that is orthogonal to wave front

$S(x, y, z) = \text{const}$. If Fresnel approximation is accomplished, then the distance z_O (and z_I) between object and aperture (or aperture and output) plane is much larger than the maximum linear dimension of the aperture. Therefore,

$$r_O = z_O\left[1 + \frac{1}{2}\left(\frac{x - x_O}{z_O}\right)^2 + \frac{1}{2}\left(\frac{y - y_O}{z_O}\right)^2\right],$$

$$r_I = z_I\left[1 + \frac{1}{2}\left(\frac{x - x_I}{z_I}\right)^2 + \frac{1}{2}\left(\frac{y - y_I}{z_I}\right)^2\right].$$

The optical path along the ray connecting the source point $P_O\,(x_O, y_O)$ with the point $P_I\,(x_I, y_I)$ in the output plane is

$$S(x_O, y_O; x_I, y_I) = z_0 + z_I + \frac{(x - x_O)^2}{2(z_O + z)} + \frac{(x - x_I)^2}{2(z_I - z)} + \frac{(y - y_O)^2}{2(z_O + z)} + \frac{(y - y_I)^2}{2(z_I - z)}$$

where the spherical substrate of HOE [4] can be expressed in the form

$$z(x, y) = \frac{x^2 + y^2}{2\rho} + \frac{(x^2 + y^2)^2}{8\rho^3}.$$

Now, the eikonal function defines the optical distance along the ray (see Fig. 1), and the transmission function evaluated at the observation plane due to the source point describes the optical field in the form

$$h(x_O, y_O; x_I, y_I) = \tilde{A}(x_I, y_I)\exp\left[i\frac{2\pi}{\lambda}S(x_O, y_O; x_I, y_I)\right].$$

Therefore, diffraction integral (2) determining the amplitude distribution of field in the observation plane is given by

$$U_I\,(x_I, y_I) = \tilde{A}(x_I, y_I)\int\int_{-\infty}^{\infty} U_O(x_O, y_O)\exp\left[i\frac{2\pi}{\lambda}S(x_O, y_O; x_I, y_I)\right]\mathrm{d}x_O\mathrm{d}y_O. \tag{4}$$

Thus, Equation (4) shows the relationship between field distribution and the optical distance along the ray connecting the input and the output planes of an optical system.

3. Grating equation for ray transfer

The analysis of holographic optics is in many ways similar to that of conventional refractive optics. In this paper, we describe the ray tracing through a diffraction surface and derive an equation for holographic optical element in the matrix form, analogously as for ray tracing through refraction surface [1]. The most general approach considers propagation of the incident and diffracted waves the direction of which is determined by the local grating spacing and its orientation. It is relatively simple to trace rays through a holographic element, since one can easily compute the exact directional cosines of each beam incident and diffracted at a given point of the diffractive surface. A method of ray tracing through a curved holographic element has been first considered by WELFORD [5], and the vector equation applicable to those elements formed on substrate of any shape is represented for the first order of diffraction by

$$\mathbf{n}\times\left[\mathbf{r}_I-\mathbf{r}_C-\frac{\lambda}{\lambda_0}(\mathbf{r}_O-\mathbf{r}_R)\right] = 0, \tag{5}$$

where $\mathbf{n}$ is a unit vector along the local normal to the holographic surface at an incident point $P_H(x, y)$; analogously $\mathbf{r}_O$, $\mathbf{r}_R$, $\mathbf{r}_C$, $\mathbf{r}_I$ are the respective unit vectors along the rays, as shown in Fig. 2. From Equation (5) the scalar product of the two following vectors is not equal to zero

$$\mathbf{n}\cdot\left[\mathbf{r}_I-\mathbf{r}_C-\frac{\lambda}{\lambda_0}(\mathbf{r}_O-\mathbf{r}_R)\right] = \Gamma, \tag{6}$$

but can be rewritten in the form

$$\Gamma = \cos\alpha_I-\cos\alpha_C-\frac{\lambda}{\lambda_0}(\cos\alpha_O-\cos\alpha_R).$$

We remember that the grating equations for the incident ray with the direction cosines (ξ_C, η_C, ζ_C) impinging on the optical element are

$$\xi_I = \xi_C+\frac{\lambda}{\lambda_0}(\xi_O-\xi_R),$$

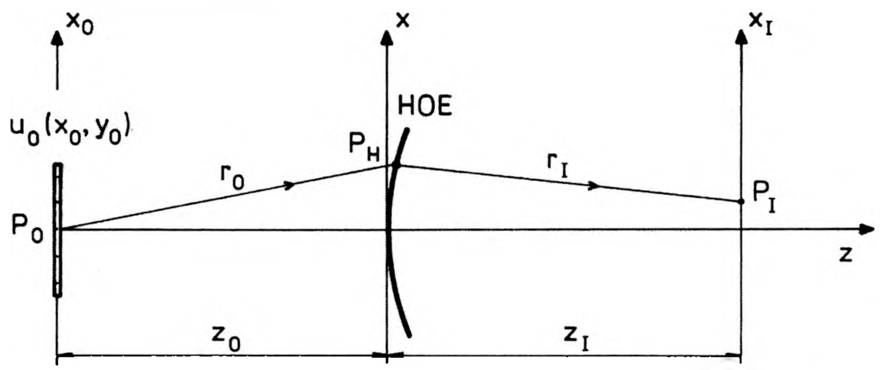

Fig. 2. Unit vectors along the reconstruction and diffraction rays and the local normal to holographic surface.

