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Propagation of a low frequency electromagnetic pulse generated by an electric dipole in seawater

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An exact analytic solution for the propagation in seawater of a low frequency electromagnetic pulse generated by an electric dipole is investigated. The dipole is excited by a rectangular current pulse with a finite, nonzero rise and decay time. The frequency-domain formula for the downward-travelling field of a horizontal electric dipole excited by a pulse is Fourier transformed to obtain an explicit expression for the field that is uniformly valid in distance and time. It is noted that the present analysis may be used for studying pulse propagation in any highly conducting medium besides seawater.

1. Introduction

During the last few years, considerable interest has been demonstrated by the electromagnetics community in exploring the propagation of pulses in seawater. When a pulse generated by the current in an electric dipole travels in a dissipative medium the wave number is no longer linear in frequency. As a result the shape of a pulse along with its characteristics (amplitude, duration, rise and decay time) are modified. This is mainly due to the fact that the dipole source creates a field of interest which involves the complete near, intermediate, and far fields. As a consequence, the form of the propagating pulse shifts successively from that of the excitation current and near field to spatial and its time derivatives [1]–[4]. Existing studies on the propagation of a transient electromagnetic wave in seawater are either incomplete or approximate in nature [5]–[14].

In our study, the exact solution for the propagation of a pulse with a nonzero rise and decay time is found. The signal is not modulated. Furthermore, realistic pulses do not extend from $-\infty$ to $+\infty$ in time [1] nor do they exhibit step discontinuities as does the ideal rectangular pulse. Therefore, to study the effect of a nonzero rise and decay time on the transient response is worthwhile attempt. Such a consideration results in the elimination of the delta functions as a useful pulse [4]. The low-frequency approximation is mainly based on the condition $\sigma/\omega\varepsilon \gg 1$, valid for all frequencies of interest in seawater.

2. Definition of the current pulse and its transform

A normalized rectangular pulse with a nonzero rise and decay time can be represented in terms of the step function $U(t)$ as follows:

$$\mathscr{F}(t) = \frac{1}{2t_1}\left\{(1-e^{-\omega_p(t+t_1)})U(t+t_1)-(1-e^{-\omega_p(t-t_1)})U(t-t_1)\right\}. \tag{1}$$

In Equation (1), $2t_1$ is the width of the original rectangle, $\tau_p = 1/\omega_p$ is the rise time, which is taken to be equal to the decay time and $U(.)$ is the unit step function.

Consider an electric dipole immersed in seawater. The electric dipole is excited by a current pulse in A/s of the form

$$I_z(t) = I_0\mathscr{F}(t) = \frac{I_0}{2t_1}\left\{(1-e^{-\omega_p(t+t_1)})U(t+t_1)-(1-e^{-\omega_p(t-t_1)})U(t-t_1)\right\}. \tag{2}$$

The Fourier transform of this pulse is

$$\tilde{I}_z(\omega) = \int_{-\infty}^{\infty} I_z(t)e^{i\omega t}\mathrm{d}t = \frac{I_0 i\omega_p}{\omega+i\omega_p}\frac{\sin(\omega t_1)}{(\omega t_1)}. \tag{3}$$

It is interesting to note that when $\omega_p \to +\infty$, Eq. (3) becomes

$$\tilde{I}_z(\omega) = \frac{I_0}{\omega t_1}\sin(\omega t_1), \tag{4}$$

which is the Fourier transform of the normalized ideal rectangular envelope

$$I_z(t) = \frac{I_0}{2t_1}[U(t+t_1)-U(t-t_1)]. \tag{5}$$

Note that when $t_1 \to 0^+$, $\tilde{I}_z(\omega)$ reduces to

$$\tilde{I}_z(\omega) = \frac{I_0 i\omega_p}{\omega+i\omega_p}, \tag{6}$$

which is the Fourier transform of the normalized exponential pulse,

$$I_z(t) = I_0\omega_p e^{-\omega_p(t+t_1)}U(t+t_1). \tag{7}$$

3. Electric field of a current pulse

The $\hat{z}$-directed, frequency dependent electric field generated by an electrically short dipole with its axis along the z-axis and an electric moment $2h_eI_0$ [Am] is given by [3], (see Fig. 1)

$$\tilde{E}_z(\rho,\omega) = \frac{\mu_0 a h_e \tilde{I}_z(\omega)}{2\pi}\left[\frac{i\omega}{a\rho}+\frac{(i-1)\sqrt{\omega}}{2a^2\rho^2}-\frac{1}{2a^3\rho^3}\right]e^{-a\rho\sqrt{\omega}+ia\rho\sqrt{\omega}}, \tag{8}$$

on the plane $z=0$ perpendicular to the dipole, where h_e [meter] is the effective length of an electrically short dipole with the actual half length h, $\rho = (x^2+y^2)^{1/2}$ is the distance from the center of the dipole, and

