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On a Possibility of Aberration Determination for the Electrostatic Lenses with the Help of Resistance Network

The transformation of the electron trajectory equation for the electrostatic field with a rotational symmetry is given, allowing the numerical calculation of the nonparaxial trajectory based on the potential distribution $U_0(z)$ measured along the symmetry axis of the system. The positions of the foci, focal lengths, principal planes and the constants of the spherical C_{sph}, and chromatic C_{chr} aberrations as well as distortion C_d are determined for the immersion lenses of two types. The results prsented in the form of tables and graphs are compared with those given by other authors. It is pointed out that the typical resistance network makes it possible to determine some aberrations of the electron-optical systems.

I. Introduction

A complete description of a nonrelativistic electrostatic electron-optical system with a rotational symmetry requires the solving of a nonlinear differential equation

$$r'' - \frac{1+(r')^2}{2U(z,r)}\left[\frac{\partial U(z,r)}{\partial r} - r'\frac{\partial U(z,r)}{\partial z}\right] = 0. \qquad (1)$$

where

z — coordinate along the symmetry axis,

r — radial coordinate

$U(z,r)$ — potential distribution function.

The symbol$'$ denotes differentiation with respect to the z coordinate.

The analytical solution of equation (1) is known for a few special cases only which in general are of no practical importance. The reason for that is not only the impossibility of solving the trajectory equation but also the lack of an analytical form of the actual potential distribution $U(z,r)$, the consequence of the difficulties appearing when solving analytically the boundary problems of the Laplace type.

The research centres working on optical electronics are usually equipped with the resistance networks, however the research done with the help of these networks concerns mainly the electrostatic field distribution rather than the trajectory determination. In the available literature no paper has been found by the author treating of the aberration determination on the base of a typical resistance network. The reason seems to lie in a common opinion that the errors due to discretization (unavoidable in any network) do not allow the trajectory determination with the accuracy sufficient for the aberration examination.

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II. Numerical method of the electron trajectory determination applicable for the case of discrete potential distribution in the resistance network

Assuming the Gaussian optics approximation the trajectory equation may be written in the form of the Busch equation

$$4U_0 r_0'' + 2U_0' r_0' + U_0'' r_0 = 0, \qquad (2)$$

$$r_0 \neq 0, \; U_0 \neq 0$$

where $U_0 = U_0(z)$ denotes the potential distribution along the symmetry axis,

$r_0 = r_0(z)$ is the radial coordinate of the paraxial trajectory.

In the earlier papers [5] and [6] was given transformation, which reduces the Busch equation to the form

$$u' = -(u^2+p^2), \qquad (3)$$

where the auxiliary function $u = u(z)$ and the parameter p are determined by the following relations

$$u = \frac{r_0'}{r_0} + \frac{1}{4}\frac{U'}{U_0} \qquad (4)$$

$$p = \frac{\sqrt{3}}{4}\frac{U_0'}{U_0}. \qquad (5)$$

Form (3) of the equation is particulary convenient for numerical calculations when applying the resistance network.

The method of the auxiliary function may be generalized in such a way that the nonparaxial cases can be also treated. Expending both the potential function and its derivatives into series we may reduce the general trajectory equation (1) to the form

$$2U_0r''+\frac{r}{2}U_0''+r'U_0'+r'^2\frac{r}{2}U_0''+r'^3U_0'+\frac{r^3}{8}\frac{(U_0'')^2}{U_0}+\frac{r^2}{4U_0}r'U_0''U_0'=0. \tag{6}$$

by neglecting the terms of the order greater than the second. It may be easily noticed that the first three terms in equation (6) are identical with the right hand side of the Busch equation, and thus equation (6) may be rewritten in the form

$$4U_0r''+2U_0'r'+U_0''r+\Delta=0, \tag{7}$$

where

$$\Delta=2\left(r'^2+\frac{r^2}{4}\frac{U_0''}{U_0}\right)\left(\frac{r}{2}U_0''+r'U_0'\right),$$

may be considered as a correction term. Consequently r may be replaced by its paraxial approximation r_0.

From the Busch equation

$$r_0'U_0'+\frac{r_0}{2}U_0''=-2U_0r_0'',$$

thus

$$\Delta=-4U_0r_0''\left(r_0'^2+\frac{r_0^2}{4}\frac{U_0''}{U_0}\right). \tag{8}$$

When transforming the Busch equation into the form $u'=-(u^2+p^2)$ the equation was divided by $4U_0r_0$, hence the correction term has also to be devided by that expression. Then equation (7) takes the form

$$u'=-(u^2+p^2)+a, \tag{9}$$

where:

$$a=\frac{r_0''}{r_0}\left(r_0'^2+\frac{r_0^2}{4}\frac{U_0''}{U_0}\right),$$

The aberration correction term a_k (where k numbers the network knots along the symmetry axis, the index o being ommited for the sake of simplicity) may be determined from the differential expression

$$a_k=\frac{r_{k+1}+r_{k-1}-2r_k}{r_k}\left[(r_{k+1}-r_k)^2+\frac{r_k^2(U_{k+1}+U_{k-1}-2U_k)}{4U_k}\right]. \tag{10}$$

The differential form of (10) has been derived from the forward differences by assuming the elementary mesh-length Δz to be equal to 1 (square mesh). The knowledge of the potential distribution $U_0(z)$ along the symmetry axis enables the calculation of the value of the paraxial ray r_{0k} as well as of the correction term a_k. The correction terms a_k render possible to determine the new values of the function u_k according to equation (9) and thus the new values of the trajectory coordinates r_k and the local trajectory inclinations r_k'.