$$\eta_I = \eta_C + \frac{\lambda}{\lambda_0}(\eta_O - \eta_R).$$

This result shows the reconstruction beam at a flat holographic diffraction surface and determines the direction cosines of the respective image beam. For holographic optical elements on any shaped surface we multiply both sides of Eq. (6) by the unit vector of the local normal of a curved surface, and we obtain

$$\mathbf{r}_I - \mathbf{r}_C - \frac{\lambda}{\lambda_0}(\mathbf{r}_0 - \mathbf{r}_R) = \mathbf{n}\Gamma.$$

The direction cosines of the diffracted beams in a rectangular coordinate system oriented with its z-axis along the vertex normal (optical axis), are then discribed as follows:

$$\xi_I = \xi_C + \frac{\lambda}{\lambda_0}(\xi_O - \xi_R) + \frac{x_C}{\rho}\Gamma,$$

$$\eta_I = \eta_C + \frac{\lambda}{\lambda_0}(\eta_O - \eta_R) + \frac{y_C}{\rho}\Gamma,$$

where ρ is the curvature radius of the local curved holographic substrate. In the case of flat surface, $\rho \to \infty$, and the third expression on the right-hand side of the above two equations tends to zero. For simplicity, let us consider a holographic lens in one dimension having a spatial frequency

$$\frac{1}{\Lambda} = \frac{\xi_O - \xi_R}{\lambda_0}, \tag{7}$$

which increases in the x-direction. The matrix equation relating $x(z)$ and $\xi(z)$ on either side of the holographic optical element is then

$$\begin{bmatrix} x_I \\ \xi_I \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{\lambda}{\Lambda x_C} + \dfrac{\Gamma}{\rho} & 1 \end{bmatrix} \begin{bmatrix} x_C \\ \xi_C \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} x_C \\ \xi_C \end{bmatrix}. \tag{8}$$

Analogously to a conventional glass lens, the holographic lens has rotational symmetry with its interference pattern perpendicular to plane of incidence and diffracted rays.

4. Light distribution in terms of ray matrix elements

Let us consider an object plane placed in front of a holographic lens and illuminated with normally incident monochromatic plane wave, as shown in Fig. 1. The source plane and its conjugate are located at infinity and in the back focal plane of the lens.

To find the amplitude distribution of the field across the output plane of the system, the Fresnel diffraction formula is applied. If the field amplitude transmitted by an object is represented by the function $u_O(x_O, y_O)$, the output of the system [6] may be written as

$$\begin{aligned} U_I(x_I, y_I) = {} & \frac{1}{\lambda^2 z_O z_I} \iint\limits_{-\infty}^{\infty} \mathrm{d}x_O \mathrm{d}y_O \iint\limits_{-\infty}^{\infty} u_O(x_O, y_O) \exp\left[i\frac{k}{2z_O}\left(x_O^2 + y_O^2\right)\right] \\ & \times \exp\left[-i\frac{k}{z_O}\left(x_O x + y_O y\right)\right] \exp\left[i\frac{k}{2}\left(\frac{1}{z_O} + \frac{1}{z_I} - \frac{1}{f}\right)\left(x^2 + y^2\right)\right] \\ & \times \exp\left[\frac{k}{2z_I}\left(x_I^2 + y_I^2\right)\right] \exp\left[-i\frac{k}{z_I}\left(x_I x + y_I y\right)\right] \mathrm{d}x \mathrm{d}y \end{aligned} \tag{9}$$

where the constant phase factor has been dropped since it does not affect the result in any significant way. Substituting the expression

$$\frac{1}{z_\Delta} = \frac{1}{z_O} + \frac{1}{z_I} - \frac{1}{f}$$

into Eq. (9), we obtain

$$\begin{aligned} U_I(x_I, y_I) = {} & \frac{z_\Delta}{\lambda^2 z_O z_I} \exp\left[i\frac{k z_\Delta (f - z_O)}{2 z_O z_I f}\left(x_I^2 + y_I^2\right)\right] \iint\limits_{-\infty}^{\infty} u_O(x_O, y_O) \\ & \times \exp\left[i\frac{k z_\Delta (f - z_I)}{2 z_O z_I f}\left(x_O^2 + y_O^2\right)\right] \exp\left[-i\frac{k z_\Delta}{z_O z_I}\left(x_O x_I + y_O y_I\right)\right] \mathrm{d}x_O \mathrm{d}y_O. \end{aligned}$$

