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A CONTROL CHART USING BELIEF INFORMATION FOR A GAMMA DISTRIBUTION

The design of a control chart has been presented using a belief estimator by assuming that the quantitative characteristic of interest follows the gamma distribution. The authors present the structure of the proposed chart and derive the average run lengths for in-control and a shifted process. The average run lengths for various specified parameters have been reported. The efficiency of the proposed chart has been compared to existing control charts. The application of the proposed chart is illustrated with the help of simulated data.

Keywords: *control chart, belief statistic, average run length, gamma distribution*

1. Introduction

Manufacturing high quality products requires advanced production technology. Such technology plays a significant role in improving the quality of a product [1]. However, advanced manufacturing technology depends on the early detection of any shift in the process. Therefore, control charts are one of the important tools for ensuring the high quality of a product. These tools are used to monitor the process and provide quick indication when the process is shifted due to various controllable or unpredictable factors. Timely indication about a shift in the production process minimizes the proportion

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of non-conforming products. The idea of a control chart was given by [2]. Shewhart control charts are useful for detecting larger shifts in a manufacturing process but they are unable to detect a smaller shift in the process. Therefore, some alternative control charts such as the cumulative sum (CUSUM) control chart, and exponentially weighted moving average (EWMA) chart have been developed to overcome this issue. The CUSUM chart was originally designed by [3]. Later on, [4] proved the efficiency of the CUSUM chart compared to the Shewhart control chart. Shewhart control charts only use current information to make a decision about the state of a process. On the other hand, CUSUM and EWMA charts are based on the utilization of current and past information to make decisions about a process. As suggested by [5] *EWMA control chart detects the shift of one half to one standard deviation of the process mean*. The design and applications of CUSUM and EWMA control charts are discussed by several authors including, for example [6–20].

Fuzzy approaches are very helpful when designing a control chart in the case of ambiguity or not well defined situations. A control chart using the direct fuzzy approach, which is efficient and independent of the defuzzification method applied, was designed by [21]. [22] designed a control chart using a belief estimator by assuming that quality is modeled by a normal distribution.

Control charts are designed by assuming that the quantitative trait of interest follows the normal distribution, which is not always true in practice. The variable of interest may follow some non-normal distribution such as an exponential distribution or a gamma distribution. The use of control charts designed for a normal distribution may not be workable in this situation and may cause an increase in the proportion of non-conforming products. Furthermore, the normal distribution is applied in situations where data is collected in subgroups, so that the central limit theorem can be applied when designing the control chart. Again, in practice, it is not always possible to collect data in groups. Data not collected in subgroups may be skewed and they may fit an exponential distribution or a Gamma distribution better [23]. In fact, the convolution of exponential distributions is a gamma distribution and so the gamma distribution is useful to model the sum of exponentially distributed random variables. More details about this issue can be read in [24, 7, 25, 26, 18, 19, 27].

There is an article on a control chart for data from a gamma distribution [14], but it is for monitoring the time until an event under the assumption that a particular event follows a homogeneous Poisson process. Hence, it may not be used for the general purpose of monitoring a gamma-distributed characteristic. In this paper, we will present a control chart for the gamma distribution using a belief estimator. We will develop the necessary measures for the proposed chart and discuss the application of this chart with the help of simulated data.

2. Design of the proposed chart

Suppose that the quantity of interest (such as the time between events) T follows the gamma distribution with the following cumulative distribution function (cdf)

$$P\left(T \le t\right) = 1 - \sum_{j=0}^{a-1} \frac{\mathrm{e}^{-1/b}\left(\frac{t}{b}\right)^{j}}{j!} \tag{1}$$

where a is the shape parameter and b is the scale parameter. This distribution reduces to an exponential distribution when $a = 1$. According to Wilson and Hilferty[4] if T has a gamma distribution, then $T^* = T^{1/3}$ approximately follows the normal distribution.

