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OPTIMAL TAX PROGRESSION

In the paper we try to find the optimal tax system. The government has to collect some amount of taxes. The society consists of agents with different wealth. The problem is to find such a tax rate for every level of wealth that the social welfare (measured as the sum of the personal utilities) is maximized. We consider two different models – the static one and the dynamic one. It turns out that the best tax system is progressive. However, if we consider the dynamics, then it comes out that there should be very high tax exemptions for investments.

Keywords: optimal taxation, income tax, optimal control theory

INTRODUCTION

In the paper we deal with the problem of finding a "good" tax system. The question what properties should have fair tax appears in the economic theory all the time and causes hot debates among economists and politicians. We deal with the problem using methods of optimal control theory, this article is an augmentation of Panek, Kliber, 2006. The usage of the optimal control theory differentiates our paper from the most work in this area given in the bibliography. We are sure that formulating the problem in question as an optimal control problem allows us to analyze it more profoundly. In the paper, we deal only with income-tax and do not take into account other taxes.

The paper consists of two parts. In the first one we present a static model. Society consists of many (possibly infinitely many) social groups with different wealth, and the government has to collect some amount of money. We show how it can be done in a way that maximizes social utility (measured as a sum of individual utilities). It turns out that in such a model the taxes should be very progressive. Then we consider heterogeneous society in which there are many different "classes" and each class has a different utility function. We show an example that, if the wealth is positively correlated with risk aversion (i.e., persons who are more risk-averse have on average higher incomes), then the progression should be

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lower than in the previous model. In the second part of the paper we take into account the fact that people can save their incomes and accumulate. We show how this fact changes the solutions from the first part of the paper.

1. STATIC MODEL

We begin with a model for income taxation of citizens. We suppose that the society consists of agents with different wealth. The problem of finding the best tax system consists in finding such a tax progression that maximizes the sum of utilities of all citizens. Thus, we can say that it is *de facto* a problem of optimal redistribution of wealth. In the first part of the article, we neglect the mechanism for generating the incomes and we deal with their redistribution only. We cover the problem of production of wealth in the second part of the article.

We consider two models. In the first one (basic model), all agents have the same utility function. In the second one (extended model), utility functions can vary.

1.1. Basic model

The society consists of agents with different incomes. Let $x \in [0, \infty)$ be the income of an agent. The distribution of wealth in society is described by non-negative function g. The value $g(x)$ means the fraction of the society whose incomes are in the infinitesimal interval $[x, x+dx]$. The fraction of the society with income between x_0 and x_1 is thus given by the integral

$\int_{x_0}^{x_1} g(x)dx$. Of course $\int_0^{\infty} g(x)dx = 1$.

We assume that some part $I(x)$ of the income x is immediately given away to other members of the society. It is a cost of acquiring the income. We make the following assumptions about I :

$$I(0) = 0, \quad 0 < I'(x) < 1, \quad \lim_{x \to \infty} I'(x) = \mu > 0 . \tag{1}$$

Let $f(x)$ be the tax rate for the income x . We assume that income tax is imposed on net income, so that a person with income x pays

$\left(x-I(x)\right)f(x)$. The rest, i.e., $\left(1-f(x)\right)\left(x-I(x)\right)$ remains for his or her disposal. The total amount of collected taxes is

$$N\int_0^\infty \left(x-I(x)\right)f(x)g(x)dx, \tag{2}$$

where N is the population of society. We can (and we do) assume that $N=1$.

Each agent measures his of her well-being with scalar utility function $u(x)$, which fulfils standard assumptions: it is increasing, concave, $\lim_{x\to 0} u'(x)=+\infty$ and $\lim_{x\to+\infty} u'(x)=0$. Social welfare U is measured as a sum of well-being of all agents:

$$U=\int_0^\infty u(x)g(x)dx. \tag{3}$$

The government needs a certain sum of money $A>0$ to be collected. We are looking for such a system of taxation that maximizes the social welfare and allows collecting A. Mathematically, we are looking for a function $f:[0,\infty)\to[0,1]$ which solves the following optimization problem:

