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Influence of the A Priori Information about an Object on the Direct Recovery Procedure in Incoherent Imaging

In paper [1] the direct recovery problem for incoherent imaging was discussed and a method of solution given for the case of no a prioii information about the object. The present paper is devoted to a quanti:ative analisis of the influence of the a priori information on both the recovery procedure and the results.

I. Introduction

In paper [1] the problem of the direct recovery, understood as the image and object reconstruction procedure starting with their measurement representation, has been considered for the case of no a priori information about the object. Such a formulation of the problem permits to considerably simplify the analysis on the one hand and to clearly point out the basic difference between the object and image recovery procedures on the other. As has been shown in [1] this difference lies in the intrisic impossibility of performing a pleasing recovery of the object, while the corresponding image may be, in principle, reconstructed to the reasonably arbitrary accuracy**.

However, in practive we almost always possess some knowledge about the object, prior to the measurement, though in many cases it may be difficult to represent analitically. If the a priori information may be expressed analitically, the reconstruction procedure developed in [1] may be applied after the corresponding modification.

Below, we will examine the applicability of this method to the cases when some reasonable assumption about the object may be made. The analysis will stay in close connection with that given in [1] and the notation will be strictly preserved as this paper is in fact the second part of [1].

II. Object of continuous intensity distribution

If the fact of continuity of object intensity distribution is known, as additional a priori information, independently of the given set of the observed image points $x(\boldsymbol{a}_k)$, $k = 1, \ldots, N$, then it is possible to represent the object as a series of known, regular, elementary, local functions centered at the points $\bar{\boldsymbol{a}}'_k$ or $\bar{\boldsymbol{a}}_k$ (or any other set of N points within the region σ/γ^2_{im}) with unknown coefficients. These coefficients are to be determined by the reconstruction procedure. We shall illustrate the recovery method in this case by considering again the half-tone screen approximation from a view-point different from that in [1]. After having explained that we are interested in the half-tone scrren approximation, we can represent the sought object intensity distribution in the forms

$$I_{\text{ob}}(\bar{\boldsymbol{a}}) = \sum_{n=1}^{N} c_n \operatorname{rect}(\bar{\boldsymbol{a}} - \bar{\boldsymbol{a}}_n) \tag{1a}$$

$$n = 1, \ldots, N$$

or

$$I_{\text{ob}}(\bar{\boldsymbol{a}}) = \sum_{n=1}^{N} C'_n \operatorname{rect}(\bar{\boldsymbol{a}} - \bar{\boldsymbol{a}}'_n) \tag{1b}$$

in those used in Eq. (18, a, b) in [1], where

$$\operatorname{rect}(\bar{\boldsymbol{a}} - \bar{\boldsymbol{a}}_n) = \begin{cases} 1 & \text{for } (\bar{\boldsymbol{a}} - \bar{\boldsymbol{a}}_n) \in \theta_k \qquad k = 1, \ldots, N \\ 0 & \text{otherwise} \end{cases}$$

and

$$\operatorname{rect}(\bar{\boldsymbol{a}} - \bar{\boldsymbol{a}}'_n) = \begin{cases} 1 & \text{for } (\bar{\boldsymbol{a}} - \bar{\boldsymbol{a}}'_n) \in \theta'_k \qquad k = 1, \ldots, N. \\ 0 & \text{otherwise} \end{cases}$$

The quantities c_n and c'_n are unknown, but as their determination solves our reconstruction problem, this will be our goal in this section. By substituting (1) into (1) in [1] we can relate c_n-and c'_n with the observed image points $x(\bar{\boldsymbol{a}}_k)$, $k = 1, \ldots, N$ as follows:

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**) Eventually at the expence of both the procedure and the arrangement complexity.

$$x(\bar{a}_k) = \int_P\int_{P_0} \sum_{n=1}^{N} c_n \operatorname{rect}(\bar{a}-\bar{a}_n) K_{\text{im}}\left(\frac{\bar{p}_1}{z_2}+\frac{\bar{a}}{z_1}\right)$$

$$K^*_{\text{im}}\left(\frac{\bar{p}_2}{z_1}+\frac{\bar{a}}{z_1}\right) \varphi(\bar{p}_1-\bar{a}_k, \bar{p}_2-\bar{a}_k) d\bar{p}_1 d\bar{p}_2 d\bar{a}$$

$$= \sum_{n=1}^{N} c_n R_{\text{nk}} \tag{2a}$$

and by the same argument

$$x(\bar{a}_k) = \sum_{n=1}^{N} c_n'^N R'_{\text{nk}}, \tag{2b}$$

where

$$R_{\text{nk}} = \int_P\int_{P_0} \operatorname{rect}(\bar{a}-\bar{a}_n) K_{\text{im}}\left(\frac{\bar{p}_1}{z_2}+\frac{\bar{a}}{z_1}\right) K^*_{\text{im}}\left(\frac{\bar{p}_2}{z_2}+\frac{\bar{a}}{z_1}\right)$$

$$\varphi(\bar{p}_1-\bar{a}_k, \bar{p}_2-\bar{a}_k) d\bar{p}_1 d\bar{p}_2 d\bar{a} \tag{3a}$$

and

$$R'_{\text{nk}} = \int\int_{PP_0} \operatorname{rect}(\bar{a}-\bar{a}'_n) K_{\text{im}}\left(\frac{\bar{p}_1}{z_2}+\frac{\bar{a}}{z_1}\right) K^*_{\text{im}}\left(\frac{\bar{p}_2}{z_2}+\frac{\bar{a}}{z_1}\right)$$

$$\varphi(\bar{p}_1-\bar{a}_k, \bar{p}_2-\bar{a}_k) d\bar{p}_1 d\bar{p}_2 d\bar{a}. \tag{3b}$$

