Optica Applicata, Vol. XXX, No. 2–3, 2000

Jackson cross cylinder – simple formulation of its optical principles

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The optical principles of an optometric device: the Jackson cross cylinder (JCC) is discussed. The vector representation of the dioptric power is used to visualize the action of the JCC in each step of the clinical procedure for the subjective determination of an astigmatic refractive error.

1. Introduction

The Jackson cross cylinder is an old optometric tool, which even today is widely used in the subjective refraction, mainly in the cylinder axis and power refinement. It is based on an earlier device proposed by Stokes in 1849, which consisted of two cylinders of $+F$ DC and $-F$ DC, one plano-convex and the other plano-concave, with their flat surfaces being nearly in contact with the curved ones and arranged to rotate in opposite directions. Performing in this way the resultant power can be varied from zero, when the axes of the two cylinders are parallel, to a maximum astigmatic power ($+F$ DS/$-2F$ DC), when the axes are perpendicular [1]. The JCC is a Stokes lens composed of cylinders of low power (typically of ± 0.25 DC, or ± 0.50 DC), with their axes crossed at 90°. It is designed to flip around an axis at 45° from the minus and plus axes, so that after one rotation of 180° the plus and minus axes are exchanged [2]. As a clinical tool, the use of the JCC is actually easy and, after the Jackson's description concerning determination of the power [3] and axis [4] of the cylindrical lens necessary to correct astigmatic refractive error, the basic technique has remained nearly unaltered for decades. More recently, CARTER [5] proposed some variations in the methodology for the crossed cylinder test, and later DEL PRIORE and GUYTON [6] provided a reappraisal of the method revealing some important features of the JCC. They used the method proposed by GARTNER [7] of vector addition in a two-dimensional space to support recommendations as to the situation where the JCC may fail to detect astigmatic refractive errors even if they occur in large amounts, how the JCC can be used to refine the cylinder power at an axis made purposefully incorrect, and about the appropriate step sizes in axis rotation that should be used in cylinder axis refinement.

Although the concept of the JCC is quite simple, such questions as: How does it work in the refinement of cylinder axis and power?, have not an obvious answer. The purpose of this paper is to provide an intuitive and graphical response to this question by using a novel approach. In fact, in this work a vector representation of dioptric power in a three-dimensional space [8] – [12] is used for the first time in the analysis of the principles of the JCC.

2. Dioptric power space

An astigmatic power specified by the traditional optometric formulae in terms of sphere, cylinder and axis, as $S/C \times \alpha$, can also be represented (within the limits of paraxial optics, and assuming that the elliptic arc along an arbitrary meridian could be approximated by an arc of circumference) by a continuous function P_θ

$$P_\theta = S + C\sin^2(\theta - \alpha) \tag{1}$$

that describes the power of a spherocylindrical surface along a meridian at an angle θ. This function known as meridional power has been called the curvital power [13] to distinguish it from the power perpendicular to the meridian, called torsional power [14]. Applying trigonometric identities Eq. (1) can be expressed alternatively as

$$P_\theta = S + C\left[\frac{1}{2} - \frac{\cos 2(\theta - \alpha)}{2}\right], \tag{2}$$

$$P_\theta = S + \frac{C}{2} - \frac{C}{2}[\cos 2\theta \cos 2\alpha + \sin 2\theta \sin 2\alpha]. \tag{3}$$

According to Eq. (2) the power along the meridian θ can be separated into two parts: the spherical equivalent power

$$M = S + \frac{S}{2}, \tag{4}$$

and the "pure" cylinder

$$C_\theta = -\frac{C}{2}\cos(2\theta - 2\alpha). \tag{5}$$

The form of Equation (5) suggests the use of a graphical representation of the astigmatic power which is basically the same geometrical construction applied to the addition of harmonic waves. In effect, if we consider the cylinder C_θ as a phasor of magnitude $C/2$ and phase 2α, it can be represented in a plane as a combination of two orthogonal components corresponding to $\theta = 0$ and $\theta = 45°$ (see Eq. (3)), as shown in Fig. 1. Therefore, the specification of an astigmatic power can be done using three parameters, namely the components along the axes C_0 and C_{45} of Fig. 1, and the spherical equivalent power M, which corresponds to the dioptric value of the circle of least confusion. These components, given by equations