The numerical method based on the relations (3) and (9) may be applied to determining some aberration characteristics and aberration constants. In particular the constants of spherical (C_{sph}), chromatic (C_{chr}) and distortion (C_d) aberrations can be evaluated.

III. Error estimation of the $u(z)$ method

The increase of the function $u(z)$ resulting from the corresponding increase Δz of the variable z is equal to

$$\Delta_1 u=-(u_0^2+p_0^2)\Delta z, \tag{11}$$

Representing the $u(z)$ function by the first three terms of its expansion into series

$$u(z_0+\Delta z)\approx u(z_0)+u'(z_0)\Delta z+\frac{u''(z_0)}{2!}\Delta z^2 \tag{12}$$

we get, according to (11), the following expressions for the absolute error of the function $u(z)$

$$\delta u\approx\frac{u''(z_0)}{2}\Delta z^2. \tag{13}$$

Applying a similar approximate representation of the function $U_0(z)$ we obtain

$$\delta U_0'\approx\frac{U_0''(z_0)}{2}\Delta z. \tag{14}$$

Hence

$$\delta p\approx\frac{\sqrt{3}}{4}\frac{\delta U_0'}{U_0}\approx\frac{\sqrt{3}}{8}\frac{U_0''(z_0)}{U_0}\Delta z. \tag{15}$$

Since

$$p'=\frac{\sqrt{3}}{4}\left[\frac{U_0''}{U_0}-\left(\frac{U_0'}{U_0}\right)^2\right],$$

$$\frac{U_0''}{U_0}=\frac{4}{\sqrt{3}}p'+\frac{16}{3}p^2,$$

then

$$\delta p\approx\left(\frac{1}{2}p'+\frac{2}{\sqrt{3}}p^2\right)\Delta z. \tag{16}$$

From (4) it is clear that

$$\delta\left(\frac{r'}{r}\right) = \delta u - \frac{\delta p}{\sqrt{3}},$$

Substituting into the last equation (13) and (16), we obtain the following expression for the local error

$$\delta\left(\frac{r'}{r}\right) \approx \frac{u''(z)}{2}\Delta z^2 - \left(\frac{1}{2\sqrt{3}}p' + \frac{2}{3}p^2\right)\Delta z. \quad (17)$$

To estimate the total error it is necessary to divide the local error by Δz and integrate it over the region $[0, z]$. For the case $\Delta z = 1$ we have

$$\Delta\left(\frac{r'}{r}\right) \approx \frac{1}{2}\left[\left(u' - \frac{p}{\sqrt{3}}\right)\Big|_0^z - \frac{4}{3}\int_0^z p^2 dz\right]. \quad (18)$$

The determination of the error $\Delta\left(\frac{r'}{r}\right)$ is thus reduced to the substitution of the limiting values of u' and p, followed by the evaluation of the integral in (18). An approximate estimation of the integral may be easily performed by summing up the values p^2 appreciated during the trajectory determination with the help of the function $u(z)$.

On the base of the relation

$$\frac{\Delta\left(\frac{r'}{r}\right)}{\frac{r'}{r}} = \frac{\Delta r'}{r'} - \frac{\Delta r}{r}. \quad (19)$$

we can estimate the difference of the relative errors of r and r'.

The separation of both errors requires a more complex calculation. However, it has to be noticed that, in practice $\frac{\Delta r'}{r'}$ is usually much greater than $\frac{\Delta r}{r}$ so that the right hand side of (19) gives an approximate evaluation of $\frac{\Delta r'}{r'}$.

Some considerable relative errors occur when the $u(z)$ method is used in the region closest to the axis. In that case $\frac{r'}{r} \gg \frac{p}{\sqrt{3}}$ and equation (3) takes the form $u' = -u^2$ with the solution $u = \frac{1}{z-z_0}$. The trajectory is then described by the equation $r = A(z-z_0)$, where A is a constant.

Being estimated numerically the function $u(z)$ never reaches the value ∞ at the point $z = z_0$, it tends, however, to this limit by an unlimited number of steps Δz. This is one of the sources of the error independently of the errors resulting from the application of the numerical method based on the knowledge of the function $u(z)$. Even if the exact values of the function $u(z)$ are used, the application of the numerical method in the case when

$$\frac{r'}{r} = \left(u - \frac{p}{\sqrt{3}}\right) < -1$$

results in negative values of the radius r already in the second step, while in the next steps an oscilation about the z-axis appears. This part of the calculation has no physical meaning. To avoid it the calculation should be interrupted within this region and the region should be overjumped e.g. by way of the linear extrapolation.

To check the accuracy of the $u(z)$ method the trajectories for several potential distributions have been determined for the cases when the analytical solutions of the track are known (see [2], [7]). These trajectories have been calculated for a relatively gross discretization, by dividing the examined segment of the track into ten parts. The errors evaluated in this way may be considered as the upper value estimation of the error because the discretization applied is usually much better (20 to 50 steps).

Table 1

Illustrates the results obtained

$\frac{U_f}{U_i}$	r'_i	$\frac{\Delta r_f}{r_f}$
< 2	< 0.01	$< 0.25\%$
	< 0.1	$< 1.5\%$ (except the case of exponential distribution)
< 5	< 0.01	$< 0.25\%$
	< 0.1	$< 5\%$
< 10	< 0.1	$< 15\%$

where: U_i — initial potential,
U_f — final potential,
r_i' — initial inclination of the trajectory,
r_f — final inclination of the trajectory.

The initial value of the radius r_i has been assumed to be equal to 1, while the initial inclination in all the cases has been assumed $r_i' > 0$. The potential distribution $U_0(z)$ was an increasing function. The error rose with the increase of the ratio $\frac{U_f}{U_i}$ and r_i'.