$$\max\int_0^\infty u\left(\left(1-f(x)\right)\left(x-I(x)\right)\right)g(x)dx \tag{4}$$

subject to

$$\int_0^\infty \left(x-I(x)\right)f(x)g(x)dx=A. \tag{5}$$

Let the function $y(x)$ be a solution to the differential equation:

$$\frac{dy}{dx}=\left(x-I(x)\right)f(x)g(x).$$

The problem (4)-(5) can be reformulated in this way:

$$\max \int_0^\infty u\big((1-f(x))(1-I(x))\big)g(x)dx, \tag{6}$$

subject to

$$\frac{dy}{dx} = (1-I(x))f(x)g(x), \tag{7}$$

$$y(0)=0, \;\; y(\infty)=A, \tag{8}$$

$$\forall x \;\; f(x)\in[0,1]. \tag{9}$$

The problem (6)-(9) takes the form of the classic optimal control problem, in which the decision variable is f, the state variable is y, and x is a independent variable. The Hamiltonian for this problem is

$$H(f,y,x,\lambda) = u\big((1-f(x))(1-I(x))\big)g(x) + \lambda(1-I(x))f(x)g(x), \tag{10}$$

where (according to Pontriagin Maximum Principle) λ is the solution to a differential equation:

$$\frac{d\lambda}{dx} = -\frac{\partial H}{\partial y}.$$

As we have

$$\frac{\partial H}{\partial y} = 0,$$

thus

$$\frac{d\lambda}{dx} = -\frac{\partial H}{\partial y} = 0.$$

It follows that $\lambda(x) = \lambda = const$.

According to Maximum Principle, for every x the value of f should be such that it maximizes the Hamiltonian (See Fleming and Rishel, 1975 or Leonard and Van Long 1992). The first derivative of H with respect to f equals

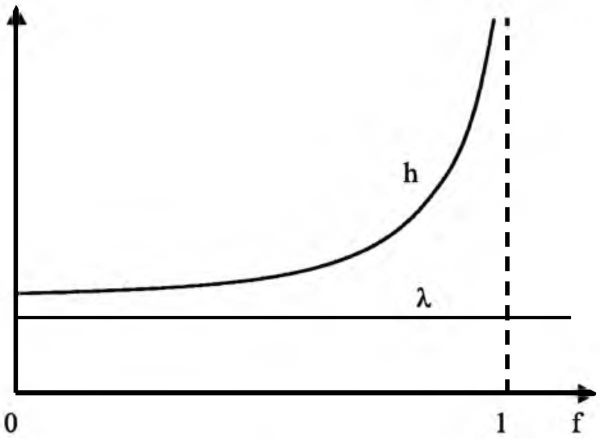
$$\frac{\partial H}{\partial f} = -\left(1 - I(x)\right)u'\left((1-f)\left(1 - I(x)\right)\right)g(x) + \lambda\left(1 - I(x)\right)g(x) =$$
$$= \left(1 - I(x)\right)g(x)\left[\lambda - u'\left((1-f)\left(1 - I(x)\right)\right)\right]$$

If the value of λ is non-positive then for each $x > 0$ and $f \in [0,1]$ we have $\frac{\partial H}{\partial f} < 0$. Thus, for $x \geq 0$ the Hamiltonian attains maximum for $f = 0$ and according to (7)

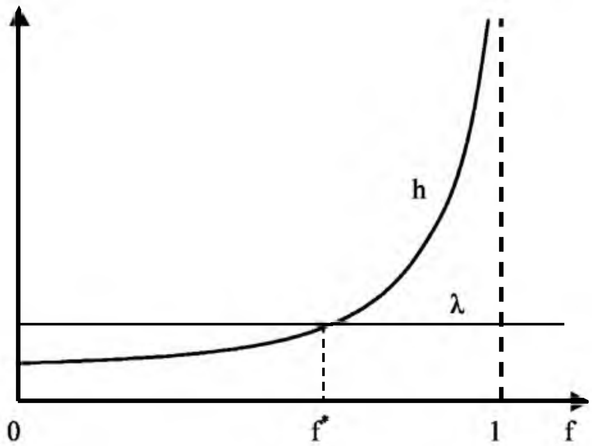
$$\forall x > 0 \quad \frac{dy}{dx} = 1 .$$

As $y(0) = 0$ we obtain that $y(x) = 0$ for all $x > 0$, but we have $y(\infty) = A$ (see (8)). Thus λ must be positive.