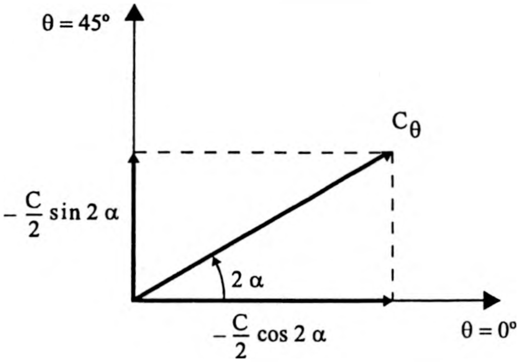

Fig. 1. Phasor diagram for the astigmatic decomposition of the cylindrical power C_θ.

$$
\begin{aligned}
M &= S+\frac{C}{2},\\
C_0 &= -\frac{C}{2}\cos 2\alpha, \qquad (6)\\
C_{45} &= -\frac{C}{2}\sin 2\alpha,
\end{aligned}
$$

have also been derived by DEAL and TOOP [11] and THIBOS *et al.* [12], employing different approaches. They have been used to define a uniform 3-D space where a single point represents the spherocylindrical power P_θ (see Fig. 2). Moreover, a single scalar magnitude characterizing P_θ can be defined as the length of the oblique hypotenuse from the origin to P_θ

$$|P_\theta| = \sqrt{M^2+C_0^2+C_{45}^2}. \qquad (7)$$

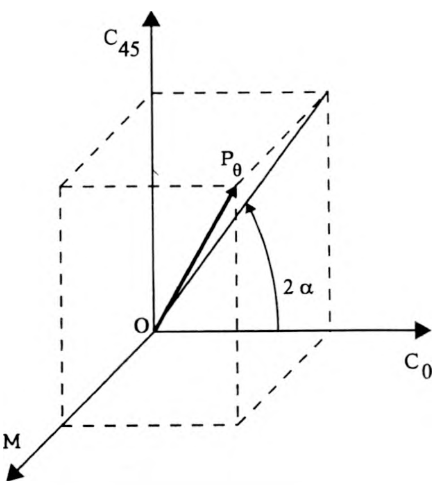

Fig. 2. Spherocylindrical power P_θ represented in the dioptric power space.

The result summarized by Eqs. (6) and (7) is an elegant and useful way to express the classical clinical representation of sphere, cylinder and axis. A complete and formal statement on the nature and mathematical and geometrical representations of dioptric power, as well as a historical overview, can be found in [13].

3. The JCC technique in cases of small or null astigmatic error

BORISH [15] developed the JCC technique for cases in which the objective refractive findings, or other starting points for the subjective procedure, were indeterminate with respect to the cylindrical component of the refractive error. The first step of this technique is to place the circle of least confusion at the outer limiting membrane of the retina. This is done by placing a lens of an adequate spherical power in front of the eye. Next, the patient is requested to compare the legibility of a test chart in the two views provided by flipping the JCC lens. The JCC lens is presented first with the minus-cylinder axis at 180°, and then with the axis at 90°. The visual acuity between the views is compared. If the patient reports no preference, the JCC lens is rotated by 45° and a comparison is then made with the minus-cylinder axis at 45° and at 135°. If the patient again reports no difference between the views, the amount of astigmatism that can be noted subjectively is either zero or is so slight that a cylindrical component to the refractive correction is unnecessary. If, on the contrary, one of the positions of the JCC is reported as providing better acuity than the other, a -0.25 DC correcting cylinder is placed in front of the eye with its axis aligned with the minus-cylinder axis of the JCC. Once located the cylinder, its axis and power must be refined.

This clinical procedure is easily understood by means of the dioptric power representation. To that aim we will only deal with two thin lenses, namely, the lens