For the exponential distribution a very great error $\frac{\Delta r_f}{r_f} = 344\%$ was established. In that case the end of the track was very close to the focus and this great value was connected with the very small value of r_f (compare the assumption $r \neq 0$ and $U_0 \neq 0$ in eq. (2)). A testing was also made in the opposite direction (when the distribution $U_0(z)$ along the track was diminishing and $r_i' < 0$). The relative error of r in the opposite direction was on average 1.5 times greater than in the original direction. The relative inclination error was on average several times greater that the relative error of the radius r (about 5 times in the normal and about 2.5 times in the opposite direction). It can be concluded that the relative error of inclination in the opposite direction is on average about 1.3 times less than that for the original direction.

When calculating the said average error ratios some limiting cases were not taken into account.

IV. Examples of both the parameters and aberrations determination for electrostatic lenses

The exact estimation of the location errors of the foci, principal planes, focal lengths, aberration constants C_{sph}, C_{chr} and C_d is difficult as the separation of the errors $\frac{\Delta r'}{r'}$ and $\frac{\Delta r}{r}$ is then required. The purpose of the testing was to evaluate the errors of the numerical method suggested above and it may be considered as a basis for some qualitative statements. The total error of the calculated magnitude consist of 1) the error of the $u(z)$ method, 2) the error of the discretization introduced by the resistance network and 3) the error of the potential measurement.

Unfortunately, in the available literature no estimation procedure has been found, which could be used for the accuracy evaluation of the result obtained by the author. In this situation it has been decided to compare the author's own result with those of the other authors. For this purpose two kinds of lenses, thoroughly examined in the literature (see [1], [3], [4], [8]), have been chosen. These are the immersion two-tube lenses with the radia satisfying the condition $\frac{R_2}{R_1} = 1$ or $\frac{R_2}{R_1} = 1.5$ and of several different voltages ratio.

The measurements of the voltage distribution $U_0(z)$ in the examined lenses were made with the help of resistance network, which are the property of the

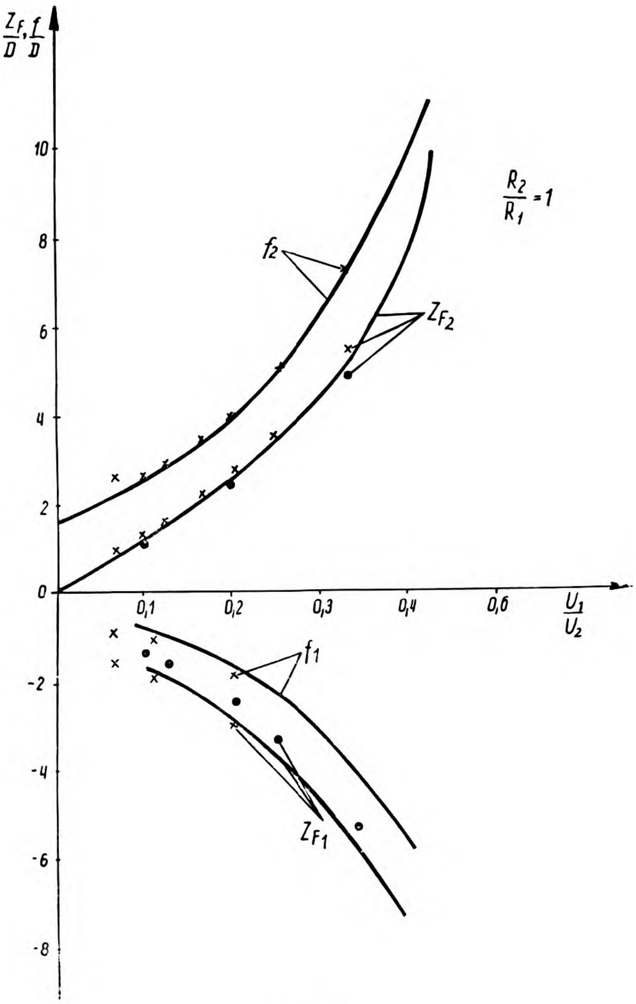

Fig. 1

—— ARDENNE • • KIELMAN × × MULAK

Institute of Electronic Technology Wrocław, Technical University (Poland). The number of loops in the measuring part of the network was equal to 20 in the radial direction and 50 in the direction of the symmetry axis, respectively. The network is made of the composition resistors (the resistance accuracy is 2 per cent). An average error of the potential distribution measurements for testing the network with the help of a plane condenser amounts to 0.05 per cent, while the accuracy of the testing with a cylindric condenser is 0.15 per cent. The voltage measurement is a relative one. A four decade voltage divider was used with the accuracy of 0.01 per cent.

Figures 1-4 present the location of the foci and the focal lengths of the immersion lenses depending on the potential ratio on the electrodes. The results happen to be in accordance with those obtained by other authors. It can be seen in table 2 where the corresponding results both numerical and experimental are shown.