The mean and the variance of the variable T^* are given as follows:

$$\mu_{(T^*)} = E(T^*) = b^{1/3} \frac{\Gamma\left(a + \frac{1}{3}\right)}{\Gamma\left(a\right)} \quad (2$$

$$\sigma_{T^*} = b^{1/3} \sqrt{\frac{\Gamma\left(a+\frac{2}{3}\right)}{\Gamma(a)} - \left(\frac{\Gamma\left(a+\frac{1}{3}\right)}{\Gamma(a)}\right)^2} \tag{3}$$

Hence, the distribution of T^* is approximately normal as follows:

$$T^* \sim N\left(\frac{b^{1/3}\Gamma\left(a+\frac{1}{3}\right)}{\Gamma(a)}, \frac{b^{2/3}\Gamma\left(a+\frac{2}{3}\right)}{\Gamma(a)} - \mu_{T^*}^2\right) \tag{4}$$

In the data collection process, it is assumed that a single observation ($n = 1$) of the quantity of interest is collected at each iteration or subgroup. Suppose that T_k and $O_k = \left(T_1, T_2, ..., T_k\right)$ are the k-th observation and vector of observations up to the k-th iteration, respectively. Let $O_k = (T_k, O_{k-1})$, $B(O_k)$ be the posterior belief and $B(O_{k-1})$ be the prior belief. The goal is to update $B(O_k)$ using $B(O_{k-1})$ and a new observation T_k.

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[4]Wilson E.B., Hilferty M.M., *The distribution of chi-squares*, Proc. of the National Academy of Sciences, 1931, 17, 684.

For a gamma distribution, using the transformed variable $T = T^{1/3}$, the posterior belief will be updated using the following equation:

$$B(O_k) = B(T_k, O_{k-1}) = \frac{B(O_{k-1})\mathrm{e}^{(T^* - \mu_{T^*})/\sigma_{T^*}}}{B(O_{k-1})\mathrm{e}^{(T^* - \mu_{T^*})/\sigma_{T^*}} + \left(1 - B(O_{k-1})\right)} \tag{5}$$

Note that for simplicity, the subscript k is omitted in the variable T^*.

We define the statistic suggested by [22] as

$$Z_k = \frac{B(O_k)}{1 - B(O_k)} \tag{6}$$

The recursion relation is given as

$$Z_k = Z_{k-1}\mathrm{e}^{(T^* - \mu_{T^*})/\sigma_{T^*}} \tag{7}$$

We set the initial value, Z_0 to be equal to 1 and $B(O_0) = 0.5.$ According to [22], the statistic given in Eq. (6) follows the normal distribution with mean 0 and variance k.

Hence, the upper control limit (UCL) and lower control limit (LCL) of the proposed control chart are given as follows:

$$UCL = L\sqrt{k} \tag{8}$$

$$LCL = -L\sqrt{k} \tag{9}$$

The control coefficient L will be determined by specifying the type-1 error rate or the average run length for online control of a process. In summary, the proposed control chart uses the following procedure:

Step 1. At the k-th subgroup, select an item randomly and measure its quantitative characteristic T_k. Then, calculate $T_k^* = T_k^{1/3}$ and calculate

$$\ln(Z_k) = \ln(Z_{k-1}) + \frac{T_k^* - \mu_{T_k^*}}{\sigma_{T_k^*}} \tag{10}$$

Step 2. Declare the process as under-control if $LCL \le \ln(Z_k) \le UCL$. Declare the process as out-of-control if $\ln(Z_k) > UCL$ or $\ln(Z_k) < LCL$.

To derive the necessary measures for an under-control process and an out-of-control process, it is assumed that the scale parameter of the gamma distribution will change and the shape parameter will remain unchanged during the process. Let b_0 and b_1 denote the scale parameter for an under-control process and an out-of-control process, respectively. For an in-control process, the probability of incorrectly declaring that the process is out-of-control is given as follows:

$$\begin{aligned} P_{\text{out}}^{0} &= P\left\{\ln(Z_k) < LCL \mid b = b_0\right\} + P\left\{\ln\left(Z_k\right) > UCL \mid b = b_0\right\} \\ &= 1 - \Phi\left(\frac{UCL}{\sqrt{k}}\right) + \Phi\left(\frac{LCL}{\sqrt{k}}\right) \end{aligned} \tag{11}$$

Finally, Eq. (11) reduces to

$$P_{\text{out}}^{0} = 1 - \Phi(L) + \Phi(-L) \tag{12}$$

It is noted that P_{out}^{0} is independent of k.