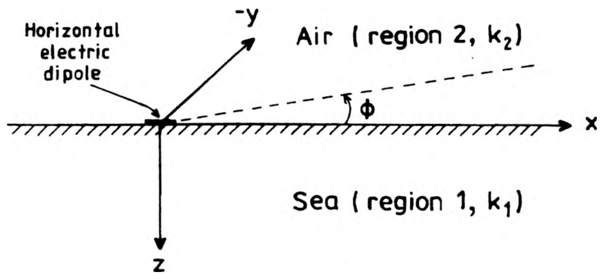

Fig. 1. Horizontal electric dipole on the surface of the sea.

$$a = \left(\frac{\mu_0 \sigma}{2}\right)^{1/2} = (4\pi \times 10^{-7} \times 4/2)^{1/2} = 1.585 \times 10^{-3}. \tag{9}$$

Here, $\sigma \sim 4$ S/m is the conductivity of the seawater in which the dipole is immersed and $\varepsilon_r \approx 80$, the condition $\sigma \gg \omega\varepsilon$ on the frequency is $f \ll \sigma/2\pi\varepsilon_r\varepsilon_0 = 9.0 \times 10^8$ Hz. Since frequencies of 0 – 100 Hz are of interest for the carrier frequency, this condition imposes no practical restriction. The time-dependent field is

$$E_z(\rho,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{E}_z(\rho,\omega) e^{-i\omega t} \mathrm{d}\omega = \frac{\mu_0 a h_e I_0}{8\pi^2 t_1}\int_{-\infty}^{\infty}\left[\frac{i\omega}{a\rho} + \frac{(i-1)\sqrt{\omega}}{2a^2\rho^2} - \frac{1}{2a^3\rho^3}\right]$$
$$\times \left[\frac{\omega_p}{\omega + i\omega_p}\right]\frac{1}{\omega}(e^{2i\omega t_1} - 1)e^{-a\rho\sqrt{\omega}} e^{-i\omega t_1} e^{-i(\omega t - a\rho\sqrt{\omega})} \mathrm{d}\omega. \tag{10}$$

The variables and parameters in expression (10) are conveniently expressed in terms of the dimensionless quantities ρ', t', ω' as follows:

$$\left.\begin{aligned} &a = a'\sqrt{t_1}, \quad t' = t/t_1, \quad \omega = \omega'/t_1, \quad \omega_p = \omega_p'/t_1, \\ &\rho = \rho'/a', \quad a\rho = \rho'\sqrt{t_1}, \quad \omega t = \omega' t'. \end{aligned}\right\} \tag{11}$$

With these changes in variables and notation, Eq. (10) becomes

$$E_z(\rho',t') = \frac{\mu_0 a' h_e I_0}{8\pi t_1^2} A(\rho',t'), \tag{12}$$

where

$$A(\rho',t') = \frac{A_1(\rho',t')}{\rho'} + \frac{A_2(\rho',t')}{\rho'^2} + \frac{A_3(\rho',t')}{\rho'^3}, \tag{13}$$

$$A_j = I_j(\rho', t'-1) - I_j(\rho', t'+1), \quad j = 1,\ 2,\ 3, \tag{14}$$

$$I_1(\rho',\tau') = \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\omega_p'}{(\omega' + i\omega_p')} e^{-\rho'\sqrt{\omega'}} e^{-i(\omega'\tau' - \rho'\sqrt{\omega'})} \mathrm{d}\omega', \tag{15}$$

$$I_2(\rho',\tau') = \frac{i-1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega'_p}{\sqrt{\omega'}(\omega'+i\omega'_p)}\, e^{-\rho'\sqrt{\omega'}} e^{-i(\omega'\tau'-\rho'\sqrt{\omega'})}\mathrm{d}\omega', \tag{16}$$