Calling the matrices

$$\{R_{\text{nk}}\} \text{ and } \{R'_{\text{nk}}\}, \tag{4}$$

respectively, the "upper" and "lower" bound reconstruction matrices involving the half-tone screen approximation, it is worth noticing that the matrices (4) represent certain modifications of the upper and lower bound matrices as determined by Eq. (15) in [1] for the case of absolute a priori ignorance. From the formal point of view the modification consists in replacing delta functions in Eq. (15) in [1] by rectangular functions. From the reconstruction view-point, the modification consists in the inserting of all the a priori information about the object structure in the matrix elements R_{nk} and R'_{nk}. Thus, reconstruction matrices now contain both all the knowledge about the imaging and observing systems, necessary for reconstruction, and the a priori structural information about the object. This increase of information contained in R_{nk} and R'_{nk} results in their greater complexity, when compared with B_{nk} and B'_{nk}. The last fact is somewhat typical and will be observed in the next example, too. Furthermore, note that the matrices R_{nk} and R'_{nk} have the property of representing the "upper" and "lower" bound reconstruction procedures only in a limited sense, which will be clear from the following.

Solving in a routine way the systems (1 a, b) of the linear equations with respect to c_n and c'_n, we get

$$c_n^N = \frac{\begin{vmatrix} R_{1,1}, \ldots, R_{1,n-1}, x_1, R_{1,n+1}, \ldots, R_{1,N} \\ R_{N,1}, \ldots, R_{N,n-1}, x_N, R_{N,n+1}, \ldots, R_{N,N} \end{vmatrix}}{|\{R_{nk}\}|} \tag{5a}$$

$$n = 1, \ldots, N$$

and

$$c_n'^N = \frac{\begin{vmatrix} R'_{11}, \ldots, R'_{1,n-1}, x_1, R'_{1,n+1}, \ldots, R'_{1,N} \\ R'_{N,1}, \ldots, R'_{N,n-1}, x_N, R'_{N,n+1}, \ldots, R'_{N,N} \end{vmatrix}}{|\{R'_{n,k}\}|}; \tag{5b}$$

$$n = 1, \ldots, N$$

Substituting the recovered values c_n^N and $c_n'^N$ into the equations (1), we obtain two representations of the object intensity distribution

$$I_{\text{ob}}^{\max}(\bar{a}) = \sum_{n=1}^{N} c_n^N \operatorname{rect}(\bar{a}-\bar{a}_n) \tag{6a}$$

$$I_{\text{ob}}^{\min}(\bar{a}) = \sum_{n=1}^{N} c_n'^N \operatorname{rect}(\bar{a}-\bar{a}'_n), \tag{6b}$$

each of them being the recovered object intensity distribution consistent within the half-tone screen approximation, with the given set of observed points $x(\bar{a}_k)$, $k = 1, \ldots, N$. If we assume

$$I_{\text{ob}}(\bar{a}) = \frac{1}{2}[I_{\text{ob}}^{\max}(\bar{a})+I_{\text{ob}}^{\min}(\bar{a})]$$

$$= \frac{1}{2}\sum_{n=1}^{N}[c_n^N \operatorname{rect}(\bar{a}-\bar{a}_n)+c_n'^N \operatorname{rect}(\bar{a}-\bar{a}'_n)], \tag{7}$$

as the final object representation involving the half-tone screen approximation, then

$$\Delta I_{\text{ob}}(\bar{a}) = \pm\frac{1}{2}[I_{\text{ob}}^{\max}(\bar{a}) - I_{\text{ob}}^{\min}(\bar{a})]$$

$$= \pm\frac{1}{2}\sum_{n=1}^{N}[c_n^N \operatorname{rect}(\bar{a}-\bar{a}_n) - c_n'^N \operatorname{rect}(\bar{a}-\bar{a}'_n)] \tag{8}$$

represents the recovery error in the following sense:

After having decided to reconstruct the object with the half-tone screen approximation, we are still free to arbitrarily localize the middle points of the cell texture with respect to the object region θ to be recovered. Equations (1 a, b) represent somewhat limiting positions of these middle points; (1a) representing the case when the middle points are located at the points $(\bar{a}_k)$ mostly contributing to the corresponding observed image points $x(\bar{a}_k)$, while (1b) represents the case of locating them at the points $(\bar{a}'_k)$ of the least contributions. Thus, if (7) describes the recovered object distribution involving the half-tone screen approximation, (6a) and (6b) represent the extremal possible a posteriori departure from (7) within this approximation, due to extremal positions of the cell texture still consistent with the measurement results.

The error of the object intensity value within one fourth of each cell θ_k $k = 1, \ldots, N$ due to this uncertainty of texture localization may be evaluated from the formula (8). This does not mean that the real intensity distribution in the object remains within these limits. Therefore, the reconstruction matrices (4) may be considered as the matrices of upper and lower reconstruction procedure only in the sense limited to the object structures given by the half-tone screen approximation.

Finally, it is clear that the representation (1a, b) of the object is only one of many possible for the objects known a priori as being continuous. Generally, we can represent the unknown object intensity distribution in the form of the series

$$I(\bar{a}) = \sum_{n=1}^{N} c_n f(a - a_n)$$

and $\qquad n = 1, \ldots N$

$$I'(\bar{a}) = \sum_{n=1}^{N} c_n f(a - a'_n)$$

and call the continuous functions $f(\bar{a}-\bar{a}_n)$ elementary structural functions of the recovered object intensity distribution. The form and analytical properties of these functions must be determined by the a priori knowledge about the object. For instance, in the up-to-now considered cases the structural functions $f(\bar{a}-\bar{a}_n)$ were respectively equal to $\delta(\bar{a}-\bar{a}_n)$ for the case of no a priori information (see [1]) and to rect $(\bar{a}-\bar{a}_n)$ when the half-tone screen approximation was involved.