Figure 1 contains the graph of the function $h(f) = u'\left((1-f)\left(1 - I(x)\right)\right)$ against $f \in [0,1]$ (given $x > 0$). As we can see, there are two possible cases. In the first one (Fig. 1(a)) the curve $h(f)$ and the line λ do not intersect which means that for all $f \in [0,1]$ we have $u' > \lambda$. The value $\partial H / \partial f$ is negative for $f \in [0,1]$ and thus, the Hamiltonian attains its maximum in the point $f = 0$. The function u' goes from $+\infty$ to 0, so the condition $u'(x) > \lambda$ is fulfilled for small x. This means that low incomes should be tax-free.



(a) Optimal solution $f^* = 0$



(b) Optimal solution $f^* \in (0,1)$

Figure 1. Optimal taxation f^*

Source: own computations

In the second case (Fig. 1(b)), the curve $h(f)$ intersects with λ in the point f^*. The value $\partial H / \partial f$ is positive for $f < f^*$ and negative for $f > f^*$, so the Hamiltonian attains its maximum in f^*. Let ϑ be a reciprocal of marginal utility $\vartheta = (u')^{-1}$, i.e.,

$$\vartheta(y) = x \Leftrightarrow u'(x) = y .$$

In the point f^* we have

$$\lambda = u'\left((1 - f^*)(x - I(x))\right)$$

and thus,

$$f^* = 1 - \frac{\vartheta(\lambda)}{x - I(x)} . \tag{11}$$

As λ is constant, $\vartheta(\lambda) = c > 0$ is also a constant.

The optimal taxation has the following form:

$$f(x) = \begin{cases} 0 & \text{for} \quad x - I(x) \le c, \\ 1 - \dfrac{c}{x - I(x)} & \text{for} \quad x - I(x) > c. \end{cases} \tag{12}$$

The constant $c > 0$ should be such that total amount of taxes equals A, i.e.,

$$\int_{\{x: x - I(x) > c\}} (x - I(x)) \left(1 - \frac{c}{x - I(x)}\right) g(x) dx = A$$

(see (5)).

1.2. Extended model

In the basic version of the model we have assumed that all agents have the same utility function u. Now we drop this assumption. We consider a model in which utility functions can vary. We suppose that the society can be divided into different groups of people indexed by $\theta \in \Theta$, where Θ is the set of all groups. Every agent from the group θ has utility function $u(\theta,\cdot)$. The function $g(\theta,x)$ describes the joint distribution of agents' types and incomes. Of course, $\int_0^\infty \int_\Theta g(\theta,x)d\theta dx = 1$.

The equivalent of the problem (4)-(5) can be formulated as follows:

$$\max_f \int_0^\infty \int_\Theta u\big(\theta,(1-f(x))(x-I(x))\big)g(\theta,x)d\theta dx, \tag{13}$$

subject to

$$\int_0^\infty \int_\Theta (x-I(x))f(x)g(\theta,x)d\theta dx = A. \tag{14}$$

Let us define the function $G(x)$:

$$G(x) = \int_\Theta g(\theta,x)d\theta. \tag{15}$$

As one can see it is marginal distribution of incomes, and the function $U(x,f)$:

$$U(x,f) = \int_\Theta u\big(\theta,(1-f)(x-I(x))\big)g(\theta,x)d\theta. \tag{16}$$