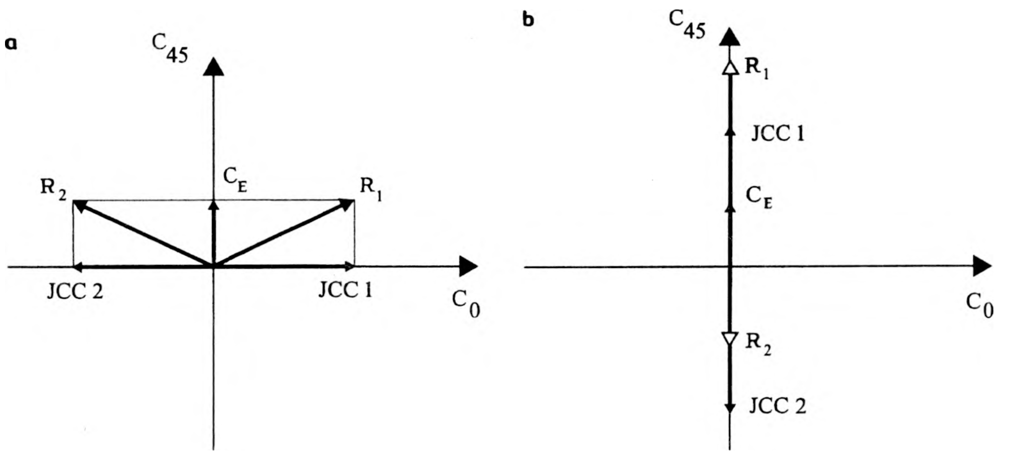

Fig. 3. Detection of a small astigmatic error C_E with the Jackson cross cylinder represented in the JCCs plane of the dioptric power space. JCC1 and JCC2 represent the Jackson cross cylinder. R_1 represents the resultant of C_E+JCC1 and R_2 is the resultant of C_E+JCC2.

that represents the undetected astigmatic refractive error ($C_E \times \alpha$), and the JCC lens. In the framework of the dioptric power space, the first step is equivalent to selection of a plane where the analysis of the JCC operation can be performed by considering a null spherical equivalent power ($M = 0$, in Fig. 2). In other words, in our analysis of the JCC we can ignore the spherical parts of the spherocylindrical powers (M) and consider only the astigmatic parts (C_0 and C_{45}). Let us consider a cylinder C_E with its axis at 45°. In the next step of the procedure, the superposition of the JCC with its axis is established at 180° (JCC1), with the JCC being flipped (JCC2). The resultant *residual* refractive error at each position of the JCC is then the addition of the two vectors: C_E and JCC1, represented as R_1 in Fig. 3a; and C_E and JCC2, depicted as R_2 in the same figure. In our example, as the modulus of the residual refractive error R_1 equals R_2, the patient will not report better acuity for one JCC position upon the other. On the other hand, if the JCC is rotated by 45°, then the patient will report better acuity for the minus-cylinder axis at 135° because $|R_2| < |R_1|$, see Fig. 3b. Consequently, the correcting minus cylinder must be placed with its axis at 135°. It is evident that, only checking the visual acuity for two different JCC positions (90°/180° and 45°/135°) any residual cylinder would be detected.

4. Analysis of the JCC procedure in the axis and power refinement

The use of vector representation of power clarifies the steps embraced in the refinement of both axis and power of an astigmatic correction. To this end we will deal now with three astigmatic powers: the one corresponding to the astigmatic refractive error (correcting cylinder), the trial cylinder lens in front of the eye (erroneous correcting cylinder) and the JCC lens. As before, this JCC procedure is performed, too, maintaining the circle of least confusion on the receptor layer at the outer limiting membrane of the retina, performing the necessary adjustments in sphere power throughout the procedure to maintain a null spherical equivalent power. Then, the graphical analysis is again restricted to the plane of null spherical equivalent power.

As in clinical practice, let us first consider the procedure of refining the cylinder axis. The cylinder axis is determined by placing the $\pm$axes of the JCC lens at an angle of 45° with the correcting cylinder in the refractor.

Suppose that the astigmatic refractive error is ($C_E \times \alpha$), and the trial cylinder lens in front of the eye is ($C_C \times \beta$), that is, the error to be detected with the JCC is $(C_C \times \beta) - (C_E \times \alpha) = (C_C \times \beta) + (-C_E \times \alpha)$. The representation in the plane of the JCCs of the dioptric power space is given in Fig. 4, where it is also represented as an action of the JCC in the two flip positione, *i.e.*, when the negative axis of the JCC is at +45° to the trial cylinder lens, position 1, and when it is at −45°, position 2. Therefore the resultant residual refractive error in each position of the JCC is the addition of the three vectors: $-C_E$, C_C, JCC1, named R_1 in Fig. 5; and $-C_E$, C_C, JCC2, named R_2 in the same figure. If the modulus of the residual refractive error R_1 is lower than R_2, the patient reports better vision when the negative axis of the JCC is in position 1. The optometric procedure of subjective refraction states that the