The average run length (*ARL*) has been widely applied to control charts. This indicates how long on average it takes before a process is classified as out-of-control. The *ARL* for an in-control process is given as follows:

$$ARL_0 = \frac{1}{P_{\text{out}}^{0}} \tag{13}$$

Now, we derive the necessary measures for the shifted process. We suppose that the scale parameter of the gamma distribution has been shifted from b_0 to $b_1 = sb_0$, where s is a shift constant.

The mean and the variance of T^* for the shifted process are given by

$$E\left(T^* \mid b_1\right) = s^{1/3} b_0^{1/3} \frac{\Gamma\left(a + \frac{1}{3}\right)}{\Gamma(a)} \tag{14}$$

$$\operatorname{Var}\left(T^* \mid b_1\right) = s^{2/3} b_0^{2/3} \left[\frac{\Gamma\left(a + \frac{2}{3}\right)}{\Gamma(a)} - \left(\frac{\Gamma\left(a + \frac{1}{3}\right)}{\Gamma(a)}\right)^2\right] \tag{15}$$

So given b_1, $\ln(Z_k)$ follows an approximate normal distribution with mean and variance given as follows:

$$E\ln\left(Z_k\right)|\,b_1\big) = k\,\frac{\dfrac{\Gamma\left(a+\frac{1}{3}\right)}{\Gamma(a)}(s^{1/3}-1)}{\sqrt{\dfrac{\Gamma\left(a+\frac{2}{3}\right)}{\Gamma(a)}-\left(\dfrac{\Gamma\left(a+\frac{1}{3}\right)}{\Gamma(a)}\right)^2}} \tag{16}$$

$$\operatorname{Var}\ln\left(Z_k\right)|\,b_1\big) = ks^{2/3} \tag{17}$$

Therefore, the probability of declaring the shifted process as being out of control for the k-th subgroup is given as follows:

$$\begin{aligned} P^1_{\text{out},k} &= P\left\{\ln\left(Z_k\right) < LCL\,|\,b = b_1\right\} + P\left\{\ln\left(Z_k\right) > UCL\,|\,b = b_1\right\} \\ &= P\left\{\ln\left(Z_k\right) < -L\sqrt{k}\,|\,b = b_1\right\} + P\left\{\ln\left(Z_k\right) > L\sqrt{k}\,|\,b = b_1\right\} \end{aligned} \tag{18}$$

Finally, we have

$$\begin{aligned} P^1_{\text{out},k} &= 1-\Phi\left(\left(L\sqrt{k}-\frac{k\,\dfrac{\Gamma\left(a+\frac{1}{3}\right)}{\Gamma(a)}(s^{1/3}-1)}{\sqrt{\dfrac{\Gamma\left(a+\frac{2}{3}\right)}{\Gamma(a)}-\left(\dfrac{\Gamma\left(a+\frac{1}{3}\right)}{\Gamma(a)}\right)^2}}\right)\left(\sqrt{ks^{2/3}}\right)^{-1}\right) \\ &+\Phi\left(\left(-L\sqrt{k}-\frac{k\,\dfrac{\Gamma\left(a+\frac{1}{3}\right)}{\Gamma(a)}(s^{1/3}-1)}{\sqrt{\dfrac{\Gamma\left(a+\frac{2}{3}\right)}{\Gamma(a)}-\left(\dfrac{\Gamma\left(a+\frac{1}{3}\right)}{\Gamma(a)}\right)^2}}\right)\left(\sqrt{ks^{2/3}}\right)^{-1}\right) \end{aligned} \tag{19}$$

The probability of declaring the process as being out of control for the $(k + j)$-th subgroup when the shift in the process occurs at k is expressed as

$$P\left\{RL = j\right\} = \left(1 - P^{1}_{\text{out},k+1}\right)\left(1 - P^{1}_{\text{out},k+2}\right)\ldots\ (1 - P^{1}_{\text{out},k+j-1})P^{1}_{\text{out},k+j} \tag{20}$$

where RL is a random variable representing the out-of-control run length.