To perform the graphs of the longitudinal spherical

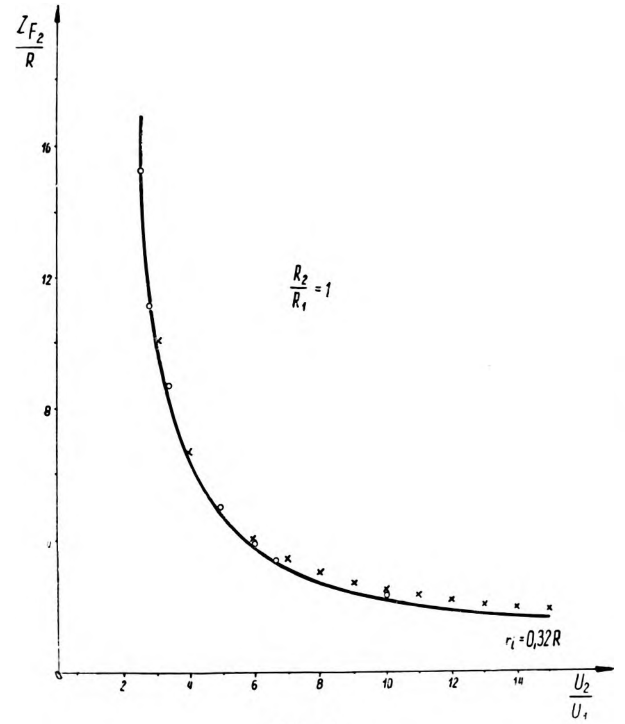

Fig. 2

o o ARDENNE ——— KIELMAN x x MULAK

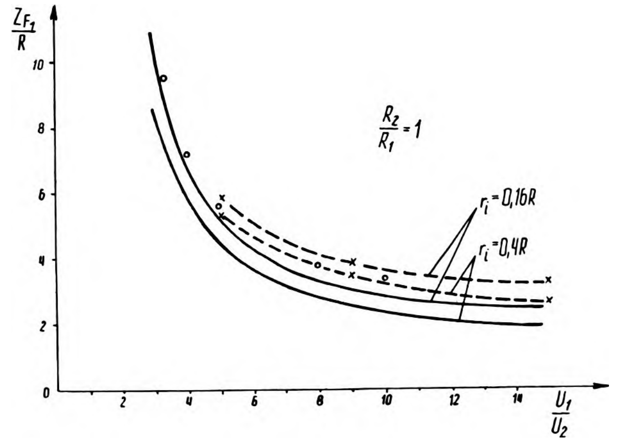

Fig. 3

o—o ARDENNE ——— KIELMAN x - - x MULAK

aberration (see Figs 5-8) the calculation of the nonparaxial trajectory for different initial values r_i of the ray were used. The curves in Fig. 5, 6 and 8 are almost parallel to one other. The curve given by Motz and Klanfer [4] (see Fig. 6) was calculated numerically from the exact equation (1) of the electron trajectory basing on potential distribution evaluated by the relaxation method, while the compared curve was estimated by means of the $u(z)$

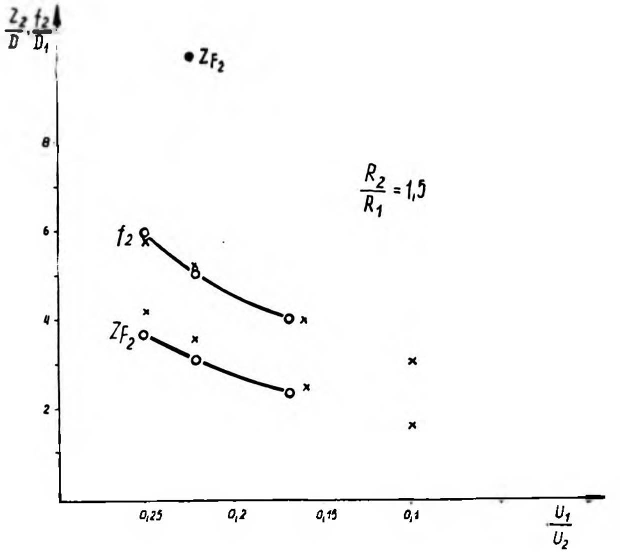

Fig. 4

o o ARDENNE • KIELMAN x x MULAK

method. In the last case the aberration correction was made by means of the potential distribution measured along the axis of the system simulated on the resistance network. When comparing both curves it is clear that as far as the third-order aberrations are concerned the accuracy is sufficiently good. The almost parallel shift of the curves means that a systematic error was introduced mainly by the numerical method used to determine the trajectory.

Table 2

$\frac{U_2}{U_1}$	Author	z_{F_1}	z_{H_1}	z_{F_2}	z_{H_2}
	Maloff, Epstein	−43.2	−17,6	40.0	−26.4
	Spangenberg, Field	−43.36	—	36.22	—
	Klanfer, Motz				
6	numer., paraxial.	−37.12	−12.74	44.0	−24.02
	$r_0 = 4h$ nonparaxial.			38.50	
	Ardenne	−43.2	−16.0	39.3	−26.3
	Mulak numer., paraxial.	−45.61	−16.0	42.62	−22.28
	$r_0 = 4h$ nonparaxial.			37.2	
4	Ardenne			60	95
	Mulak			68	93.85

