OPERATIONS RESEARCH AND DECISIONS

No. 2 2016

DOI: 10.5277/ord160204

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ORDERS OF CRITICALITY IN VOTING GAMES

The authors focus on the problem of investigating the blackmail power of players in simple games, which is the possibility of players of threatening coalitions to cause them loss using arguments that are (apparently) unjustified. To this purpose, the classical notion of the criticality of players has been extended, in order to characterize situations where players may gain more power over the members of a coalition thanks to collusion with other players.

Keywords: *voting game, blackmailing power, semivalue*

1. Introduction

We consider a parliament that has produced a majority coalition. If this majority corresponds to a minimal winning coalition, then all the coalition parties are critical, i.e. each of them is able to destroy the majority by leaving. However, we may face a different situation in which not all the parties are critical, i.e. the majority corresponds to a quasi-minimal winning coalition. A similar situation was typical in the eighties when the Italian governments included five parties, namely Christian Democracy (*Democrazia Cristiana* – DC), Italian Socialist Party (*Partito Socialista Italiano* – PSI), Italian Social-Democratic Party (*Partito Socialista Democratico Italiano* – PSDI), Italian Republican Party (*Partito Repubblicano Italiano* – PRI), Italian Liberal Party

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(*Partito Liberale Italiano* – PLI); for some years, only DC was critical (in the last years of the eighties PSI also became critical) but all the parties received ministries and/or departments, and in 1981 the premiership of the government was given to Giovanni Spadolini, the leader of PRI.

At a first glance, this situation may seem unusual, because all the parties received a quota of the power, even if the non-critical parties should have received nothing. On the other hand, it is possible to notice that the total number of seats of the coalition parties meant that a minimal winning coalition would still exist even after a non-critical party left. The presence of non-critical parties can be explained by their role in making other parties marginal. For instance, in the case of a minimal winning coalition of four parties, each of them is critical and should receive a quarter of the power; but if the majority is formed by a quasi-minimal winning coalition including five parties, only one of which is critical, each non-critical party may reclaim some power from the unique critical party for its role in keeping other parties non-critical. In fact, its exit from the governing coalition would enlarge the group of critical parties, thus reducing the power of the original unique critical party. This situation was considered from a different viewpoint in [3]; they accounted for the possibility of parties forming a different majority coalition that excludes another party, which, in its turn, may propose another majority coalition that does not include the party that started the process. In this way, the hypotheses on which the bargaining set [1] relies are satisfied, so the elements in the bargaining set are suitable for measuring the power of the parties.

In the present Italian political situation, we may consider the group *Alleanza Liberal-popolare-Autonomie* (ALA) that is represented only in the Senate, where the majority supporting the Italian government is very unstable, as opposed to the Lower Chamber, *Camera dei Deputati*, where the majority is stable. The ALA group had in mind to support the approval of some reforms proposed by the government, thus acting as a critical party. However, the presence of this group had the consequence that other members of the majority, from the Democratic Party, decided to support the reforms in order to avoid the ALA group becoming critical. We can say that critical parties have a first order of criticality, while non-critical ones have a higher order of criticality.

The aim of this paper is to provide a formal definition of second order critical players, which may be extended to higher orders, and analyze some properties, resulting in a proposal for the allocation of power.

The paper is organized as follows. We start by recalling some notation and general definitions in Section 2. Section 3 deals with higher orders of criticality, where player i becomes critical for coalition M only if other players (not critical for the same coalition) leave M before i. In Section 4, we strengthen the notion of the criticality of player i to coalition M adding the further constraint that the threat of i to leave M is made "credible" only if i has an opportunity to form a winning coalition with players outside of M. Section 5 concludes.

2. Preliminaries

A cooperative game with transferable utility (TU-game) is a pair (N, v), where $N = \{1, 2, ..., n\}$ denotes the finite set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, with $v(\emptyset) = 0$. $v(S)$ is the worth of coalition $S \subseteq N$, i.e. what players in S may obtain by standing alone.

A TU-game (N, v) is simple when $v : 2^N \to \{0, 1\}$, with $S \subseteq T \Rightarrow v(S) \leq v(T)$ [4] and $v(N) = 1$. If $v(S) = 0$ then S is a losing coalition, while if $v(S) = 1$ then S is a winning coalition. Given a winning coalition S, if $S \setminus \{i\}$ is losing then $i \in N$ is a critical player for S. When a coalition S contains at least one critical player, S is a quasi-minimal winning coalition; when all the players of S are critical, it is a minimal winning coalition. A simple game may also be defined by giving the set of winning coalitions or the set of minimal winning coalitions.

A particular class of simple games is represented by weighted majority games. A vector of weights $(w_1, w_2, ..., w_n)$ is associated to the players that leads to the following definition of the characteristic function of the corresponding weighted majority game (N, w):

$$w(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}, \; S \subseteq N$$

where q is the majority quota. A weighted majority situation is often denoted as $[q; w_1, w_2, ..., w_n]$. Usually, we require that the game is proper or N-proper, i.e. if S is winning then $N \setminus S$ is losing; for this aim, it is sufficient to choose $q > \frac{1}{2} \sum_{i \in N} w_i$. Note that a simple game may not correspond to any weighted majority situation.