Therefore, ARL for the shifted process under the proposed control chart is given as follows:

$$ARL_1 = P^{1}_{\text{out},k+1} + 2(1 - P^{1}_{\text{out},k+1})P^{1}_{\text{out},k+2} + 3(1 - P^{1}_{\text{out},k+1})(1 - P^{1}_{\text{out},k+2})P^{1}_{\text{out},k+3} + \ldots \tag{21}$$

Let r_0 be the specified target ARL. The following algorithm is used to determine the control coefficient L and the ARL for the shifted process:

Step 1. Select a range for the control coefficient L.

Step 2. Determine L such that $ARL_0 \geq r_0$.

Step 3. Use Eq. (19) to calculate $P^{1}_{\text{out},k}$ for a fixed k and various shift constants s.

Step 4. Determine the values of ARL_1 for a fixed k and various shift constants s.

We reported ARL_1 for various specified parameters such as $a = 1, 5, 10$, $r_0 = 300, 370$ and $s = 1, 1.1, 1.2, 1.5, 2, 3, 3.5, 4$. In Tables 1, 2, the values of ARL_1 are reported for $a = 1$. In Tables 3, 4, the values of ARL_1 are reported for $a = 5$. In Tables 4–6, the values of ARL_1 are reported for $a = 10$.

From Tables 1–6, we note the following trends in ARL_1:

1. For a fixed value of a, the ARL_1 increases as r_0 increases.
2. For a fixed value of s, the ARL_1 decreases as k increases.
3. For fixed values of a and k, the ARL_1 decreases as s increases.
4. For fixed values of the remaining parameters, the ARL_1 increases as a increases.

Table 1. The ARL when $r_0 = 300$ and $a = 1$, $L = 2.9352$

s	k					
	3	10	25	50	100	500
1	300.00	300.00	300.00	300.00	300.00	300.00
1.01	290.42	289.63	287.95	285.19	279.80	242.53
1.02	280.66	277.69	271.51	261.76	244.02	154.64
1.03	270.78	264.56	252.06	233.43	202.74	92.10
1.04	260.87	250.63	230.96	203.71	163.52	55.36
1.05	250.98	236.26	209.42	175.14	129.96	34.39
1.1	203.68	167.12	118.52	77.00	42.04	5.59
1.2	129.55	78.24	38.90	18.79	7.86	1.24
1.5	38.95	13.95	4.89	2.18	1.22	1.00
2	11.17	3.37	1.44	1.05	1.00	1.00
3	3.53	1.36	1.01	1.00	1.00	1.00
3.5	2.63	1.18	1.00	1.00	1.00	1.00
4	2.15	1.09	1.00	1.00	1.00	1.00

Table 2. The *ARL* when $r_0 = 370$, $a = 1$, $L = 2.9996$

s	*k*					
	3	10	25	50	100	500
1	370.00	370.00	370.00	370.00	370.00	370.00
1.01	357.71	356.70	354.54	351.00	344.11	296.66
1.02	345.20	341.40	333.51	321.06	298.51	186.36
1.03	332.57	324.62	308.69	285.05	246.34	109.39
1.04	319.91	306.86	281.89	247.51	197.27	64.90
1.05	307.30	288.58	254.66	211.69	155.67	39.83
1.1	247.29	201.47	141.36	90.78	48.81	6.13
1.2	154.57	91.95	44.90	21.30	8.70	1.27
1.5	44.58	15.55	5.29	2.29	1.24	1.00
2	12.27	3.58	1.48	1.05	1.00	1.00
3	3.73	1.39	1.02	1.00	1.00	1.00
3.5	2.75	1.19	1.00	1.00	1.00	1.00
4	2.22	1.10	1.00	1.00	1.00	1.00