$$I_3(\rho',\tau') = \frac{-1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega'_p}{\omega'(\omega'+i\omega'_p)}\, e^{-\rho'\sqrt{\omega'}} e^{-i(\omega'\tau'-\rho'\sqrt{\omega'})}\mathrm{d}\omega', \tag{17}$$

$$\tau' = t'+1. \tag{17a}$$

In Equations (15) – (17), each integrand has a branch point at $\omega' = 0$ and a simple pole at $\omega' = -i\omega'_p$ in the lower half-plane. The branch cut is chosen to be along the negative imaginary axis and the path of integration is along the real axis with an indentation about $\omega' = 0$ in the upper half-plane as shown in Fig. 2.

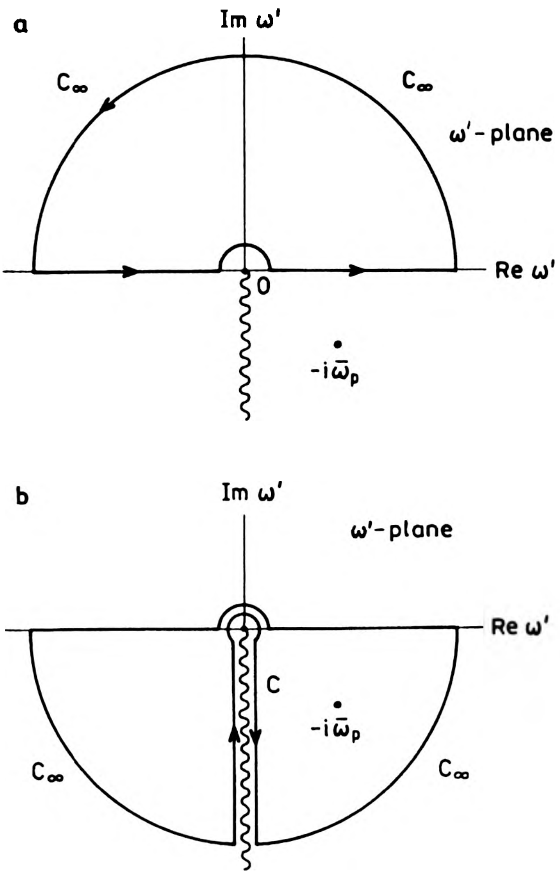

Fig. 2. Contour of integration for the integral $I_3(\rho',\tau')$ when $\tau' < 0$ (a) and $\tau' > 0$ (b). In figure b, part C of the contour encloses both sides of the branch cut in the lower half-plane.

Examination of the integrals I_1, I_2 and I_3 reveals that

$$\left.\begin{aligned} I_1(\rho',\tau') &= 2\frac{\partial I_3(\rho',\tau')}{\partial \tau'} \\ I_2(\rho',\tau') &= -\frac{\partial I_3(\rho',\tau')}{\partial \rho'} \end{aligned}\right\}, \tag{18}$$

by changing the order of integration and differentiation, since $I_1(\rho',\tau')$, $I_2(\rho',\tau')$ and $I_3(\rho',\tau')$ each exists and converges uniformly with respect to τ' and $\rho' > 0$. Therefore it is sufficient to evaluate them only. The evaluation of $I_3(\rho',\tau')$ is given in Appendix.

4. Evaluation of $E_z(\varrho,t)$

Once $I_3(\rho',\tau')$ has been obtained, $I_2(\rho',\tau')$ and $I_1(\rho',\tau')$ may readily be evaluated using Eqs. (18). Finally, by the use of Eqs. (13), (14) and (A36), the following formula has been derived:

$$\begin{aligned} A(\rho',t') = {} & \left(\frac{2}{\pi(t'-1)}\right)^{1/2}\left\{\frac{e^{-\rho'^2/2(t'-1)}}{(t'-1)} - \frac{e^{-\rho'^2/2(t'-1)}}{\rho'^2+2\omega_p'(t'-1)^2}\right. \\ & \left. - \frac{e^{-\rho'^2/2(t'-1)}\rho'^2}{[\rho'^2+2\omega_p'(t'-1)^2](t'-1)} + \frac{8e^{-\rho'^2/2(t'-1)}\omega_p'(t'-1)^2\rho'}{[\rho'^2+2\omega_p'(t'-1)^2]^2}\right\} \\ & - \left(\frac{2}{\pi(t'+1)}\right)^{1/2}\left\{\frac{e^{-\rho'^2/2(t'+1)}}{(t'+1)} - \frac{e^{-\rho'^2/2(t'+1)}}{\rho'^2+2\omega_p'(t'+1)^2}\right. \\ & \left. - \frac{e^{-\rho'^2/2(t'+1)}\rho'^2}{[\rho'^2+2\omega_p'(t'+1)^2](t'+1)} + \frac{8e^{-\rho'^2/2(t'+1)}\omega_p'(t'+1)^2\rho'}{[\rho'^2+2\omega_p'(t'+1)^2]^2}\right\} \\ & + \left(\frac{2}{\pi(t'-1)}\right)^{1/2}\left\{\frac{e^{-\rho'^2/2(t'-1)}}{\rho'^2} - \frac{e^{-\rho'^2/2(t'-1)}}{\rho'^2+2\omega_p'(t'-1)^2}\right. \\ & \left. + \frac{e^{-\frac{\rho'^2/(t'-1)}{(t'-1)}}}{[\rho'^2+2\omega_p'(t'-1)^2]\rho'^2} - \frac{2e^{-\frac{\rho'^2/2(t'-1)}{(t'-1)}}}{[\rho'^2+2\omega_p'(t'-1)^2]^2}\right\} \\ & - \left(\frac{2}{\pi(t'+1)}\right)^{1/2}\left\{\frac{e^{-\rho'^2/2(t'+1)}}{\rho'^2} - \frac{e^{-\rho'^2/2(t'+1)}}{\rho'^2+2\omega_2'(t'+1)^2}\right. \\ & \left. + \frac{e^{-\frac{\rho'^2/(t'+1)}{(t'+1)}}}{[\rho'^2+2\omega_p'(t'+1)^2]\rho'^2} - \frac{2e^{-\frac{\rho'^2/2(t'+1)}{(t'+1)}}}{[\rho'^2+2\omega_p'(t'+1)^2]^2}\right\} \\ & + (1/\rho'^3)\operatorname{erfc}(\rho'/\sqrt{2(t'-1)}) - \sqrt{2(t'-1)/\pi}\,\frac{e^{-\rho'^2/2(t'-1)}}{[\rho'^2+2\omega_p'(t'-1)^2]^2\rho'^2} \\ & - (1/\rho'^3)\operatorname{erfc}(\rho'/\sqrt{2(t'+1)}) + \sqrt{2(t'+1)/\pi}\,\frac{e^{-\rho'^2/2(t'+1)}}{[\rho'^2+2\omega_p'(t'+1)^2]^2\rho'}. \end{aligned} \tag{19}$$

Now substituting Equation (19) in Equation (12) we get the required time-dependent field.

5. Conclusions

The exact time domain solution for an electromagnetic pulse with a finite, nonzero rise and decay time in seawater has been derived. The field consists of two transients, each being the sum of two terms. Further, the results obtained may be applied to remote sensing in seawater, when low frequency pulses are used. These findings may also lead to some useful applications in an environment such as seawater or underground. A similar analysis can be allowed for any highly dissipative medium. The conclusions derived are expected to describe pulse propagation in the human body as well, provided that a proper carrier frequency is chosen for modulation of the low-frequency signal.

Appendix

The purpose of this Appendix is to evaluate $I_3(\rho', \tau')$. To this end, ω'_p is replaced by a complex quantity $\tilde{\omega}_p$, *i.e.*

$$\tilde{\omega}_p = \omega'_p + i\omega'_1 = |\tilde{\omega}_p| e^{i\vartheta}, \tag{A1}$$

where: $\omega'_1 > 0$ and $0 < \vartheta < \pi/2$. Then,

$$-i\tilde{\omega}_p = |\tilde{\omega}_p| e^{-i(\pi/2 - \vartheta)} \quad \text{or} \quad \sqrt{-i\tilde{\omega}_p} = \sqrt{\tilde{\omega}_p}\, e^{-i\pi/4} \tag{A2}$$

where

$$\sqrt{\tilde{\omega}_p} = \sqrt{|\tilde{\omega}_p|}\, e^{i\vartheta/2}. \tag{A3}$$