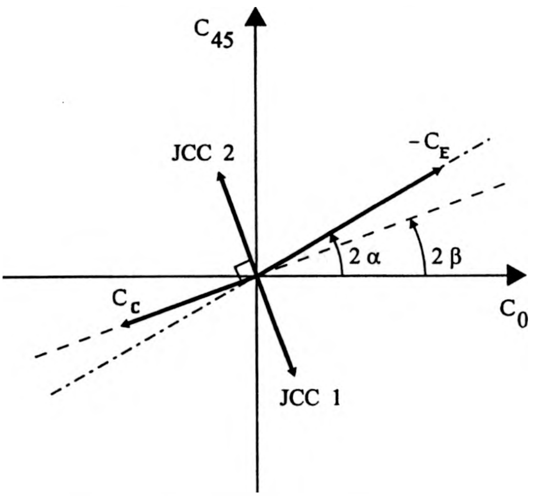

Fig. 4. Cylinder axis refinement with the Jackson cross cylinder represented in the JCCs plane of the dioptric power space. C_E is the astigmatic refractive error. C_C is the erroneous correcting cylinder. JCC1 and JCC2 represent the Jackson cross cylinder with their axes forming $\pm 45°$ with C_C.

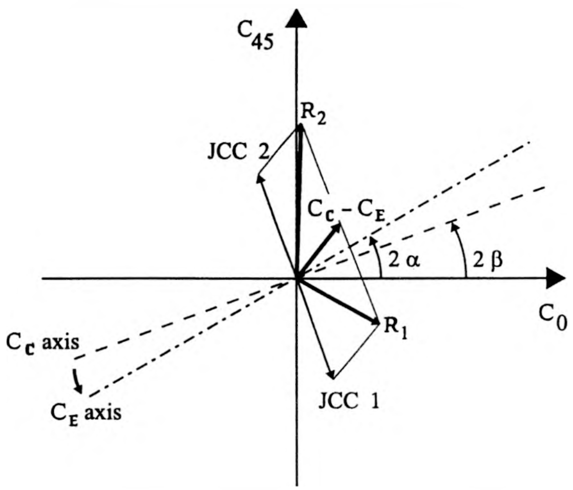

Fig. 5. Same as in Fig. 3, where R_1 represents the resultant of $-C_E+C_C+\mathrm{JCC1}$ and R_2 is the resultant of $-C_E+C_C+\mathrm{JCC2}$.

negative trial cylinder (erroneous correcting cylinder) must be rotated in the direction of the minus axis of the JCC which provided the better image, and the JCC lens is also rotated an equal amount in the same direction. The reason for such an answer is found by considering a reference situation where the axis of the trial cylinder is properly located. In this case, the action of the JCC, as illustrated in Fig. 6, will produce a residual refractive error of the same magnitude ($|R_{01}| = |R_{02}|$) for the two flip positions of the JCC. The patient will show no preference for one flip position over the other because the foci of the resultant astigmatism in each case are

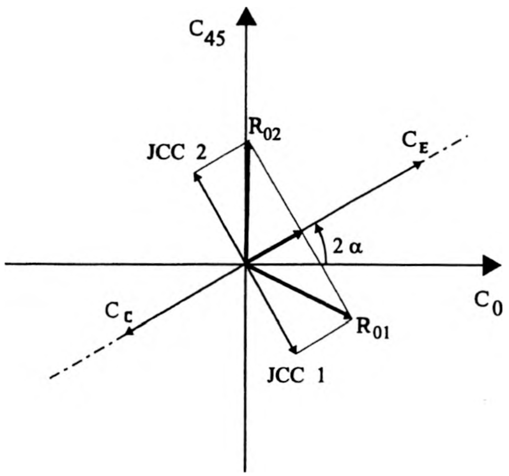

Fig. 6. Reference situation for the proper position of the trial cylinder axis. The resultant R_{01} and R_{02} of the two flip positions of the JCC produce the same blur and the patient will show no preference for one position over the other.

equidistant on each side of the retina but obliquely positioned relative to the eye's original ones. Therefore, in order to transform the situation represented in Fig. 5 into that of reference of equality, it becomes evident that the axis of C_C must be rotated towards the negative axis of the JCC in position 1 which provided a better vision. The angle of rotation of the trial cylinder depends on its magnitude: as its power increases, the suggested initial rotation required diminishes [6]. However, for simplicity, the angle of rotation can be fixed, and the final equality situation is reached after some approaching steps. Each step starts with the negative axis of the JCC at 45° from the C_C axis, so that the rotation of the trial cylinder must be accompanied by a rotation of the same angle of the JCC. In fact, in almost all modern phoroptors the rotation of the JCC is synchronized with the rotation of the trial cylinder.