h — mesh length

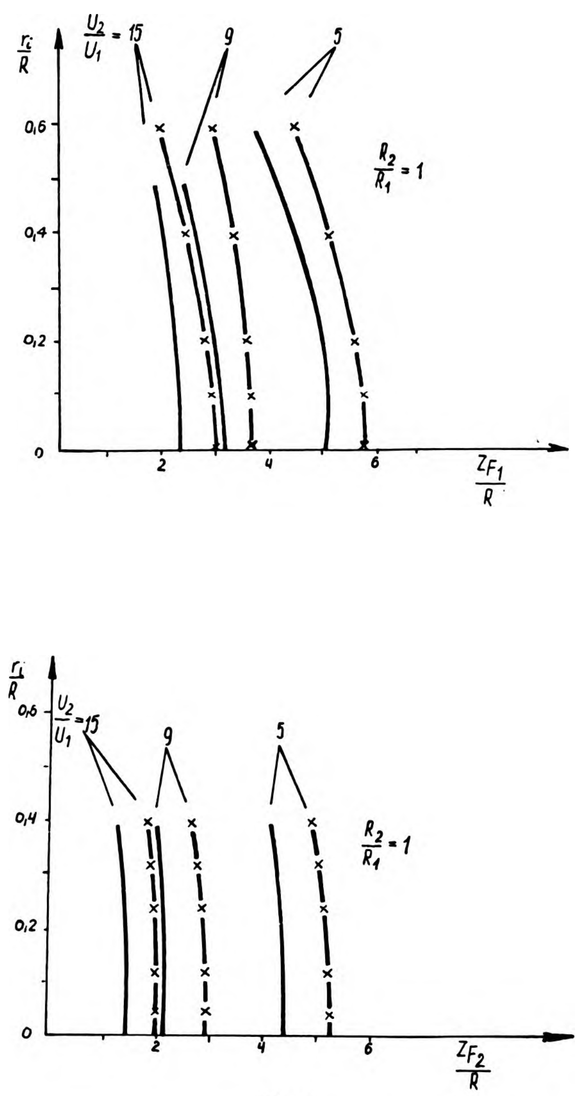

Fig. 5

——— KIELMAN x——x MULAK

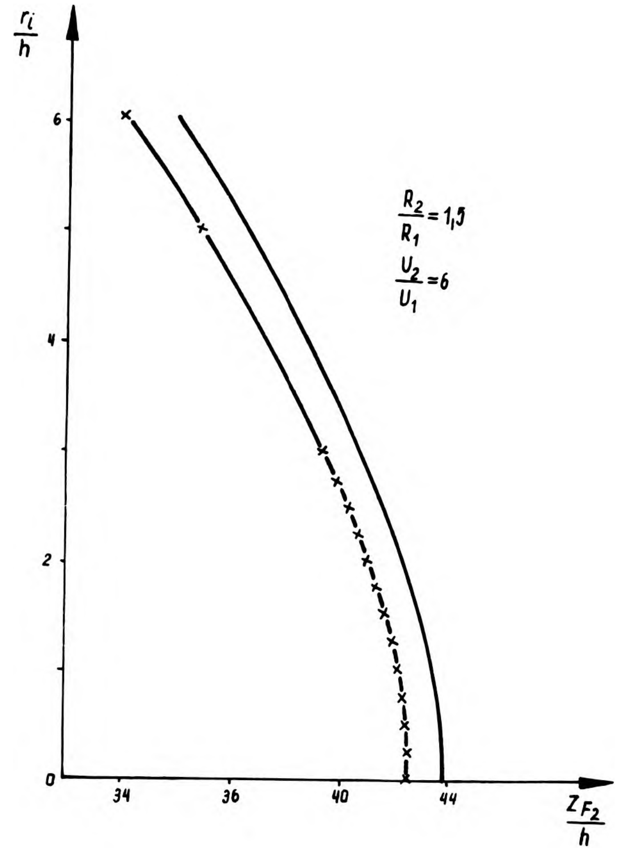

Fig. 6

——— MOTZ, KLANFER x——x MULAK

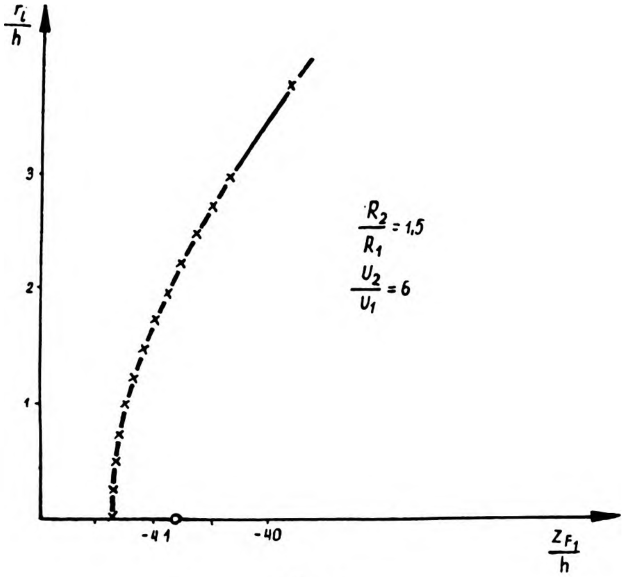

Fig. 7

o o focal point according to ARDENNEGO x——x MULAK

The comparison of all the curves in Figs. 5, 6 and 7 confirms the property described earlier in the literature (comp. [3]), that the spherical aberration of the descelerating lenses is greater than that of the same lenses applied in the accelerating direction. Figs. 9-11 illustrate the dependence of the constant spherical aberration on the focal length. The constant C_{sph} is then determined in the following way

$$C_{sph} = \frac{\Delta z_F}{r_f'^2}, \tag{20}$$

where Δz_F —difference of the paraxial and non-paraxial ray focus position.

The calculated values C_{sph} coincided with the straight lines $C_{sph}(f)$ given by Ardenne [1] and Zvorykin [8].

In Figs 12 and 13 a comparison of the calculated values of the chromatic aberration constants

$$C_{chr} = \frac{\Delta z_F}{\frac{\Delta U_0}{U_0}}, \tag{21}$$

where Δz_F — denotes the difference in the foci given by the solution of the trajectory equation corresponding to the distributions $U_0(z)$ and $U_0(z)+\Delta U_0(z)$

$U_0 = \frac{U_1+U_2}{2}$ being the average potential of the electrode, is given.