For $\tau' < 0$, the path of integration may be closed by a large semicircle in the upper half-plane as shown in Fig. 2**a**. It follows that

$$I_3(\rho', \tau') = 0, \ \tau' < 0 \tag{A4}$$

since the function is holomorphic in the upper half-plane. For $\tau' > 0$, the path of integration may be closed in the lower half-plane as shown in Fig. 2**b** and $I_3(\rho', \tau')$ can be written as

$$I_3(\rho', \tau') = I_p(\rho', \tau') + I_b(\rho', \tau') \tag{A5}$$

where I_p is the contribution of the simple pole at $\omega' = -i\tilde{\omega}_p$, namely

$$I_p = e^{-\tilde{\omega}_p \tau'} e^{i\rho'\sqrt{2\tilde{\omega}_p}}. \tag{A6}$$

In Equation (A5), $I_b(\rho', \tau')$ is the contribution of both the branch cut and the branch point

$$I_b(\rho',\tau') = -\frac{1}{2\pi}\int\limits_C \frac{\tilde{\omega}_p}{\omega'(\omega'+i\tilde{\omega}_p)} e^{-\rho'\sqrt{\omega'}} e^{-i(\omega'\tau'-\rho'\sqrt{\omega'})} \mathrm{d}\omega' \tag{A7}$$

ere the contour C encloses the branch cut, upward on the left-hand side and vnward on the right-hand side, and encircles the branch point with a small circle radius δ. In order to simplify the expressions for $I_b(\rho',\tau')$ in Eq. (A7), let $= e^{3i\pi/2}\xi$ on the left-hand side of the branch cut and $\omega' = e^{-i\pi/2}\xi$ on the ıt-hand side of the branch cut. Then, it follows that

$$I_b = -\frac{1}{2\pi}\left\{\lim_{\delta\to 0^+} \int\limits_{C_\delta} \frac{\tilde{\omega}_p}{\omega'(\omega'+i\tilde{\omega}'_p)} e^{-\rho'\sqrt{\omega'}} e^{-i(\omega'\tau'-\rho'\sqrt{\omega'})} \mathrm{d}\omega' \right.$$
$$- \int\limits_0^\infty \frac{\tilde{\omega}_p}{\xi(-i\xi+i\tilde{\omega}_p)} e^{-\xi\tau'} \mathrm{d}\xi \left(e^{-\rho'\sqrt{\xi}(-1+i)/\sqrt{2}} e^{i\rho'\sqrt{\xi}(-1+i)/\sqrt{2}} \right.$$
$$\left. - e^{-\rho'\sqrt{\xi}(1-i)/\sqrt{2}} e^{i\rho'\sqrt{\xi}(1-i)/\sqrt{2}} \right). \tag{A8}$$

Using Cauchy-integral formula for the first integral in Eq. (A8) and on plifying, we obtain

$$I_b = 1 + \frac{1}{\pi}\bar{I}_b(\rho',\tau') \tag{A9}$$

ere

$$\bar{I}(\rho',\tau') = \int\limits_0^\infty \frac{\tilde{\omega}_p e^{-\xi\tau'}}{\xi(\xi-\tilde{\omega}_p)} \sin(\rho'\sqrt{2\xi})\mathrm{d}\xi. \tag{A10}$$

ıltiplying Eq. (A10) by $e^{\tilde{\omega}_p\tau'}$ and taking derivative with respect to τ' of the resulting ression, we obtain

$$\frac{\partial}{\partial\tau'}\left[e^{\tilde{\omega}_p\tau'} \bar{I}_b(\rho',\tau') \right] = -e^{\tau'\tilde{\omega}_p}\tilde{\omega}_p I(\rho',\tau') \tag{A11}$$

ere

$$I(\rho',\tau') = \int\limits_0^\infty \frac{\sin(\rho'\sqrt{2\xi})}{\xi} e^{-\tau'\xi}\mathrm{d}\xi. \tag{A12}$$

evaluate the integral in Eq. (A12), we proceed as follows. Taking the derivative of (A12) with respect to ρ', we obtain

$$\frac{\partial I(\rho',\tau')}{\partial \rho'} = \sqrt{2}\int_0^\infty \frac{\cos(\rho'\sqrt{2\xi})}{\sqrt{\xi}}\,e^{-\tau'\xi}\mathrm{d}\xi$$

$$= \sqrt{2}\,e^{-\rho'^2/2\tau'}\,\mathrm{Re}\int_0^\infty \frac{e^{-(\sqrt{\tau'\xi}-i\rho'/\sqrt{2\tau'})^2}}{\sqrt{\xi}}\mathrm{d}\xi. \tag{A13}$$