Table 3. The *ARL* when $r_0 = 300$, $a = 5$, $L = 2.9352$

s	*k*					
	3	10	25	50	100	500
1	300.00	300.00	300.00	300.00	300.00	300.00
1.01	288.83	284.43	275.38	261.39	236.94	130.19
1.02	274.74	259.15	230.59	193.81	144.81	38.99
1.03	258.51	228.67	181.87	133.21	83.17	14.04
1.04	240.97	197.12	139.11	89.86	48.64	6.21
1.05	222.85	167.31	105.28	61.11	29.61	3.31
1.1	140.51	69.66	28.83	12.19	4.66	1.04
1.2	53.93	16.10	4.89	2.04	1.15	1.00
1.5	7.69	1.91	1.06	1.00	1.00	1.00
2	1.97	1.03	1.00	1.00	1.00	1.00
3	1.08	1.00	1.00	1.00	1.00	1.00
3.5	1.03	1.00	1.00	1.00	1.00	1.00
4	1.01	1.00	1.00	1.00	1.00	1.00

Table 4. The *ARL* when $r_0 = 370$, $a = 5$, $L = 2.9996$

s	*k*					
	3	10	25	50	100	500
1	370.00	370.00	370.00	370.00	370.00	370.00
1.01	355.68	350.03	338.45	320.62	289.58	156.19
1.02	337.63	317.74	281.50	235.20	174.18	45.35
1.03	316.91	279.03	220.20	159.79	98.54	15.89
1.04	294.59	239.24	167.02	106.62	56.84	6.85
1.05	271.61	201.93	125.38	71.78	34.15	3.57
1.1	168.43	81.91	33.16	13.70	5.08	1.05

s	*k*					
	3	10	25	50	100	500
1.2	62.77	18.18	5.33	2.15	1.17	1.00
1.5	8.44	1.99	1.07	1.00	1.00	1.00
2	2.06	1.03	1.00	1.00	1.00	1.00
3	1.09	1.00	1.00	1.00	1.00	1.00
3.5	1.03	1.00	1.00	1.00	1.00	1.00
4	1.01	1.00	1.00	1.00	1.00	1.00

Table 5. The *ARL* when r_0 = 300, a = 10, L = 2.9352

s	*k*					
	3	10	25	50	100	500
1	300.00	300.00	300.00	300.00	300.00	300.00
1.01	286.85	278.08	260.83	236.00	197.23	76.93
1.02	267.57	238.69	192.49	143.23	91.14	16.04
1.03	244.37	194.23	131.66	81.78	42.44	5.05
1.04	219.43	153.30	88.45	47.62	21.43	2.32
1.05	194.54	119.39	59.95	28.89	11.80	1.45
1.1	98.33	35.97	11.85	4.52	1.86	1.00
1.2	27.98	6.42	1.99	1.14	1.00	1.00
1.5	3.28	1.13	1.00	1.00	1.00	1.00
2	1.18	1.00	1.00	1.00	1.00	1.00
3	1.00	1.00	1.00	1.00	1.00	1.00
3.5	1.00	1.00	1.00	1.00	1.00	1.00
4	1.00	1.00	1.00	1.00	1.00	1.00

Table 6. The *ARL* when r_0 = 370, a = 10, L = 2.9996

s	*k*					
	3	10	25	50	100	500
1	370.00	370.00	370.00	370.00	370.00	370.00
1.01	353.13	341.91	319.91	288.39	239.54	91.05
1.02	328.48	291.75	233.54	172.22	108.28	18.22
1.03	298.93	235.66	157.88	96.84	49.45	5.54
1.04	267.32	184.58	104.90	55.62	24.52	2.46
1.05	235.95	142.67	70.38	33.31	13.28	1.50
1.1	116.67	41.61	13.31	4.93	1.96	1.00
1.2	32.04	7.06	2.09	1.16	1.00	1.00
1.5	3.50	1.15	1.00	1.00	1.00	1.00
2	1.20	1.00	1.00	1.00	1.00	1.00
3	1.00	1.00	1.00	1.00	1.00	1.00
3.5	1.00	1.00	1.00	1.00	1.00	1.00
4	1.00	1.00	1.00	1.00	1.00	1.00

The *ARL*s for various values of k from Table 4 are plotted in Fig. 1 as a function of s. From Figure 4, it can be noted that the values of *ARL* for k = 3 are larger than those for

k = 500. This figure clearly indicates that, using the proposed chart, having a larger k leads to a greater probability of detecting a shift of the process.