The conformity with the results given by Zvorykin [8] is in general good with the exception of the cases of great values of the ratio $\frac{U_2}{U_1}$ and in the case of $\frac{U_3}{U_1} \approx 2$. The constancy of $C_{chr\,2}$ in that region is difficult to explain.

Because of the lack of the complete data the distortion constant

$$C_d = \frac{\Delta M}{M_0 r_i^2}, \tag{22}$$

where M — linear magnification, was determined for the lens with $\frac{R_2}{R_1} = 1$ and $\frac{U_2}{U_1} = 5$ for the object position $z_{obj} = 160$. Then $C_d \approx 0.0134$ and its small changes in the region $0 < r_i \leqslant 0.3\,R$ mean that in this range the deciding role is played by the aberrations of the third order. The constant values of C_{sph} and C_{chr} behave similarly by 1 per cent change of the average potential of the electrod.

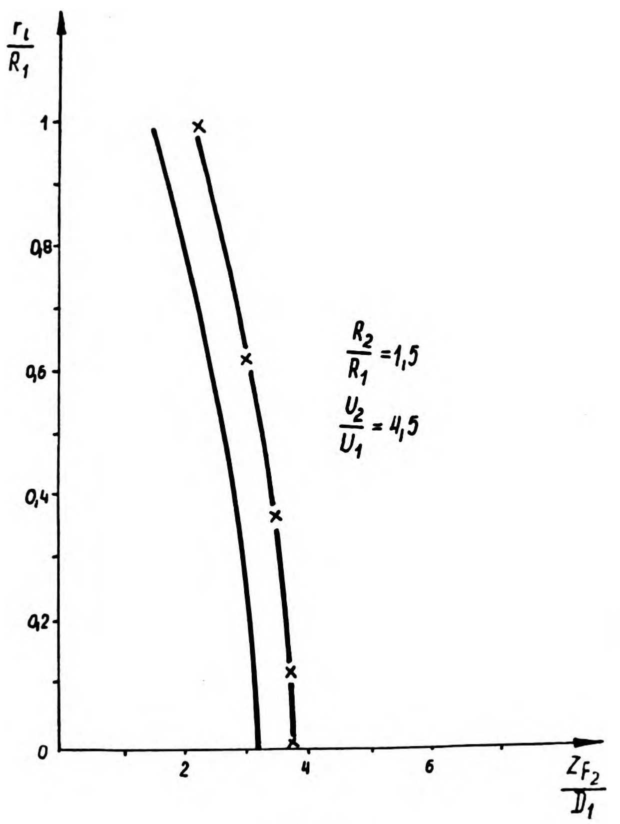

Fig. 8

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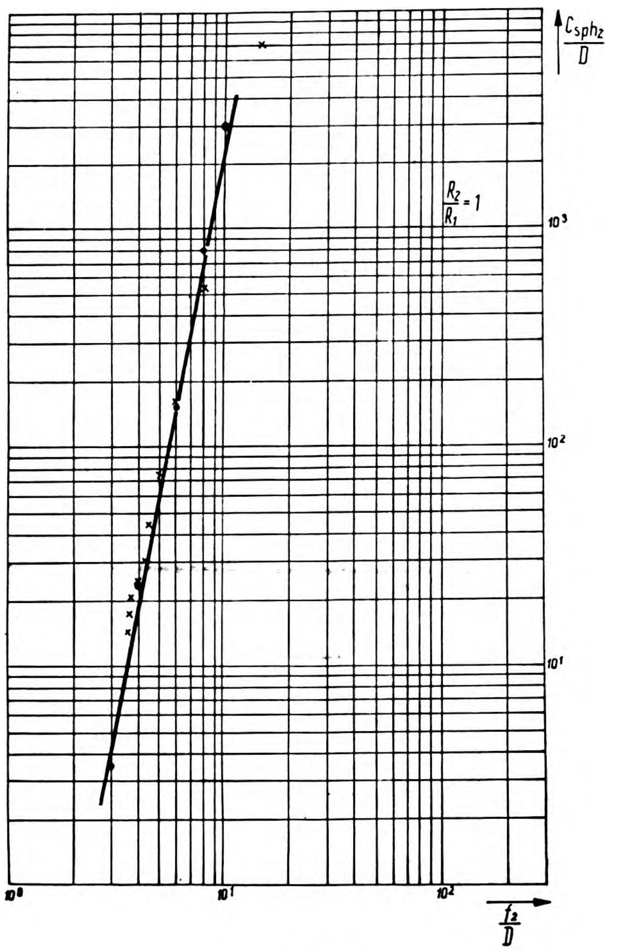

Fig. 9

•——• ZWORYKIN x x MULAK

V. Other application of the $u(z)$ method

In the literature no data concerning the electron-optical aberration of different types have been found. Because of the lack of information the aberrations are usually assumed to be independent and the total aberration is calculated from the r.m.s. of the component aberrations. The trajectory method based on the function $u(z)$ allows us to examine the mutual interdependence and rules of superposition of particular aberrations. The spherical and chromatic aberrations for not too great values r_i and ΔU add to each other. For instance the double-tube lens $\frac{R_2}{R_1} = 1.5$ for $\Delta U = 8$ V($U_1 = 200$ V, $U_2 = 1200$ V) and $r_i = 3$h the total aberration is about 5 per cent greater than the sum of component aberrations, while the combining according to r.m.s. gives the estimation with the error exceeding 50 per cent. The method $u(z)$ may be applied also to determine the divergence of the electron beamthe aberration unknown in optics of light bundles. To estimate the accuracy