Using $\zeta = (\sqrt{\tau'\xi} - i\rho'/\sqrt{2\tau'})$ in Eq. (A13) we get

$$\frac{\partial I(\rho',\tau')}{\partial \rho'} = \frac{2\sqrt{2}}{\sqrt{\tau'}}e^{-\rho'^2/2\tau'}\mathrm{Re}[\bar{I}(\rho',\tau')], \tag{A14}$$

where

$$\bar{I}(\rho',\tau') = \int_{-i\rho'/\sqrt{2\tau'}}^{\infty} e^{-\zeta^2}\mathrm{d}\zeta = \frac{\sqrt{\pi}}{2}\,\mathrm{erfc}\left(\frac{-i\rho'}{\sqrt{2\tau'}}\right), \tag{A15}$$

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\mathrm{d}t = 1 - \mathrm{erf}(z), \tag{A16}$$

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}e^{-s^2}\mathrm{d}s. \tag{A17}$$

Now, for $z = \left(\frac{-i\rho'}{\sqrt{2\tau'}}\right)$ which is pure imaginary and a change of variable $s = \left(\frac{-i\rho'}{\sqrt{2\tau'}}\right)\xi$ in the integral Eq. (A17), it becomes obvious that erf(z) is pure imaginary. Thus, $\mathrm{Re}[\mathrm{erf}(z)] = 0$. Consequently,

$$\frac{\partial I(\rho',\tau')}{\partial \rho'} = \frac{\sqrt{2\pi}}{\sqrt{\tau'}}e^{-\rho'^2/2\tau'}. \tag{A18}$$

From Eq. (A12), $I(\rho' = 0, \tau') = 0$. Hence,

$$I(\rho',\tau') = \frac{\sqrt{2\pi}}{\sqrt{\tau'}}\int_0^{\rho'} e^{-u^2/2\tau'}\mathrm{d}u = \pi\,\mathrm{erf}(\rho'/\sqrt{2\tau'}). \tag{A19}$$

With Eq. (A11), it follows directly that

$$\frac{\partial}{\partial \tau'}\left[e^{\tilde{\omega}_p\tau'}\bar{I}_b(\rho',\tau')\right] = -e^{\tau'\tilde{\omega}_p}\tilde{\omega}_p\pi\,\mathrm{erf}(\rho'/\sqrt{2\tau'}). \tag{A20}$$

By the use of Eq. (A10), the initial value of $\bar{I}_b(\rho',\tau')$ reads

$$\bar{I}_b(\rho' = 0, \tau') = \int_0^{\infty} \frac{\tilde{\omega}_p \sin(\rho'\sqrt{2\xi})}{\xi(\xi - \tilde{\omega}_p)} d\xi. \tag{A21}$$

Let $\sqrt{\xi} = x$. Then,

$$\bar{I}_b(\rho', \tau' = 0) = 2\int_0^{\infty} \frac{\tilde{\omega}_p \sin(\rho'\sqrt{2}x)}{x(x^2 - \tilde{\omega}_p)} dx$$

$$= (\tilde{\omega}_p/2i)\left\{\int_{-\infty}^{\infty} \frac{e^{-i\rho'\sqrt{2}x}}{x(x^2 - \tilde{\omega}_p)} dx - \int_{-\infty}^{\infty} \frac{e^{i\rho'\sqrt{2}x}}{x(x^2 - \tilde{\omega}_p)} dx\right\}, \tag{A22}$$

and the path of integration is properly indented about $x = 0$. These integrals are elementary and can be evaluated by contour integration in the complex plane, where each integrand has three simple poles at 0, $\pm(\tilde{\omega}_p)^{1/2}$

$$\bar{I}_b(\rho', \tau' = 0) = -\pi + \pi e^{i\rho'(2\tilde{\omega}_p)^{1/2}}. \tag{A23}$$

With Eq. (A20), $\bar{I}_b(\rho', \tau')$ is readily evaluated in terms of a new integral

$$\bar{I}_b(\rho', \tau') = \bar{I}_b(\rho', \tau' = 0)e^{-\tilde{\omega}_p\tau'} - \pi\tilde{\omega}_p \int_0^{\tau'} e^{-\tilde{\omega}_p(\tau' - \xi)} \operatorname{erf}(\rho'/\sqrt{2\xi}) d\xi. \tag{A24}$$