Continuing with our analysis, let us choose as a first step the rotating angle

$$\beta' = \frac{\alpha - \mathrm{B}}{2}.$$

In this case, the residual refractive error is represented in Fig. 7. Again the position JCC1 provides better vision, which indicates that the axis of the trial cylinder should be rotated even more towards the negative axis of the JCC in position 1. However, this situation is closer to the "reference" situation in Fig. 6 than the previous one in Fig. 5, *i.e.*, $|R_1|$ and $|R_2|$ are more similar. Therefore, it is clear that in a subsequent step equality will be reached, so that the axis of the trial cylinder will be aligned with the axis of the astigmatic refractive error after another rotation of an angle β' of the trial cylinder.

The analysis of the procedure of achieving the trial cylinder power in the dioptric power space is even easier to understand. If the axis is properly determined, the

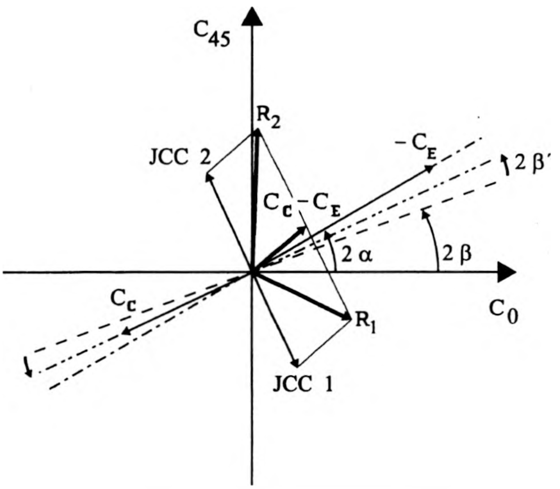

Fig. 7. Same as in Fig 4, but as a result of the new axis setting β' of the trial cylinder. The trial cylinder has been rotated from the position in Fig. 4 towards the minus axis of the JCC position that provided better vision.

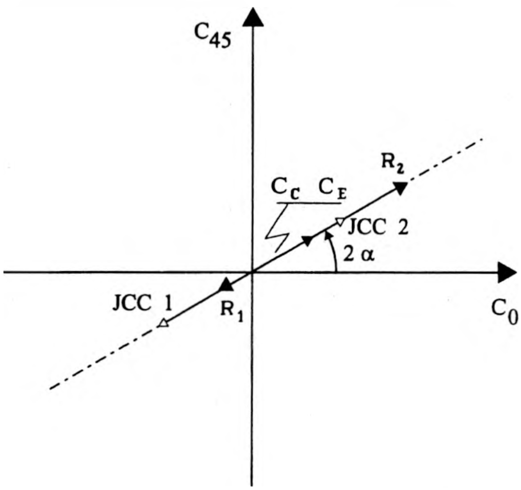

Fig. 8. JCCs plane in the trial cylinder power refinement. The residual refractive errors in the two flip positions are $R_1 = -C_E + C_C + \mathrm{JCC}\,1$ and $R_2 = -C_E + C_C + \mathrm{JCC2}$.

procedure will indicate that the JCC must be rotated until one of its principal meridians is aligned with the C_C axis, and the JCC is flipped. The two flip positions are represented in Fig. 8. In this case, as $|R_1| < |R_2|$, the patient prefers the position 1, where the negative axis of the JCC coincides with the axis of C_C. The power of the negative trial cylinder must then be increased in magnitude. Otherwise, if the subject prefers the position 2, where the negative axis of the JCC is at 90° to the axis of C_C, then the trial cylinder power should be reduced until $|R_1| = |R_2|$.

Therefore, the end point of the JCC test is reached when the modulus of the residual refractive error is equal for the two flip positions.

5. Conclusions

In conclusion, the logical basis of the subjective procedure that uses the JCC in determination of the astigmatic refractive error has been discussed in terms of the astigmatic power decomposition in a 3-D dioptric space. The method used to explain the basic principles of the JCC is simple and intuitive, so it could be asserted that there is no other analysis of the performance of the JCC that shows the results in such an obvious manner that the present study does.

Acknowledgments — This work was supported by the project GV99-100-1-01 of the Conselleria de Cultura, Educació i Ciència, Generalitat Valenciana, Spain. L. Muñoz-Escrivá gratefully acknowledges financial support from the "Cinc segles" Program, Universitat de València, Spain.

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Received February 2, 2000