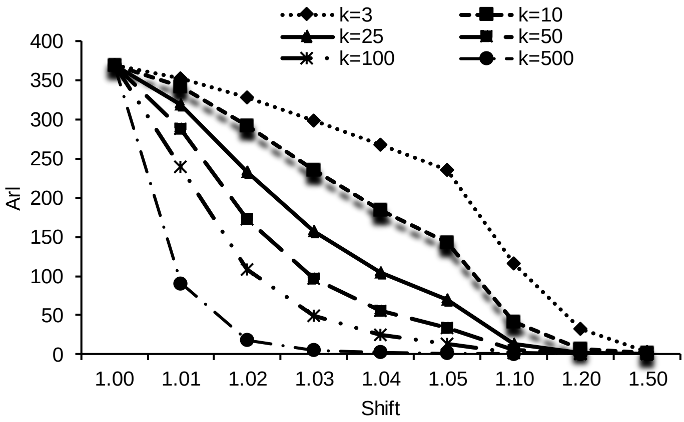

Fig. 1. The *ARL* curve for various *k*

3. Advantages of the proposed control chart

In this section, we will compare the advantages of the proposed control chart over the existing control chart. In each comparison, we will use the same values for all the specified parameters.

3.1. Proposed chart versus *t*-chart proposed by [23]

To compare the efficiency of both control charts, it is assumed that $r_0 = 300$.

Table 7. The average run length when ARL_0 = 399, L = 2.9352

s	[28]			Proposed chart when a = 5			[23]
	K						
	3	10	25	3	10	25	
1	300	300	300	300	300	300	300
1.1	203.68	167.12	118.52	140.51	69.66	28.83	227.57
1.2	129.55	78.24	38.90	53.93	16.10	4.89	170.82
1.5	38.95	13.95	4.89	7.69	1.91	1.06	77.62
2	11.17	3.37	1.44	1.97	1.03	1.00	30.02
3	3.53	1.36	1.01	1.08	1.00	1.00	10.39

The *ARL* values for both control charts are presented in Table 7. We note from Table 7 that the proposed control chart provides smaller *ARL*s as compared to the chart defined by [23] for all values of k and s. For example, when $s = 1.1$, the *ARL* value from the chart defined in [23] is 227.57 and from our proposed chart it is approximately 140 when $k = 3$ and 29 when $k = 25$.

3.2. Proposed chart versus chart proposed by [28]

The *ARL* values for the chart presented in [28] are presented in Table 7. It can be seen from Table 7 that the proposed chart once again provides smaller values of the *ARL* as compared to the chart presented in [28]. For example, when $s = 1.1$, the *ARL* value from the chart presented in [28] is approximately 204 when $k = 3$ and 119 when $k = 25$ and from our proposed chart it is approximately140 when $k = 3$ and 29 when $k = 25$.

4. Simulation study

In this section, the application of the proposed control chart in industry will be illustrated with the help of simulated data.