$U(x,f)$ is thus an average utility for agents with income x if the tax rate for these agents equals f. The problem can be reformulated in the following way (see also (6)-(9)):

$$\max \int_0^{\infty} U(x, f(x))\,dx, \tag{17}$$

subject to

$$\frac{dy}{dx} = (1 - I(x))f(x)G(x), \tag{18}$$

$$y(0) = 0,\ y(\infty) = A, \tag{19}$$

$$\forall x\ f(x) \in [0,1]. \tag{20}$$

The Hamiltonian for this problem equals $H = U(x, f) + \lambda(1 - I(x))fG(x)$, where (as in (6)-(9)) λ is a positive constant, and the optimal value of the control variable $f \in [0,1]$ maximizes the Hamiltonian H (given $x > 0$). The first derivative of the Hamiltonian with respect to f equals

$$\frac{\partial H}{\partial f} = \frac{\partial U}{\partial f} + \lambda(1 - I(x))G(x).$$

By the definition of U we have

$$\frac{\partial U}{\partial f} = \int_{\Theta} -(1 - I(x))u'_2(\theta, (1 - f)(1 - I(x)))g(\theta, x)d\theta = -(1 - I(x))AMU(x, f)$$

where $AMU(x, f)$ is the average marginal utility for the agents with income x if the tax rate for this group equals f:

$$AMU(x, f) = \int_{\Theta} u'_2(\theta, (1 - f)(1 - I(x)))g(\theta, x)d\theta =$$

The derivative of the Hamiltonian with respect to f can be now written as

$$\frac{\partial H}{\partial f} = (1 - I(x))[\lambda G(x) - AMU(x, f)].$$

We have thus obtained the following rules for taxation:

(1) The incomes of the people with the lowest average marginal utility AMU should be tax-free (it is possible that these people do not have the lowest incomes). For these agents, for $f = 0$ the derivative $\partial H / \partial f$ could be less then 0 and the optimal solution is $f^* = 0$.

(2) For the rest of the agents, the tax rate should be such that average marginal utility is proportional to the number of agents with specific income:

$$AMU(x, f) \propto G(x). \tag{21}$$

To illustrate these rules let us consider a society which consists of agents with different risk aversion and wealth. The utility function for the agent of type θ is

$$u(\theta, x) = \frac{1}{1-\theta} x^{1-\theta}, \tag{22}$$

so that his or her relative risk aversion equals θ. We also assume that the distribution of risk aversion and wealth in the society follows two-dimensional normal distribution. There is a problem that neither risk aversion nor wealth cannot be negative (and risk aversion in our model should be less then 1). In fact, we should take truncated normal distribution, which would make computation very cumbersome. However, if their variations are sufficiently small, then we can accept the normal distribution as a good approximation - the probability that the value of variable is out of suitable range is negligible. We denote the means of risk aversion and wealth respectively by μ_θ and μ_x and by σ_θ^2 and σ_x^2 we denote their variances. The correlation between θ and x is ρ. As G is marginal distribution of wealth, so $G \propto \mathrm{N}(\mu_x, \sigma_x^2)$. Thus,

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{1}{2} \frac{(x-\mu_x)^2}{\sigma_x^2} \right]. \tag{23}$$

We have to compute AMU for our model. Let us denote by z the income after taxation, $z = x(1 - f(x))$. The marginal utility of income equals $u'_2(z) = z^{-\theta}$. Thus, the AMU is

$$AMU(x,f) = \mathrm{E}\left[(1-f(x))^{-\theta} \mid x\right] = \int_R z^{-\theta} \frac{1}{\sqrt{2\pi}\sigma_{\theta,x}} \exp\left[-\frac{(\theta-\mu_{\theta,x})^2}{\sigma^2_{\theta,x}}\right] =$$

$$= \exp\left[-\mu_{\theta,x} \ln z + \frac{1}{2}\sigma^2_{\theta,x} (\ln z)^2\right],$$

where $\mu_{\theta,x}$ and $\sigma^2_{\theta,x}$ are conditional mean and variations

$$\mu_{\theta,x} = \mu_\theta + \rho \frac{\sigma_\theta}{\sigma_x} (x - \mu_x), \tag{24}$$

$$\sigma^2_{\theta,x} = \sigma^2_\theta (1 - \rho^2) \tag{25}$$

and the last integral is easy to compute with the moment generating function for normal distribution if we only notice that $z^\theta = e^{-\theta \ln z}$.