If $\tilde{\omega}_p \to \omega'_p$, from Eqs. (A5), (A6), (A9), (A23) and (A24), it follows that

$$I_3(\rho', \tau') = \omega'_p \int_0^{\tau'} e^{-\omega'_p\eta} \operatorname{erfc}[\rho'/\sqrt{2(\tau' - \eta)}] d\eta, \tag{A25}$$

since the result is independent of the position of the branch cut in the lower half-plane.

Next, let $\eta = \tau'\zeta$. Then, Eq. (A25) yields

$$I_3(\rho', \tau') = \omega'_p\tau' \int_0^1 e^{-\omega'_p\tau'\zeta} \operatorname{erfc}[\rho'/\sqrt{2\tau'(1 - \zeta)}] d\zeta \tag{A26}$$

where $\omega'_p\tau' = \omega_p\tau$ and $\rho'/\sqrt{2\tau'} = a\rho/\sqrt{2\tau}$.

Now, let $\Omega = (\omega'_p\tau')^{1/2}$, $R = \rho'/\sqrt{2\tau'}$ and after using integration by parts, we have from Eq. (A26)

$$I_3(\rho', \tau') = \operatorname{erfc}(R) - (R/\sqrt{\pi}) \int_0^1 (1 - \zeta)^{-3/2} \exp\left(-\Omega^2\zeta - \frac{R^2}{1 - \zeta}\right) d\zeta. \tag{A27}$$

Making change of variable $\zeta = (1 - \zeta)^{-1/2}$, we get

$$I_3(\rho', \tau') = \operatorname{erfc}(R) - (2R/\sqrt{\pi}) \exp(-\Omega^2) \bar{I}_3(\Omega, R) \tag{A28}$$

where

$$\bar{I}_3(\Omega, R) = \int_1^\infty \exp(\Omega^2/\xi - R^2\xi^2)\mathrm{d}\xi. \tag{A29}$$

After some straightforward algebra,

$$\exp(\Omega^2/\xi^2 - R^2\xi^2) = \frac{1}{2R}\frac{\mathrm{d}}{\mathrm{d}\xi}\left[e^{-2i\Omega R}\int_0^{R\xi - i\Omega/\xi} e^{-t^2}\mathrm{d}t + e^{2i\Omega R}\int_0^{R\xi + i\Omega/\xi} e^{-t^2}\mathrm{d}t\right],$$

resulting in

$$\bar{I}_3(\Omega, R) = \frac{\sqrt{\pi}}{2R}\mathrm{Re}\left(e^{2i\Omega R}\,\mathrm{erfc}(R + i\Omega)\right). \tag{A30}$$

With Eqs. (A28) and (A30),

$$\bar{I}_3(\rho', \tau') = \mathrm{erfc}(R) - e^{-R^2}\mathrm{Re}[e^{Z^2}\mathrm{erfc}(Z)] \tag{A31}$$

where

$$Z = R + i\Omega. \tag{A32}$$

The locus of $Z = Z(\tau')$ in the complex plane is the hyperbola defined by the equation

$$\mathrm{Re}(Z)\,\mathrm{Im}(Z) = \left(\frac{\omega_p'}{2}\right)^{1/2}\rho' \tag{A33}$$

where $\mathrm{Re}(Z)$, $\mathrm{Im}(Z) > 0$. The minimum distance of this hyperbola from the origin equals

$$|Z|_{\min} = [(2\omega_p')^{1/2}\rho']^{1/2}. \tag{A34}$$

If $|Z|_{\min} \gg 1$, *i.e.*,

$$a\rho(2\omega_p)^{1/2} \gg 1, \tag{A35}$$

then, by the use of Eq. (7.1.23) of [15], $I_3(\rho', \tau')$ can be approximated by the leading term

$$I_3(\rho', \tau') \sim \mathrm{erfc}\left(\frac{\rho'}{\sqrt{2\tau'}}\right) - \left(\frac{2\tau'}{\pi}\right)^{1/2} e^{-\rho'^2/2\tau'}\frac{\rho'}{\rho'^2 + 2\omega_p'\tau'^2}. \tag{A36}$$

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