Good examples of other elementary structural functions, which may be applied in the recovery procedure, are the following

$$f(\bar{a}-\bar{a}_n) = \frac{\sin^k(\bar{a}-\bar{a}_n)}{|\bar{a}-\bar{a}_n|^m}, \qquad m = 1, 2, k, \ldots \quad (9)$$

or

$$f(\bar{a}-\bar{a}_n) = \frac{J_0^k(\bar{a}-\bar{a}_n)}{|\bar{a}-\bar{a}_n|^m},$$

J_0 — denotes a Bessel function of first kind and first order and or similar, where the exponent values k and m may be adjusted to the concrete reconstruction problem according to the a priori information about the object. The developed reconstruction procedure holds for all those cases and the only modification consists in appearing of the corresponding elementary structural function in the elements of the upper and lower bound reconstruction matrices. Thus the general formulas for R_{nk} and R'^k_n are respectively

$$R_{nk} = \int_P \int_{P_0} f(\bar{a}-\bar{a}_n) K_{im}\left(\frac{\bar{p}_1}{z_2} + \frac{\bar{a}}{z_1}\right) K^*_{im}\left(\frac{\bar{p}_k}{z_2} + \frac{\bar{a}}{z_1}\right)$$

$$\varphi(\bar{p}_1-\bar{a}_k, \bar{p}_2-\bar{a}_k) d\bar{p}_1 d\bar{p}_2 d\bar{a}. \qquad (10)$$

An important example of that kind of modification will be discussed in the following paragraph.

III. Band limited objects

If the object is known to be band limited and the spatial frequency spectrum width is given, then on the basis of the sampling theorem, the object intensity distribution may be uniquely represented in the form

$$I_{ob}(\alpha, \beta) = \sum_{nm=-\infty}^{\infty} I\left(\frac{m}{2\mu_0} \frac{n}{2\nu_0}\right)$$

$$\operatorname{sinc}\left[2\mu_0\left(\alpha - \frac{m}{2\mu_0}\right)\right] \operatorname{sinc}\left[2\nu_0\left(\beta - \frac{n}{2\nu_0}\right)\right],$$

where μ_0 and ν_0 are the limits of the frequency spectrum in the α and β directions respectively*. The complete determination of the object requires estimation of the infinite number of the intensity values $I\left(\frac{m}{2\mu_0}, \frac{n}{2\nu_0}\right)$. As this requirement is unrealistic from the practical point of view, usually an approximate representation is used

$$I_{ob}(\alpha, \beta) = \sum_{m=-M_0}^{M_0} \sum_{n=-N_0}^{N_0} I\left(\frac{m}{2\mu_0}, \frac{n}{2\nu_0}\right)$$

$$\operatorname{sinc}\left[2\mu_0\left(\alpha - \frac{m}{2\mu_0}\right)\right] \operatorname{sinc}\left[2\nu_0\left(\beta - \frac{n}{2\nu_0}\right)\right]. \qquad (11)$$

In the last formulation the problem can be treated by our recovery procedure in the following way:

Let us define the set of points (α_m, β_n) in the object plane by letting $\alpha_m = \frac{m}{2\mu_0}$, $\beta_n = \frac{n}{2\nu_0}$: and,

* We resign here for a moment, of abbreviated notation $\bar{a} = (\alpha, \beta)$ to be able to consider the general case when $N \neq M$ in (11) below.

find the corresponding points ($a_m = \gamma_{im} a_m$, $b_n = \gamma_{im}\beta_n$) in the mage plane*. The locating of the observing system at those points (a_m, b_n) results in determination of the observed image points x (a_m, b_n) $n = -N_0, \ldots, +N_0$, $m = -M_0, \ldots, +M_0$ being the product of the transformation

$$x(a_k, b_k) = \sum_{-M_0}^{M_0} \sum_{-N_0}^{N_0} I(a_m, \beta_n) \iint_{P_0 P} \operatorname{sinc}[2\mu_0(a-a_m)]$$

$$\operatorname{sinc}[2\nu_0(\beta-\beta_n)] K_{im}\left(\frac{p_1}{z_2}+\frac{a}{z_1}, \frac{q_1}{z_2}+\frac{\beta}{z_1}\right). \quad (12)$$

$$K^*_{im}\left(\frac{p_2}{z_2}+\frac{a}{z_1}, \frac{\bar{q}_2}{z_2}+\frac{\beta}{z_1}\right)$$

$$\varphi(p_1-a_t, q_1-b_t, p_2-a_t, q_2-b_t)$$

$$dp_1 dq_1 dp_2 dq_2 da d\beta = \sum_{-M_0}^{M_0} \sum_{-N_0}^{N_0} I(a_m, \beta_n) R_{tkmn}$$

$$t = -M, \ldots, M \quad k = -N, \ldots, N. \quad (13a)$$

By the same arguments applied to the intermediate points (a'_m, β'_n), we have

$$x(a_t, b_k) = \sum_{-M_0}^{M_0} \sum_{-N_0}^{N_0} I(a_m, \beta_n) R'_{tkmn}. \quad (13b)$$