Using the rule (21) we obtain that

$$\frac{\sigma^2_{\theta,x}}{2} (\ln z)^2 - \mu_{\theta,x} \ln z = -D_x,$$

where

$$D_x = C + \frac{(x-\mu_x)^2}{2\sigma_x^2}. \tag{26}$$

Solving this equation we obtain

$$z = \exp\left[\frac{\mu_{\theta,x} + \sqrt{\mu_{\theta,x} - 2D_x\sigma^2_{\theta,x}}}{\sigma^2_{\theta,x}}\right]. \tag{27}$$

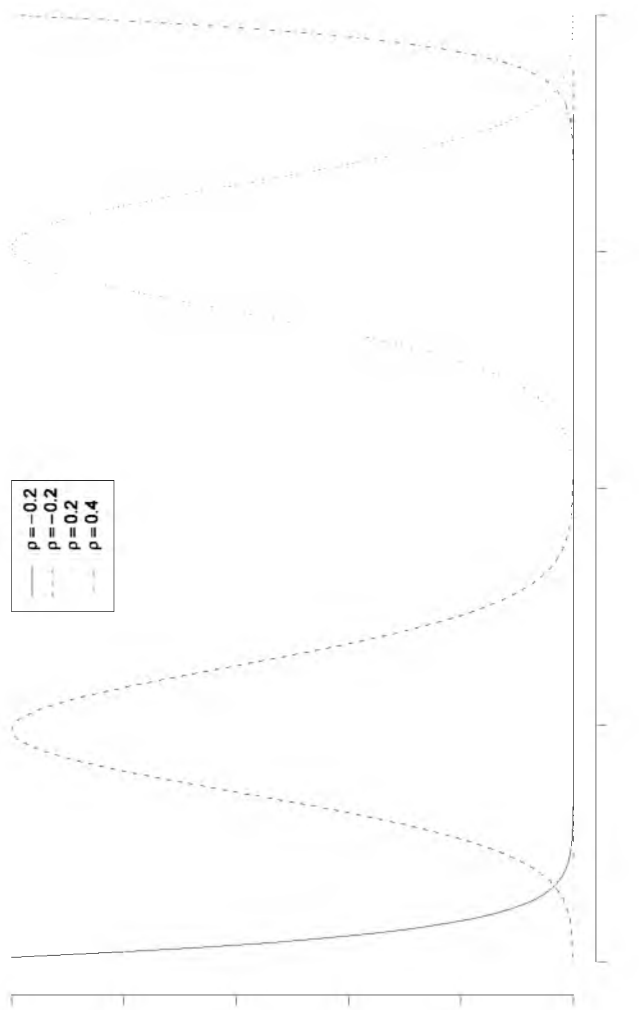


Figure 2. The income after taxation z for normally distributed wealth

Source: authors' own

The figure 2 illustrates this result. It presents the income after taxation for $\mu_\theta = 0.5$, $\sigma_\theta = 0.08$, $\mu_x = 100$, $\sigma_x = 20$ and for different correlations between income and risk aversion.

2. DYNAMICAL MODEL

In the previous section we have derived optimal taxation for the static economy. It is however possible that the short-term solution is not optimal in the long run. The rule (11) gives very high tax progression. It is often claimed that such a solution is not good in the long run, because it lowers growth rate. The agents with higher incomes can invest more and if they have to pay too high taxes, the investments are lower. It can be harmful for long-run welfare. To examine this possibility we construct two models and we form the problem of optimizing social welfare in the long run by choosing tax rates. In the first model we assume that there are a finite number of agents with different incomes. The tax rate for every agent can be different. In the second model we analyze how the distribution of income changes with time. In this model the tax rate depends only on income, not on an agent.

2.1. Simplified version of dynamic model

The society consists of l agents. We consider some fixed period $[0,T]$. Time is a continuous variable and we denote it by t. Let $x_i(t)$ be the income of agent i at the moment t. All agents have the same utility function $u(x)$, which fulfils standard assumptions (the same as in the static model). Let s_i be the investment rate of the agent i. We assume that this value does not change with time. By $f_i(t)$ we denote the tax rate for the agent i in the moment t. Let A denote a total amount of taxes to be collected by government in the period $[0,T]$, i.e.,

$$A = \int_0^T \sum_{i=1}^{l} f_i(t) x_i(t) dt. \tag{28}$$

The wealth of the agent i grows at the rate $a_i s_i$, where a_i is the efficiency of investments of the agent i.