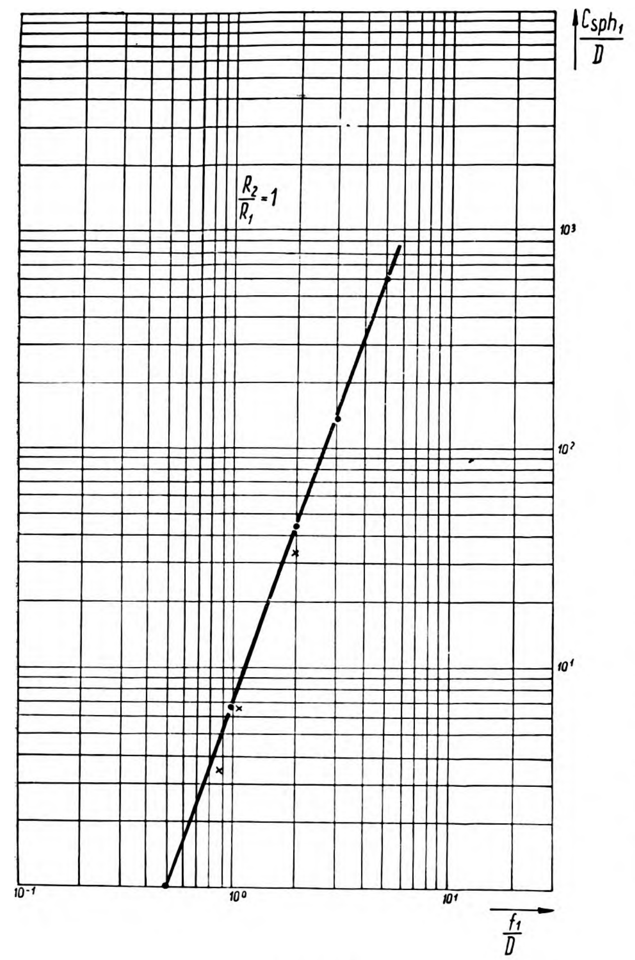

Fig. 10

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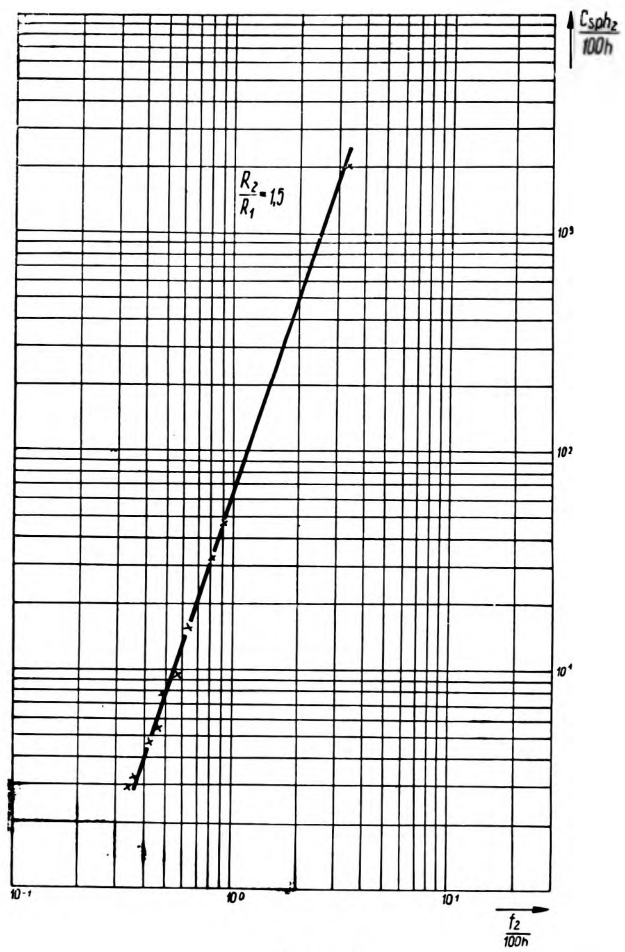

Fig. 11

of our numerical method the results have been compared with those obtained from the integral of divergence in a drift space. The relative deviations $\frac{\Delta r}{r}$ for a parallel beam occurred to be equal $+0.064$ per cent and -0.96 per cent for the perveance values 10^{-8} P and 10^{-6} P, respectively. The method of direct numerical intergrating of the trajectory equation

$$4U_0r''+2U_0'r'+(U_0''+\frac{\varrho}{\varepsilon_0})r = 0, \tag{23}$$

where ϱ — space charge density, results in errors greater by one order.

VI. Conclusions

In the face of the divergency of the data given by other authors it seems to be clear on the base of the analysis of the tables and graphs presented in this paper, that the results obtained by the author are sufficiently accurate for practical purposes. The values of parameters of the examined lenses are usually contained in the results (both experimental and numerical) given by other authors. The relative deviations (calculated with respect to the results given by other authors) for the determination of the principal planes, foci and focal length oscillate on average within the limits from a few to several per cent while in the case of constants C_{sph} and C_{chr} they amount up to 30 per cent.

The numerical method based on formulas (3) and (9) with the correction term (10) can not be used without limitations. For determining the track of the rays, passing considerably far from the axis for lenses with great ratio $\frac{U_2}{U_1}$, the further terms in the expansion of the potential and its derivatives into series have to be taken into account or some of the constant coefficient methods [5] have to be applied. There is also a possibility of further generalization of the $u(z)$ method.