Here, the corresponding expressions for the upper and lower bound reconstruction matrix elements R_{tkmn} and R'_{tkmn} are respectively

$$R_{tkmn} = \int_{P_0} \operatorname{sinc}[2\mu_0(a-a_m)] \operatorname{sinc}[2\nu_0(\beta-\beta_n)] \quad (14a)$$

$$B(a, \beta, a_t, b_k) da d\beta,$$

$$R'_{tkmn} = \int_{P_0} \operatorname{sinc}[2\mu_0(a-a'_m] \operatorname{sinc}[2\nu_0(\beta-\beta'_n)]$$

$$B(a\beta, a_t a_k) da d\beta, \quad (14b)$$

where

$$B(a, \beta, a_t\, b_k) = \int_P K_{im}\left(\frac{p_1}{z_2}+\frac{a}{z_1}, \frac{q_k}{z_2}+\frac{\beta}{z_1}\right) \times$$

$$\times K^*\left(\frac{p_2}{z_2}+\frac{a}{z_1}, \frac{q_2}{z_2}+\frac{\beta}{z_1}\right)$$

$$\varphi(p_1-a_t, q_1-b_k, p_2-a_t, q_2-b_k) dp_1 dp_2 dq_1 dq_2 \quad (15)$$

(see (9) in [1]). Thus, the reconstruction problem is reduced to solving the systems (13a, b) of linear equations with the reconstruction matrices (14a, b) containing the elementary structural functions sinc $[2\mu_0(a-a_m)]$ and sinc $[2\nu_0(\beta-\beta_m)]$ of the object. Again, the matrix (14a) may be considered as the "upper" bound reconstruction matrix, while the matrix (14b) as the "lower" bound reconstruction matrix both in the limited sense discussed above. Substituting the required values

$$I_{ob}^{N_0M_0}(a_m, \beta_n) \quad \text{and} \quad I_{ob}^{N_0M_0}(a'_m, \beta'_n), \quad (16)$$

into (11), we get

$$I_{ob}^{(\max)}(a, \beta) = \sum_{-M_0}^{M_0} \sum_{-N_0}^{N_0} I_{ob}^{M_0N_0}(a_m, \beta_n) \times$$

$$\times \operatorname{sinc}[2\mu_0(a-a_m)] \operatorname{sinc}[2\nu_0(\beta-a_n)] \quad (16a)$$

and

$$I_{cb}^{(\min)}(a, \beta) = \sum_{-M_0}^{M_0} \sum_{-N_0}^{N_0} I_{ob}^{N_0M_0}(a'_m, \beta'_n) \times$$

$$\times \operatorname{sinc}[2\mu_0(a-a'_m)] \operatorname{sinc}[2\nu_0(\beta-a'_m)], \quad (16b)$$

as upper and lower bound object estimations and finally

$$I_{ob}(a, \beta) = \tfrac{1}{2}[I_{ob}^{(\max)}(a, \beta)+I_{ob}^{(\min)}(a, \beta)] \quad (17)$$

as reconstructed object distribution.

For the sake of comparison with the formerly obtained results, it will be convenient to restrict our considerations to the case, where $M_0 = N_0$ and to shorten the notation by renumerating the scanning point coordinates in the image and object planes as follows:

$$(a_k, b_t) = (\bar{a}_s) \qquad s = 1, \ldots, N$$

$$(a_m, \beta_n) = \bar{a}_r) \qquad N = (2M_0+1)^2$$

$$(a'_m, \beta'_n) = (a'_r) \qquad r = 1, \ldots, N$$

Then the formulae (13)-(17) may be written in a more compact form

$$x(\bar{a}_s) = \sum_{r=1}^{N} I(\bar{a}_r) R_{rs} = \sum_{r=1}^{N_0} c_r^N R_{rs} \quad (13'a)$$

$$x(\bar{a}_s) = \sum_{r=1}^{N} I(\bar{a}'_r) R_{rs} = \sum_{r=1}^{N_0} c_r^N R_{rs} \quad (13'b)$$

$$R_{rs} = \int_P \operatorname{sinc}[2\bar{\mu}_0(\bar{a}-\bar{a}_r)] B(\bar{a}, \bar{a}_s) da \quad (14')$$

$$I_{ob}^{(\max)}(\bar{a}) = \sum_{r=1}^{N} I(\bar{a}_r) \operatorname{sinc}[2\bar{\mu}_0(\bar{a}-\bar{a}_r)] \quad (16'a)$$

$$I_{ob}^{(\min)}(\bar{a}) = \sum_{r=1}^{N} I(\bar{a}'_r) \operatorname{sinc}[2\bar{\mu}_0(\bar{a}-\bar{a}'_r)] \quad (16'b)$$

$$I_{ob}(\bar{a}) = \tfrac{1}{2}[I_{ob}^{(\max)}(\bar{a})+I_{ob}^{(\min)}(\bar{a})] \quad (17')$$

* This definition of (a_m, β_n) implies rectangularity of the recovered object area.

$$\Delta I(\bar{a}) = \pm\tfrac{1}{2}[I_{ob}^{(max)}(\bar{a}) - I_{ob}^{(min)}(\bar{a}')]\; r = 1, \ldots, N, \quad (18)$$

where

$$\operatorname{sinc}[2\bar{\mu}_0(\bar{a}-\bar{a}_r)] = \operatorname{sinc}[2\mu_0(a-a_r)]\operatorname{sinc}[2\nu_0(\beta-\beta_r)]$$

$$\bar{\mu}_0 = (\mu_0, \nu_0).$$

IV. Remarks on the reconstruction accuracy

All the analysis developed so far consists as matter-of-factly in different realizations of the same recovery method with modifications due to the a priori information about the object. It may be interesting to compare the accuracy of each particular reconstruction procedure for both the image and object intensity distributions in all the considered cases. To do so, let us return for a moment to the image recovery problem. Having determined the object intensity distribution in the form (7) for the case of the half-tone screen approximation or in the form (17) for the case of band limited objects respectively, we can readily obtain the corresponding formula for the reconstructed images by substituting successively (7) or (17) into (2) and (10) in [1], and then estimating the reconstruction error from Eq. (11) in [1]. This yields:

(A) For half tone screen approximations:

$$I_{im}^{(max)}(\bar{p}) = \int_{P_0} I_{ob}^{(max)}(\bar{a})\varphi(\bar{p},\bar{a})d\bar{a} \quad (19)$$

$$= \sum_{n=1}^{N} c_n^N \int_{P_0} \operatorname{rect}(\bar{a}-\bar{a}_n)\varphi(\bar{p},\bar{a})d\bar{a}$$

$$= \sum_{n=1}^{N} c_n^N A_n(\bar{p}),$$

$$I_{im}^{(min)}(\bar{p}) = \int_{P_0} I_{ob}^{(min)}(\bar{a})\varphi(\bar{p},\bar{a})d\bar{a}$$

$$= \sum_{n-1}^{N} c_n'^N \int_{P_0} \operatorname{rect}(\bar{a}-\bar{a})\varphi(\bar{p},\bar{a})d\bar{a}$$

$$= \sum_{n=1}^{N} c_n'^N A_n'(\bar{p}), \quad (20)$$

$$I_{im}(\bar{p}) = \tfrac{1}{2}[I_{im}^{(max)}(\bar{p}) + I_{im}^{(min)}(\bar{p})]$$

$$= \tfrac{1}{2}\sum_{n=1}^{N} c_n^N A_n(\bar{p}) + c_n'^N A_n'(\bar{p}), \quad (21)$$

$$\Delta I_{im}(\bar{a}_k) = \pm\tfrac{1}{2}\left|\sum_{n=1}^{N} c_n^N A_n(\bar{a}_k) - c_n'^N A_n'(\bar{a}_k)\right| \quad (22)$$

$$k = 1, \ldots, N,$$

where

$$A_n(\bar{p}) = \int_{P_0} \operatorname{rect}(\bar{a}-\bar{a}_n)\varphi(\bar{p},\bar{a})d\bar{a}$$

$$A_n'(\bar{p}) = \int_{P_0} \operatorname{rect}(\bar{a}-\bar{a}_n')\varphi(\bar{p},\bar{a})d\bar{a},$$

(B) For band limited objects:

From (14′ a, b)

$$I_{im}^{(max)}(\bar{p}) = \int_{P_0} I_{ob}^{(max)}(\bar{a})\varphi(\bar{p},\bar{a})d\bar{a}$$

$$= \sum_{n=1}^{N} c_n^N \int_{P_0} \operatorname{sinc}[2\bar{\mu}_0(\bar{a}-\bar{a}_n)]\varphi(\bar{p},\bar{a})d\bar{a} \quad (23)$$

$$= \sum_{n=1}^{N} c_n^N B_n(\bar{p}),$$

$$I_{im}^{(min)}(\bar{p}) = \int_{P_0} I_{ob}^{(min)}(\bar{a})\varphi(\bar{p},\bar{a})d\bar{a}$$

$$= \sum_{n=1}^{N} c_n'^N \int_{P_0} \operatorname{sinc}[2\bar{\mu}_0(\bar{a}-\bar{a}_r')\varphi(\bar{p},\bar{a})d\bar{a}$$

$$= \sum_{n=1}^{N} c_n'^N B_n'(\bar{p}), \quad (24)$$

$$I_{im}(\bar{p}) = \tfrac{1}{2}[I_{im}^{(max)}(\bar{p}) + I_{im}^{(min)}(\bar{p})]$$

$$= \tfrac{1}{2}\sum_{n=1}^{N} [c_n^N B_n(\bar{p}) + c_r'^N B_n'(\bar{p})], \quad (25)$$

$$\Delta I_{im}(\bar{a}_k) = \pm\frac{1}{2}\left|\sum_{n=1}^{N} c_n^N B_n(\bar{a}_k) - c_n'^N B_n'(\bar{a}_k)\right| \quad (26)$$

$$k = 1, \ldots, N,$$

where

$$B_n(\bar{p}) = \int_{P_0} \operatorname{sinc}[2\bar{\mu}_0(\bar{a}-\bar{a}_n)]\varphi(\bar{p},\bar{a})d\bar{a},$$

$$B_n'\bar{p} = \int_{P_0} \operatorname{sinc}[2\bar{\nu}_0(\bar{a}-\bar{a}_n')]\varphi(\bar{p},\bar{a})d\bar{a}.$$

For the sake of comparison, let us recall the corresponding expressions valid for the case, when no information about the object is available before measurement.

(C) Objects of no a priori information:

$$I_{im}^{(max)}(\bar{p}) = \sum_{n=1}^{N} c_n^N \varphi(\bar{p},\bar{a}_n), \quad (27)$$

$$I_{im}^{(min)}(\bar{p}) = \sum_{n=1}^{N} c_n'^N \varphi(p,a_n'), \quad (28)$$

$$I_{\text{im}}(\bar{p}) = \tfrac{1}{2}\sum_{n=1}^{N} c_n^N \varphi(\bar{p},\bar{a}_n) + c'^N_n \varphi(\bar{p},\bar{a}'_n) \tag{29}$$

$$\Delta I_{\text{im}}(\bar{a}_k) = \pm\tfrac{1}{2}\left|\sum_{n=1}^{N} \{c_n^N \varphi(\bar{a}_k,\bar{a}_n) - c'^N_n \varphi(\bar{a}_k,\bar{a}'_n)\}\right|,$$

$$k = 1, \ldots, N \tag{30}$$

Now, imagine that an object has been imaged by an imaging system and then scanned by an observing system within the region δ. As a result, we obtain a set of observed image points $x(\bar{a}_k)$, $k = 1, \ldots, N$. If we apply to this object the reconstruction procedures (C), (A) and (B) and thus treat it as absolutely unknown (case C), as known to be a half-tone screen structure (case B), and finally as known to be band limited (case A), respectively, we can obtain some qualitative expressions being a good measure of the a priori information about the object from the reconstruction viewpoint. There are, at least, two ways of defining these measures.