The agent i spends $(1-f_i-s_i)x_i$ on his or her consumption.

Our goal is to choose tax rates f_i for all agents so that they optimize social welfare measured as the sum of individual utilities in the period $[0,T]$ subject to the restriction that government should collect the sum of money A. We can use the same trick as in the static model and introduce the variable y. We can reformulate the problem in the following way:

$$\max_{f_1,\dots,f_l} \int_0^T \sum_{i=1}^{l} u\left((1-f_i(t)-s_i)x_i(t)\right)dt, \tag{29}$$

subject to

$$\dot{x}_i(t) = a_i s_i x_i(t) \qquad \text{for } i = 1,\dots,l, \tag{30}$$

$$\dot{y}(t) = \sum_{i=1}^{l} f_i(t) x_i(t), \tag{31}$$

$$0 \le f_i(t) \le 1 - s_i \qquad \text{for all } t \in [0,T],\ i = 1,\dots,l, \tag{32}$$

$$x_i(0) = x_i^0 > 0, \tag{33}$$

$$y(0) = 0, \quad y(T) = A. \tag{34}$$

The value x_i^0, $i = 1,\dots,l$ denotes the initial income of the agent i.

The Hamiltonian for the problem (29)-(34) is

$$H(\boldsymbol{\lambda},\mathbf{x},y,\mathbf{f}) = \sum_{i=1}^{l} u\left((1-f_i-s_i)x_i\right) + \sum_{i=1}^{l}\left(\lambda_i a_i s_i x_i + \lambda_{l+1} f_i x_i\right), \tag{35}$$

where $\mathbf{x}$ is l-dimensional vector of incomes of the agents, $\mathbf{x} = (x_1, x_2,\dots,x_l)$, $\mathbf{f} = (f_1, f_2,\dots,f_l)$ is the vector of tax rates, and $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\dots,\lambda_l)$ is an $(l+1)$-dimensional vector of dual variables. The dynamics of the later is described by the following system of differential equations:

$$\dot{\lambda}_i(t) = -\frac{\partial H}{\partial x_i} = -(1 - f_i(t) - s_i)u'((1 - f_i(t) - s_i)x_i) - a_i s_i \lambda_i(t) - \lambda_{l+1} f_i(t) \quad . \quad (36)$$

The dual variable λ_{l+1} is a positive constant.

According to Maximum Principle, $\mathbf{f}$ maximizes the Hamiltonian (35). The derivative of the Hamiltonian with respect to f_i equals:

$$\frac{\partial H}{\partial f_i} = -x_i u'((1 - f_i - s_i)x_i) + \lambda_{l+1} x_i = x_i[\lambda_{l+1} - u'((1 - f_i - s_i)x_i)]. \quad (37)$$

The sign of the derivative depends on the expression in the square brackets. It is easy to obtain optimal tax rate f_i^* by analyzing the graph of λ_{l+1} and $h(f) = u'((1 - f_i - s_i)x_i)$. The graphs were shown in Fig. 1. If the income of agent i is low or his/her investment rate s_i is high, then the derivative $u'((1 - s_i)x_i)$ is greater then λ_{l+1}. In this case, the optimal tax rate is $f_i^* = 0$. On the other hand, if $u'((1 - s_i)x_i)$ is lower then λ_{l+1}, then the optimal tax rate is in the interval $(0,1)$. It is the point in which the derivative (37) equals zero. Notice that this tax interval is the same for all agents and it depends on the consumption share in income. Let ϑ be reciprocal of u'. Then we can reformulate this condition in the following way:

$$(1 - f_i^* - s_i)x_i = \vartheta(\lambda_{l+1}). \quad (38)$$

The $\vartheta(\lambda_{l+1})$ on the right-hand side is constant. We denote it by c. Modifying (38) we obtain:

$$f_i^*(t) = \begin{cases} 0 & \text{for} \quad (1 - s_i)x_i \le c, \\ 1 - \dfrac{c}{x_i} - s_i & \text{for} \quad (1 - s_i)x_i > c. \end{cases} \quad (39)$$

The optimal taxation has the following characteristics. Agents with lowest income or with very high investment rate should be free of taxes. Crucial is what part of the income of the agent is devoted to consumption. The agents with a higher income or lower investment rate should pay taxes according to the rule (39).