Table 8. Simulated data

No.	$B(Q_k)$	Z_k	$\ln Z_k$	No.	$B(Q_k)$	Z_k	$\ln Z_k$
1	0.7664	3.2817	1.1884	21	0.9713	33.8526	3.5220
2	0.0636	0.0679	–2.6894	22	0.7691	3.3303	1.2031
3	0.0239	0.0245	–3.7083	23	0.5513	1.2288	0.2060
4	0.0046	0.0046	–5.3817	24	0.9990	1046.0980	6.9528
5	0.1067	0.1194	–2.1254	25	0.8586	6.0703	1.8034
6	0.9259	12.4903	2.5250	26	0.6036	1.5225	0.4203
7	0.9977	443.0305	6.0936	27	0.9987	756.0622	6.6281
8	0.3436	0.5235	–0.6473	28	0.7059	2.4002	0.8756
9	0.9197	11.4597	2.4388	29	0.8458	5.4832	1.7017
10	0.2450	0.3246	–1.1253	30	0.9996	2781.2170	7.9306
11	0.7712	3.3708	1.2151	31	0.9316	13.6223	2.6117
12	0.2075	0.2619	–1.3398	32	0.8908	8.1616	2.0994
13	0.3970	0.6584	–0.4179	33	0.9517	19.6902	2.9801
14	0.0100	0.0101	–4.5971	34	0.9621	25.3831	3.2341
15	0.7039	2.3771	0.8659	35	0.6879	2.2044	0.7905
16	0.1876	0.2309	–1.4659	36	0.9955	222.3957	5.4045
17	0.9986	739.2333	6.6056	37	1.0000	23556.4900	10.0672
18	0.0580	0.0616	–2.7873	38	1.0000	94899.7300	11.4606
19	0.2423	0.3198	–1.1401	39	0.9248	12.2922	2.5090
20	0.0603	0.0641	–2.7471	40	0.9674	29.6365	3.3890

We generated the first 20 observations in the control process from a gamma distribution with $a = 5$ and $b_0 = 2$, and the next 20 observations from the shifted process with $s = 1.2$. The simulated data are reported in Table 8. The values of $B(O_k)$ and $\ln(Z_k)$ are also reported in Table 8.

Let $r_0 = 370$ and $k = 10$. The control chart coefficient L is 2.9996. The $UCL = 9.49$ and $LCL = –9.49$ for these simulated data. We plotted the statistic $\ln(Z_k)$ on the control chart in Fig. 2.

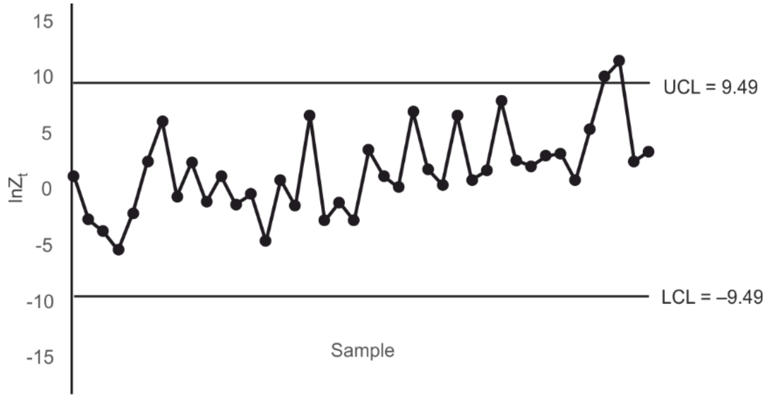

Fig. 2. Realization of the proposed chart for simulated data

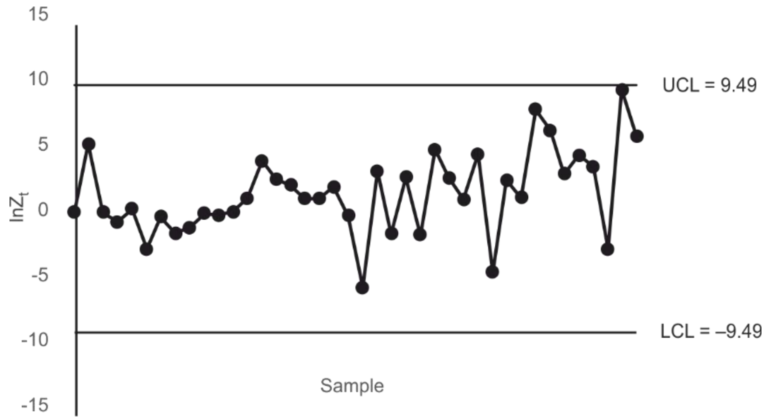

Fig. 3. The corresponding realization for the existing chart

The tabulated value of $ARL_1 = 18$. From Figure 2, it can be seen that the proposed chart detects the shift at the 38 observation. The data are also plotted in Fig. 3 for the chart proposed by [28] using the same parameters. The chart defined by [28] does not detect the shift in the process.