For an established scanning point $(\bar{a}_s)$ we can write down the formulas (22), (26) and (30).

$$|\Delta I_{\text{im}}(\bar{a}_s)|_a = \tfrac{1}{2}\left|\sum_{n=1}^{N} \{c_{na}^N A_n(\bar{a}_s) - c_{na}^N A'_n(\bar{a}_s)\}\right|$$

$$|\Delta I_{\text{im}}(\bar{a}_s)|_b = \tfrac{1}{2}\left|\sum_{n=1}^{N} \{c_{nb}^N B_n(\bar{a}_s) - c_{nb}^N B'_n(\bar{a}_s)\}\right|$$

$$|\Delta I_{\text{im}}(\bar{a}_s)|_c = \tfrac{1}{2}\left|\sum_{n=1}^{N} \{c_{nc}^N \varphi(\bar{a}_s,\bar{a}_n) - c_{nc}^N \varphi(\bar{a}_s,\bar{a}'_n)\}\right|,$$

where c_{na}^N, c_{nb}^N and c_{nc}^N and the corresponding primed values refer to the reconstruction procedures (A), (B), (C). Now defining the ratios

$$\gamma_{ac}(\bar{a}_s) = \frac{|\Delta I_{\text{im}}(\bar{a}_s)|_c}{|\Delta I_{\text{im}}(\bar{a}_s)|_a} \quad \text{and} \quad \gamma_{bc}(\bar{a}_s) = \frac{|\Delta I_{\text{im}}(\bar{a}_s)|_c}{|\Delta I_{\text{im}}(\bar{a}_s)|_b}, \tag{31}$$

we can accept them as qualitative measures of the usefulness of the a priori information about the object for our reconstruction procedure at the point $(\bar{a}_s)$, with the following meaning. As seen from (12), (13) in [1] as well as from the formulas (1 a, b) and (6 a, b) of the present paper our a priori knowledge about the object was introduced into consideration by accepting a particular set of elementary structural functions to represent the object. This acceptance by itself does not give any quantitative measure allowing to judge, which a priori information is greater for the recovery purposes, except for some qualitative feeling. The assumption of (31) solves the problem quantitatively, because each choice of elementary structural function results in the value of the relative image reconstruction error (31) taken with respect to the corresponding error in the case of absolute ignorance. It is obvious that $\gamma_{ac}(\bar{a}_s)$ and $\gamma_{bc}(\bar{a}_s)$ are the greater the more valuable the a priori information is.

For the whole reconstructed region, the corresponding measures for problems (A) and (B) could be defined respectively by:

$$\gamma_{ac} = \frac{\sum_{n=1}^{N} |\Delta I_{\text{im}}(\bar{a}_s)|_c}{\sum_{n=1}^{N} |\Delta I_{\text{im}}(\bar{a}_s)|_a} \quad \text{and} \quad \gamma_{bc} = \frac{\sum_{n=1}^{N} |\Delta I_{\text{im}}(\bar{a}_s)|_c}{\sum_{n=1}^{N} |\Delta I_{\text{im}}(\bar{a}_s)|_b}. \tag{32}$$

Thus, γ_{ac} and γ_{bc} are measures of value of the a priori information for the general reconstruction procedure when compared with the state of complete ignorance.

On the other hand, the formulas (31) and (32) giving the quantitative answer to the discussed problem for each concrete reconstruction procedure are very difficult to handle with on the up-to-now accepted level of generalization. For example, it cannot be generally stated which of the quantities $\gamma_{ac}(\bar{a}_s)$ or $\gamma_{ab}(\bar{a}_s)$ is greater, since it depends, in a complex way, on the relation of scanning step to the degree of concentration of the elementary structural functions around the object scanning points.

As far as the object is concerned, the relationship between the said three reconstruction situations is even more complex. For example, if we have no a priori information about the object (case C), we have no reasonable measure of reconstruction error as the latter can be infinitely great in all the reconstructed object points. The formulas (8) for the case (A) and (17) for the case (B), give the measure of object reconstruction errors at the object scanning points under the assumption that the elementary structural functions are given. The errors estimated from these formulas can be compared with each other but no quantitative relation to the case (C) may be obtained.

V. Generalization of the recovery procedure

In the up-to-now considerations we have implicitly assumed that 1) imaging system and observing systems are stationary; 2) configuration of scanning points creates a regular, rectangular lattice; 3) scanning points in the image and the recovery points in the object plane are related with each other by the formula

$$(\bar{a}_r) = \gamma_{\text{im}}(\bar{a}_r). \qquad r = 1, \ldots, N \tag{33}$$

and 4) we have assumed monochromacy of the light.

In the following we will get rid of those restrictive assumptions.