For some types of single lenses greater errors (than those for immersial lenses) may be expected as a result of the fact that the electron trajectory may intersect the symmetry axis twice. As the assumption $r \neq 0$

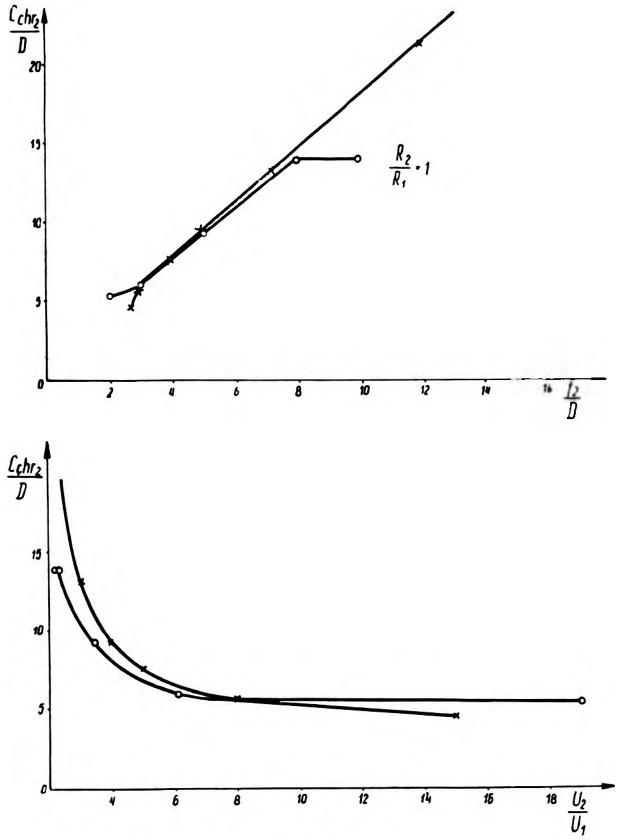

Fig. 12

o——o ZWORYKIN x——x MULAK

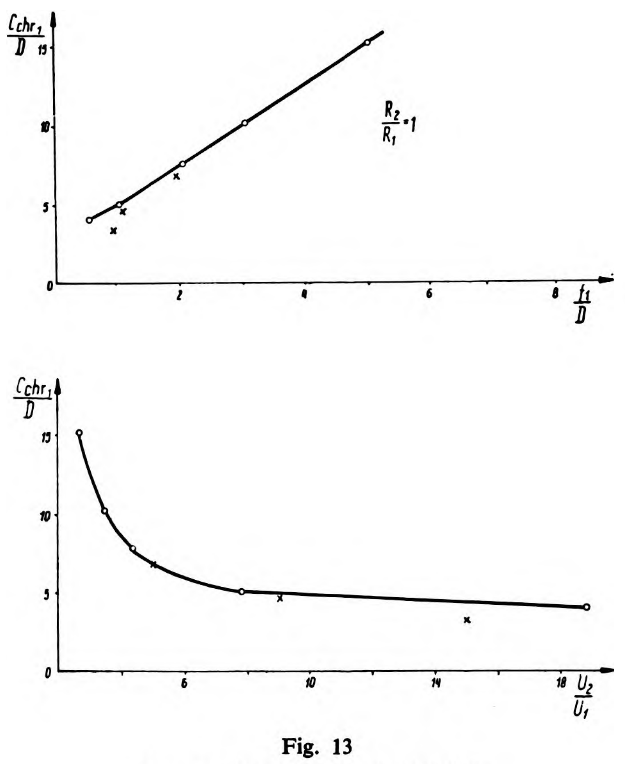

Fig. 13

x x MULAK o——o ZWORYKIN

is no more satisfied, the passage through the axis has to be done by an extrapolation procedure. This step results in an additional error.

Nevertheless, the method may compete successfully with the tedious, expensive and not too accurate experimental methods as well as with some numerical methods. To measure the potential distribution a typical resistance network has been used and typical simulation methods applied. The most obvious numerical method, consisting in the calculation of the derivatives from the forward differences, has been used for solving the trajectory equation with the corresponding correction terms; the equation being transformed with the help of the $u(z)$ function.

Thus, in spite of the common opinion it has been pointed out, that the typical resistance network may be used for the aberration determination. The main advantage of this method is not so much the value of the trajectory error (being determined among others by the discretization errors of the typical resistance network) as its stability by passing to the neighbouring trajectories.

Sur les possibilités de détermination de certaines aberrations des lentilles électrostatiques à l'aide d'un réseau de résistance

Nous avons établi une transformation de l'équation de la trajectoire de l'électron dans un champ électrique de symétrie axiale qui permet de calculer numériquement la trajectoire non paraxiale en utilisant la distribution du potentiel $U_0(z)$ suivant l'axe de symétrie du système modélé à un réseau de résistance. On a calculé les positions des foyers et des plans principaux, les longueurs focals et aussi les constantes des aberrations: sphérique C_{sf}, chromatique C_{chr} et de la distorsion C_d pour les lentilles à immersion de deux types. Les résultats obtenus sont comparés avec les données des autres auteurs. On a démontré que le réseau de résistance typique permet de déterminer certaines aberrations des systèmes électronooptiqu es.

Возможность определения некоторых аберрации электростатических линз с помощью резистивной сети

Приводится преобразование уравнения траектории электрона в электрическом поле с вращательной симметрией, позволяющее расчитать непараксиальную траекторию, опираясь на распределение потенциала $U_0(z)$, снятого на оси симметрии системы, сделанной на резистивной сети. Определены положения фокусов, кардинальных плоскостей, фокусной длины, а также постоянные сферической аберрации C_{sf}, хроматической C_{chr} дисторсии C_d для иммерсионных линз двух типов. Сопоставлены полученные результаты с соответствующими данными других авторов. Установлено, что типовая резистивная сеть позволяет определить некоторые аберрации электронооптических систем.

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