Introducing the following denotations

$$A = 1 - \frac{z_I}{f}, \quad B = \frac{z_O z_I}{z_\Delta}, \quad C = -\frac{1}{f}, \quad D = 1 - \frac{z_O}{f},$$

the general quadratic phase transform [7] of an input distribution $u_O(x_O, y_O)$ to an output plane (x_I, y_I) realized by the rotational symetrical holographic system to be described by a ray matrix (see Eq. (1)), can be written as

$$\begin{aligned} U_I(x_I, y_I) = {} & \frac{1}{B\lambda^2} \iint\limits_{-\infty}^{\infty} u_O(x_O, y_O) \exp\left\{i\frac{k}{2B}\left[A\left(x_O^2 + y_O^2\right) - 2\left(x_O x_I + y_O y_I\right)\right]\right\} \\ & \times \exp\left\{i\frac{k}{2B}\left[D\left(x_I^2 + y_I^2\right)\right]\right\} \mathrm{d}x_O \mathrm{d}y_O. \end{aligned} \tag{10}$$

The coefficients A, B, C, D are the elements of the ray matrix, where the element A is the factor affecting the phase in the integration variables at the input plane (x_O, y_O), while the element D affects the phase at the output plane (x_I, y_I); the matrix element C defines the optical power of the system and does not appear explicitly in Eq. (10), but element B describes deviation of the optical power (axial aberration). The optical system under consideration corresponding to $ABCD$ matrix realizing ray transformation between the input and output planes can be implemented in free space by using a set of optical elements, either refractive or diffractive lenses. As we remember, the ray matrix of an optical system is the product of the transfer matrices describing the free space propagation and optical elements such as lenses, diffraction gratings, mirrors, *etc.*

When the condition $f - z_I = 0$ is satisfied, then the first order Fourier transform of $u_O(x_O, y_O)$ occurs in the back focal plane of the holographic lens as an expression of field distribution

$$U_f(x_f, y_f) = \frac{1}{\lambda^2 f}\exp\left[i\frac{k}{2f}D(x_f^2 + y_f^2)\right]\int\limits_{-\infty}^{\infty}\int u_O(x_O, y_O)$$

$$\times\exp\left[-i\frac{k}{f}(x_O x_f + y_O y_f)\right]\mathrm{d}x_O\mathrm{d}y_O.$$

Analogously, the p-th order fractional Fourier transform of the function $u_O(x_O, y_O)$ is defined as

$$U_p(x_I, y_I) = \frac{1}{\lambda^2 f}\exp\left[i\frac{k(x_I^2 + y_I^2)}{2f_1\tan\Phi}\right]\int\limits_{-\infty}^{\infty}\int u_O(x_O, y_O)\exp\left[i\frac{k(x_O^2 + y_O^2)}{2f_1\tan\Phi}\right]$$

$$\times\exp\left[-i\frac{k}{f_1\sin\Phi}(x_O x_I + y_O y_I)\right]\mathrm{d}x_O\mathrm{d}y_O$$

where the rotation angle Φ of the Wigner distribution function is connected with the fractional order, *viz.*: $\Phi = \pi p/2$. For a special case $p = 1$, and we obtain the conventional Fourier transform relation. The parameter $f_1 = f\sin\Phi$ is an arbitrary focal length, and f is the focal length of the lens. The ray matrix for a fractional Fourier transform setup can then be written in the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos\Phi & f\sin^2\Phi \\ -\frac{1}{f} & \cos\Phi \end{bmatrix}.$$

In this case the system matrix is obtained by multiplying the matrices that represent the various optical elements within the system. The holographic lens that has a focal length f and straight propagation sections of length z_O and z_I are represented by the diffraction and translation matrices, respectively. Therefore we have

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & z_I \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & z_O \\ 0 & 1 \end{bmatrix}.$$

But for a lens: $a = d = 1$, $b = 0$, and $c = -1/f$; hence

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1-\dfrac{z_I}{f} & z_O+z_I-\dfrac{z_O z_I}{f} \\ -\dfrac{1}{f} & 1-\dfrac{z_O}{f} \end{bmatrix}.$$

In the optical system that realizes a fractional Fourier transform the ray matrix elements A and D are always equal ($A = D = \cos\Phi$). Therefore the input and output distances: z_O, z_I are equal, too; namely $z_O = z_I = f(1-\cos\Phi)$. If the input and output planes are conjugate planes, then the matrix element B is equal to zero making the integral (10) undefined. In this case, we can show only that the field distribution $U_I(x_I, y_I)$ is proportional to distribution $u_O(x_O, y_O)$ in the input plane.

5. Conclusion

In this paper, an insight into optical implementation of Fourier transform is provided. It has been discussed how to simplify the calculations of relations between the amplitude distributions across the input and output planes of an optical system whose ray matrix is known. The diffraction integral is presented as a function of the elements of the ray tracing matrix for an optical system of diferent configurations.

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