As to the stationarity of the imaging system, it has been assumed by derivation of the formula (1) in [1] as the latter involves Fourier transformation operations. However, it may be easily seen that in developing the recovery procedure we have never made use of the assumption. Thus, if we get rid of it in the starting expression (1) in [1], we obtain more general reconstruction problems. The simplest, though not quite consistent method of doing it is by replacing

$$K_{\mathrm{im}}\left(\frac{\bar{\boldsymbol{p}}_1}{z_1}+\frac{\bar{\boldsymbol{a}}}{z_2}\right) \quad \text{and} \quad K^*_{\mathrm{im}}\left(\frac{\bar{\boldsymbol{p}}_2}{z_1}+\frac{\bar{\boldsymbol{a}}}{z_2}\right),$$

by the general amplitude spread functions

$$K_{\mathrm{im}}(\bar{\boldsymbol{p}}_1,\bar{\boldsymbol{a}}) \quad \text{and} \quad K^*_{\mathrm{im}}(\bar{\boldsymbol{p}}_2,\bar{\boldsymbol{a}}), \tag{34}$$

of the imaging system. Thus, we can introduce in our consideration the non-stationary effects in the image plane. A similar operation may be performed on the formula (6) in [1] by replacing

$$K_{\mathrm{obs}}\left(\frac{\bar{\boldsymbol{u}}}{z_1^0}+\frac{\bar{\boldsymbol{p}}_1-\bar{\boldsymbol{a}}}{z_2^0}\right) \quad \text{and} \quad K^*_{im}\left(\frac{\bar{\boldsymbol{u}}}{z_1^0}+\frac{\bar{\boldsymbol{p}}_2-\bar{\boldsymbol{a}}}{z_2^0}\right),$$

by the general amplitude spread functions

$$K_{\mathrm{obs}}(\bar{\boldsymbol{u}},\bar{\boldsymbol{p}}_1-\bar{\boldsymbol{a}}) \quad \text{and} \quad K^*_{\mathrm{obs}}(\bar{\boldsymbol{u}},\bar{\boldsymbol{p}}_2-\bar{\boldsymbol{a}}). \tag{35}$$

In the last case, however, if the integrating element is located on the axis of the observing system and is small in comparison with the scanned area, the assumption of stationarity is well justified, and the gain in accuracy due to this replacement is usually small.

It may be interesting to notice that the generalization in this direction may be consistently introduced within the strictly incoherent approximation of the recovery problem, consisting in neglecting the partial coherence in the image plane. This has been done in [4].

As to the scanning points configuration, it is obvious that usually we are not restricted to the rectangular lattice. In particular, the a priori knowledge about the object may be used also to determine the optimal configuration adequate to that information. For example, if we can distinguish the areas of greater and smaller interest within the examined region, then the scanning point density should be proportional to the numerical measure of the interest. It is obvious that such redistributing of scanning points will influence the reconstruction error distribution in favor of the areas of greater interest.

Next, we can readily replace the assumed relationship (34) by a more exact one, if the imaging system is given by its amplitude spread function (35). The most natural, though complicated, way of doing it is the following.

Suppose for a moment that the imaging system transforms an incoherent object of uniform intensity into the image plane. Then, for each given scanning point $(\bar{\boldsymbol{a}}_k)$ we can find the object point $(\bar{a}_k)$, which contributes mostly to the observed image point $x(\bar{\boldsymbol{a}}_k)$. This may be done in the following way. From (7) in [1] by denoting $I(\bar{a}) = I_0$ and $G''I_0 = C$ we have:

$$x(\bar{\boldsymbol{a}}_k) = C\iint\limits_{P_0P} K_{\mathrm{im}}\left(\frac{\bar{\boldsymbol{p}}_1}{z_2}+\frac{\bar{a}}{z_1}\right)K^*_{\mathrm{im}}\left(\frac{\bar{\boldsymbol{p}}_2}{z_2}+\frac{\bar{a}}{z_1}\right)$$

$$x = \varphi(\bar{\boldsymbol{p}}_1-\bar{\boldsymbol{a}}_k, \bar{\boldsymbol{p}}-\bar{\boldsymbol{a}}_k)d\bar{\boldsymbol{p}}_1\, d\bar{\boldsymbol{p}}_2\, d\bar{\boldsymbol{a}}. \qquad k = 1, 2, \ldots, N$$

Defining

$$f(a,a) = \int\limits_P K_{\mathrm{im}}\left(\frac{\bar{\boldsymbol{p}}_1}{z_2}+\frac{\bar{a}}{z_1}\right)K^*_{\mathrm{im}}\left(\frac{\bar{\boldsymbol{p}}_2}{z_2}+\frac{\bar{a}}{z_1}\right)\times$$

$$\times\varphi(\bar{\boldsymbol{p}}_1-\bar{\boldsymbol{a}}_k, \bar{\boldsymbol{p}}_2-\bar{\boldsymbol{a}}_k)d\bar{\boldsymbol{p}}_1\, d\bar{\boldsymbol{p}}_2. \tag{36}$$

$$k = 1, 2, \ldots, N$$

We want to find these points $\bar{\boldsymbol{a}} = \bar{\boldsymbol{a}}_k$, which maximize the corresponding functions (36). If analytical or tabular form of K_{im}, K^*_{im} and φ are known the task may be fulfilled in a routine way, and thus the recovery points $\bar{a}_k$ corresponding to each point $\bar{\boldsymbol{a}}_k$ are strictly determined by the criterion of maximal contribution.

Finally, the generalization to the non-monochromatic case consists of integrating all the formulas with respect to the wavelength with a weighting function describing spectral intensity distribution of the used light source.

VI. Concluding remarks

As indicated in the introduction, the aim of the present paper was to clarify the recovery procedure in the sense of finding the relationship between what is really given by the measurement and what is concluded about. The introducing of the integrating element into consideration makes the analysis more realistic and more complex at the same time. To discover the said relationship, a simplified model of measurement situation has been accepted as shown in Fig. 1 in [1]. It has been found that even for this simplified model the relation between the results of measurement given in the form of the observed image points and the intensity distribution in the image and the object, is by no means simple. In particular, lack of immediate connection between the observed image points and the image intensity distribution as shown in the paper [1] is a surprising result. Thus, the necessity of special procedure (called here direct recovery procedure) for reconstruction of both image and object is evident.