The rule for optimal taxation can be interpreted in two ways. Firstly, the optimal tax rate f_i^* depends on income x_i and on investment rate s_i. Two agents with the same income can have different tax rates if their investments are different.

Notice that the rule (39) can be written as follows:

$$w_i^*(t) = \begin{cases} s_i & \text{for} \quad (1-s_i)x_i \le c, \\ 1-\dfrac{c}{x_i} & \text{for} \quad (1-s_i)x_i > c, \end{cases} \tag{40}$$

where $w_i^*(t) = f_i^*(t) + s_i$. Thus $(1 - w_i^*(t))x_i$ is the consumption of the agent i and $1 - w_i^*(t)$ is his or her consumption rate. The rule (40) is similar to the rule (11) in the static model (with $I(x) = 0$). The optimal value of w_i^* (i.e., tax rate augmented for investment rate) depends only on income x_i.

According to the rule, the value w_i should be progressive. The taxes do not have to be progressive but the tax system should take individual investments into account.

2.2. The full version of the dynamic model

In the model in the previous section, tax rate $f_i(t)$ could change with time and it could be different for each agent. In a real tax system the tax rate should depend on incomes. It is better if the tax brackets do not change with time. In this section we analyze a model which has these features. The starting point is the basic version of a static model.

Let $g(t,x)$ be a distribution of incomes at the moment t, where $t \in [0,T]$. We assume that investment rate for the agent with income x equals $s(x)$. The income grows at the rate $as(x)$, where $a > 0$ is efficiency of investments. All agents have the same utility function $u(x)$, which fulfils standard assumptions. We are looking for taxation which maximizes social welfare in the long run on condition that government collects $A > 0$. We must first derive dynamics of the income distribution. Let $G(t,x)$ be cumulative distribution of incomes at the moment t, i.e.,

$$G(t,x) = \int_0^x g(t,y)dy. \tag{41}$$

The value $G(t,x)$ shows how many people have incomes lower than x at the moment t. Consider infinitesimal change of time dt. The income of every agent changes according to the following differential equation:

$$\dot{x}(t) = as(x(t))x(t). \tag{42}$$

In the infinitesimal interval $[t, t+dt]$ the increase of the income equals $dx = asx(t)dt$.

Take any number $x > 0$. We want to obtain $G(t+dt, x)$, i.e. the number of people whose income at the moment $t+dt$ is lower then x. Notice that if at the moment t income is greater then $x - dx$, then at the moment $t+dt$ it will be greater then x. Thus,

$$G(t+dt,x) = G(t, x-dx) = G(t, x - as(x)xdt). \tag{43}$$

Subtracting $G(t,x)$ from both sides of the equation (43) we get

$$[G(t+dt,x) - G(t,x)] = G(t, x - as(x)xdt) - G(t,x). \tag{44}$$

Dividing both sides of (44) by dt and taking limit (dt tends to zero) we obtain the following partial differential equation:

$$\frac{\partial G(t,x)}{\partial t} = -as(x)x\frac{\partial G(t,x)}{\partial x}. \tag{45}$$

According to (41) we have $g(t,x)=\frac{\partial G(t,x)}{\partial x}$. To obtain the dynamics of g we have to differentiate both sides of (45) with respect to x. Finally, we obtain the following equation for the function $g(t,x)$:

$$\frac{\partial g(t,x)}{\partial t}+as(x)x\frac{\partial g(t,x)}{\partial x}+a\left(s'(x)x+s(x)\right)g(t,x)=0.$$