5. Example

Application of the proposed control chart is illustrated for a case of healthcare monitoring. We will use urinary tract infection (UTI) data from a large hospital. Similar data have been used by [23] and [28]. The data follow the gamma distribution with $a = 5$, and $b_0 = 2$. We assume the same parameter values as in the simulation study. The calculated statistics for these data are shown in Table 9.

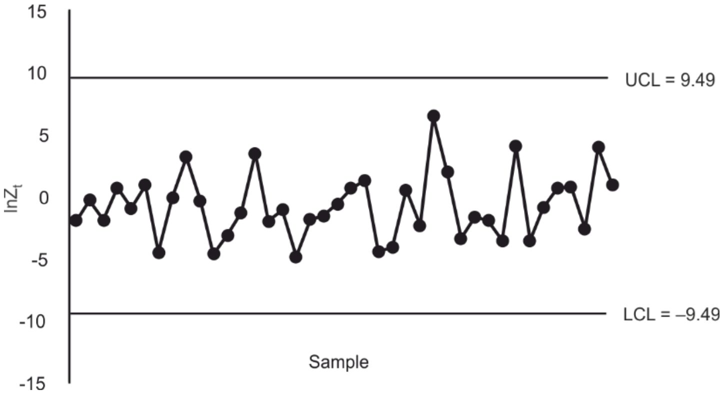

Fig. 4. Realization of the proposed chart for example data

Table 9. Data for the example

Sr#	BO_k)	Z_t	ln Z_t	Sr#	$B(O_k)$	Z_t	ln Z_t
1	0.1193	0.1355	–1.9990	21	0.6525	1.8776	0.6300
2	0.4199	0.7238	–0.3232	22	0.7815	3.5772	1.2746
3	0.1262	0.1444	–1.9349	23	0.0116	0.0117	–4.4484
4	0.6615	1.9542	0.6700	24	0.0156	0.0158	–4.1479
5	0.2703	0.3704	–0.9932	25	0.6182	1.6190	0.4818
6	0.7022	2.3579	0.8578	26	0.0845	0.0923	–2.3822
7	0.0106	0.0107	–4.5400	27	0.9984	622.6211	6.4339
8	0.4662	0.8733	–0.1355	28	0.8704	6.7174	1.9047
9	0.9576	22.5657	3.1164	29	0.0323	0.0334	–3.4000
10	0.4046	0.6795	–0.3864	30	0.1537	0.1816	–1.7060
11	0.0094	0.0095	–4.6582	31	0.1190	0.1351	–2.0015
12	0.0411	0.0429	–3.1493	32	0.0267	0.0274	–3.5957
13	0.2071	0.2612	–1.3423	33	0.9818	53.8815	3.9868
14	0.9683	30.5043	3.4179	34	0.0269	0.0276	–3.5891
15	0.1167	0.1322	–2.0237	35	0.2791	0.3872	–0.9488
16	0.2430	0.3210	–1.1363	36	0.6453	1.8190	0.5983
17	0.0076	0.0076	–4.8750	37	0.6785	2.1102	0.7468
18	0.1309	0.1506	–1.8933	38	0.0671	0.0720	–2.6314
19	0.1670	0.2004	–1.6072	39	0.9799	48.8265	3.8883
20	0.3470	0.5315	–0.6321	40	0.7148	2.5060	0.9187

The values of the statistic $\ln(Z_k)$ are plotted on the control chart in Fig. 3. From Figure 4, it can be seen that some points are near to the *UCL*.

6. Concluding remarks

A control chart has been presented when the quantitative characteristic of interest follows a gamma distribution. A belief estimator is used to derive the necessary measures for the proposed control chart. A comparison between the proposed control chart and existing control charts is given. From our comparison study, it is concluded that the proposed control chart performs better than the existing charts in terms of the *ARL*. Simulation data are used to illustrate the use of the proposed chart in industry. It is recommended to apply this control chart in industry to enable quick detection of shifts in a process, which consequently minimizes the proportion of non-conforming products. Propositions for control charts when data come from other distributions should be developed by future research.

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