The developed direct recovery method requires knowledge of the amplitude spread function $K_{\mathrm{im}}(\bar{p},\bar{a})$ of the imaging system, of the instrumental function $\varphi(\bar{p}_1-\bar{a},\ \bar{p}_2-\bar{a})$ of the observing system and the set of the observed image points* $x(\bar{a}_k)$, $k = 1, \ldots, N$. Furthermore, if any a priori information about the object is available before the measurement, it must be analytically expressible to be of any value for the recovery procedure proposed in the paper. On the base of the above knowledge, the upper and lower bound reconstruction matrices may be calculated. These matrices contain all the information about the imaging and observing systems, the scanning configuration and all the a priori knowledge about the object, and thus determine completely the recovery procedure. It is worth noticing that all the mathematical operation necessary to calculate reconstruction matrices may be easily computized, provided the necessary characteristics of imaging and observing systems are really available. Then the recovery procedure consists of the following:

The deck of input data contains the observed image points $x(\bar{a}_k)$ $k = 1, \ldots, N$. The output may consist of:

(A) As far as image reconstruction is concerned, we may obtain real image points $I_{\mathrm{im}}(\bar{a}_k)$ estimated jointly with the error of their reconstruction $\Delta I_{\mathrm{im}}(\bar{a}_k)$ and tabulated image intensity distribution $I_{\mathrm{im}}(\bar{p})$ in the recovered region. This is available for all the cases independently of the kind of the a priori knowledge about the object.

(B) As far as object recovery is concerned we must distinguish two cases.

If no a priori knowledge about the object is available, the only information after the measurement is that both the representations (18a) and (18b) in [1] and an infinite number of other in-between representations are consistent with the observed image points established by measurement. No reasonable measures of reconstruction errors may be obtained as the error is infinitely great** at all the recovered points. This is one more proof of the well-known fact that some a priori information about the object is necessary for its successful reconstruction (but not for the recovery of its image).

If some a priori information is available then, as shown in the case of half-tone screen objects and band limited objects, the reconstruction procedure results in reasonable estimations of the intensity in the recovery object points $I_{\mathrm{ob}}(\bar{a}_k)$ jointly with the error estimation formulas. It must be emphasized, however, that the error formulas are valid within the assumed object structure only the latter being determined by the acceptance of a particular set of the elementary structural functions. The error due to an incorrect choice of elementary structural functions is here not taken into account. Furthermore, the procedure provides some natural interpolation processes in the form of expressions (7) and (17) for evaluating the object intensity between the recovery points.

A serious inconvenience of the proposed method is the fact that it requires the knowledge of the amplitude spread functions of both the imaging and observing systems. As those functions are not always known, the incoherent approximation approach to the recovery problem, as formulated in [2], may be of considerable practical value. This approach ignores the partial coherence in the image plane and consequently the reconstruction procedure is described in terms of intensity spread functions of both the imaging and the observing systems.

Another essential lack of the developed theory is the over-simplification of the measurement model assumed for consideration. In reality, the image is often recorded on a photographic plane before being subjected to scanning by observing systems. In this case such properties of the emulsion like nonlinearity and noise must be taken into account. Noise may be introduced in many other places of the recovery arrangement including electric noise in the part behind the integrating element. In general, however, the developed relations are essentially true and may be a good theoretical basic for more detailed treatment.

Finally, from the theoretical point of view, the direct recovery procedure is more realistic than the methods based on the Fourier transforming, as the last operation involves assumption of stationarity of

* Note that the set of observed image points $x(\bar{a}_k)$, $k = 1, \ldots, N$ contains information on both the measurement results and the configuration of the scanning points.

** Strictly speaking the situation is slightly better, because the recovered object distribution

$$I_{\mathrm{ob}}(\bar{a}) = \sum_{k=1}^{N} c_k^N \delta(\bar{a}-\bar{a}_k)$$

may be interpreted in such a way that the functions $\delta(\bar{a}-\bar{a}_k)$ describe the position of the recovered object points while the coefficient c_k^N denote the respective values of the local intensity. Consequently, the error of reconstruction even in the case of complete ignorance is no greater than c_k^N and so finite. Reformation of the recovery procedure in this spirit might be interesting though no essentially new results may be expected.

the optical system, the requirement which is never mat strictly by the optical systems.

Influence de l'information a priori sur le procédé de la reconstruction immédiate

Résumé

Dans le travail [1] on a établi la méthode be la reconstruction immédiate dans le casoù avant la mesure aucune information sur l'objet n'est connue. Le présent travail est consacré à une analyse quantitative de l'influence de l'information a priori tant sur le procédé de la reconstruction que sur ses résultats.

Влияние информации a priori о предмете на процедуру непосредственной реконструкции в некогерентном отображении

В работе [1] приводится метод непосредственной реконструкции для случая, когда не было никакой информации о предмете перед измерением. Эта работа посвящена количественному анализу влияния информации a priori как на процедуру реконструкции, так и на её результаты.

References

[1] WILK I., *Direct Recovery Problem in Incoherent Imaging*, Optica Applicata I, 2

[2] WILK I., *Rekonstrukcja dowolnego obszaru powietrznego obrazu optycznego. Przyczynek do teorii rekonstrukcji obrazu i optymalizacji ukladu pomiarowo-obserwacyjnego IV. Prace* Nauk. Inst. Fiz. Tech. Pol. Wr., Nr 4, seria: Studia i Materiały Nr 4, 1970, p. 3.

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