We want to maximize social welfare in the time interval $[0,T]$ on condition that the government collects amount of money $A>0$ as taxes. The problem can be formulated in the following way:

$$\max_f \int_0^T\int_0^\infty g(t,x)u\left(\left(1-s(x)-f(x)\right)x\right)dxdt \tag{46}$$

subject to

$$\frac{\partial g(t,x)}{\partial t}+as(x)x\frac{\partial g(t,x)}{\partial x}+a\left(s'(x)x+s(x)\right)g(t,x)=0, \tag{47}$$

$$\int_0^T\int_0^\infty g(t,x)f(x)xdxdt=A, \tag{48}$$

$$f(x)\in\left[0,s(x)\right] \quad \text{for all } x\in[0,\infty), \tag{49}$$

$$g(0,x)=g_0(x), \tag{50}$$

where $g_0(x)$ is the initial distribution of incomes.

It can be proved that the solution to the problem (46)-(50) is the following tax rate:

$$f^*(x) = \begin{cases} 0 & \text{for} \quad x - s(x) \le c, \\ 1 - \dfrac{c}{x} - s(x) & \text{for} \quad x - s(x) > c, \end{cases}$$

where constant c depends on A. Let $I(f)$ be a value of integral (46) for function $f(x)$. Consider the difference $I(f^*) - I(f)$, where f is any feasible solution of the problem (46)-(50). We have:

$$\begin{aligned} I(f^*) - I(f) &= \int_0^T \int_0^\infty g(t,x)\left[u\left((1 - s(x) - f^*(x))x\right) - u\left((1 - s(x) - f(x))x\right)\right] dx dt \ge \\ &\int_0^T \int_0^\infty g(t,x) u'\left((1 - s(x) - f^*(x))x\right)\left(f^*(x) - f(x)\right)x \, dx dt = \\ &\int_0^T \int_{\{f^*(x)>0\}} g(t,x) u'(c)\left(f^*(x) - f(x)\right)x \, dx dt + \int_0^T \int_{\{f^*(x)=0\}} g(t,x) u'\left((1 - s(x))x\right)\left(-f(x)\right)x \, dx dt \ge \\ &\int_0^T \int_{\{f^*(x)>0\}} g(t,x) u'(c)\left(f^*(x) - f(x)\right)x \, dx dt + \int_0^T \int_{\{f^*(x)=0\}} g(t,x) u'(c)\left(-f(x)\right)x \, dx dt = \\ &\int_0^T u'(c) \int_0^\infty g(t,x) f^*(x) x \, dx dt - \int_0^T u'(c) \int_0^\infty g(t,x) f(x) x \, dx dt = A - A = 0 \end{aligned}$$

In the first inequality, we used the fact that u is concave and in the second one, we used the fact that if $f^*(x)$, then $\left(1 - s(x) - f^*(x)\right)x \le c$.

Any feasible solution f is not better then f^*. Thus f^* is the optimal solution of (46)-(50). Like in the first version of the dynamic model, the optimal tax rate has the characteristic that the sum of tax rate and investment rate grows progressively.

CONCLUSIONS

Using optimal control theory it is possible to calculate optimal taxation, which maximizes social welfare and allows government to collect a given sum of money. It turns out that if all members of the society share the same utility function then optimal (according to neoclassical rules) tax system is highly progressive. After taxation all agents have an equal income. This conclusion is weakened if we allow the possibility that utility functions vary among agents. Under this assumption the optimal taxation is determined by the rule that the average marginal utility should be proportional to the number of agents with specific income. If the agents with higher income are not numerous and if they are less risk-averse then others, then the tax rate for them can be lower then in the previous model. Nevertheless, even in this model the optimal tax system is progressive.

The results do not change significantly if we take into account the dynamics of incomes. Even then the optimal tax system is progressive, but the progression concerns the *sum* of savings and taxes. So the best tax system should strongly encourage investments. It should be noticed that this encouragement should be even stronger than tax exemptions for investment. By tax exemptions the money spend on investments are tax-free, while in the optimal taxation they are *equivalent* to taxes.

To sum it up: the optimal tax system should have following "Calvinistic" (Weber, 1976) characteristics: (1) it should favour egalitarianism - all agents should spend the same amount of money for consumption (so it limits excessive consumption), (2) it should encourage savings (investments).

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