Given a TU-game (N, v), an allocation is an n-dimensional vector $(x_i)_{i \in N} \in \mathbb{R}^N$ assigning to player $i \in N$ the amount x_i; an allocation $(x_i)_{i \in N}$ is efficient if $x(N) = \sum_{i \in N} x_i = v(N)$. A solution is a function ψ that assigns an allocation $\psi(v)$ to each TU-game (N, v) belonging to a given class of games G with player set N.

For simple games, and in particular for weighted majority games, a solution is often called a power index, as each component x_i may be interpreted as the percentage of

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[4]This property is called monotonicity.

power assigned to player $i \in N$. In the literature, several power indices have been introduced; among others, we recall the following definitions.

The Shapley-Shubik index [7], φ, is the natural version for simple games of the Shapley value [6]. It is defined as the average of the marginal contributions of player i w.r.t. all the possible orderings of players and it can be written as:

$$\varphi_i(v) = \sum_{S \subseteq N, S \ni i} \frac{(s-1)!(n-s)!}{n!} m_i(S), \ i \in N$$

where n and s denote the cardinalities of the set of players N and of the coalition S, respectively, and $m_i(S) = v(S) - v(S \setminus \{i\})$ denotes the marginal contribution of player $i \in N$ to coalition $S \subseteq N$, $S \ni i$.

The normalized Banzhaf index [2], β, is similar to the Shapley–Shubik index but it considers the marginal contributions of a player to all possible coalitions, independently of the order of the players; first, we define:

$$\beta_i^*(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N, S \ni i} m_i(S), \ i \in N$$

then, by normalization we get:

$$\beta_i(v) = \frac{\beta_i^*(v)}{\sum_{j \in N} \beta_j^*(v)}, \ i \in N$$

Let $\mathbf{p} = (p_1, \ldots, p_n)$ be a vector of n non-negative numbers such that

$$\sum_{k=1}^{n} p_k \binom{n}{k} = 1$$

with the interpretation that p_s is the probability that a coalition of size s forms. We denote by $\pi^{\mathbf{p}}$ the semivalue [4] engendered by the vector $\mathbf{p}$. Hence,

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \subseteq N, S \ni i} p_s m_i(S) \tag{1}$$

Notice that both the Shapley–Shubik index φ and the Banzhaf index β^* can be defined as particular semivalues $\pi^{\mathbf{p}^{\varphi}}$ and $\pi^{\mathbf{p}^{\beta}}$, respectively, where the probability vector $\mathbf{p}^{\varphi}$ is such that $p_s^{\varphi} = (s-1)!(n-s)!/n!$ and $\mathbf{p}^{\beta}$ is such that $p_s^{\beta} = 1/2^{n-1}$. Notice that for a (monotonic) simple game *v*, relation (1) can be simply written as

$$\pi_i^{\mathbf{p}}(v) = \sum_{S \in W_i(v)} p_s m_i(S) \tag{2}$$

where $W_i(v)$ is the set of winning coalitions containing player $i \in N$. Moreover, the value $m_i(S) = 1$ only if i is critical for S in *v*, for each $S \in W_i(v)$. Consequently, the value $\pi_i^{\mathbf{p}}(v)$ can be interpreted as the probability of player i playing a critical role in *v*. Every semivalue $\pi^{\mathbf{p}}$ satisfies the symmetry property (i.e., $\pi_i^{\mathbf{p}}(v) = \pi_j^{\mathbf{p}}(v)$ for each game v and every pair of symmetric players $i, j \in N$, i.e. such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$)) and the null player property (i.e., $\pi_i^{\mathbf{p}}(v) = 0$ for each game v and every null player $i \in N$, i.e. such that $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$).

In the following, we mainly refer to the Banzhaf index because it is more probability-oriented, as it is not based on the order in which players enter a coalition which is reasonable for majority coalitions.

3. Second and higher orders of criticality

In this section, we introduce the formal definition of order of criticality for a player. Given a winning coalition $M \subseteq N$, a critical player $i \in M$ may be called first order critical for coalition *M*. Now we deal with the other players in *M*.

Definition 1. Let $M \subseteq N$, with $|M| \geq 3$, be a winning coalition; let $i \in M$ be a player s.t. $v(M \setminus \{i\}) = 1$. We say that player i is second order critical (SOC) for coalition *M*, via player $j \in M \setminus \{i\}$ iff $v(M \setminus \{i, j\}) = 0$ with $v(M \setminus \{j\}) = 1$.

The interpretation is the following: player i is not critical for *M*, but there exists in *M* another player *j*, different from *i*, s.t. *M* becomes a losing coalition when both the players leave. From this definition, we immediately get the following proposition.

Proposition 1. If player $i \in M$ is second order critical for coalition *M*, via player $j \in M$, then player j is second order critical for coalition *M*, via player *i*.

Remark 1. Note that when there are critical players of second order, there are at least two but they can be greater in number, as shown in the following example.

Example 1. Consider the weighted majority situation [51; 40, 8, 5, 5, 5]; considering the grand coalition of all players, the first party is the unique critical one, while the other four parties are second order critical, even if the last three parties are critical only via the second party. Definition 1 can be extended to higher orders as follows.

Definition 2. Let $k \geq 2$ be an integer, let $M \subseteq N$, with $|M| \geq k+2$, be a winning coalition; let $i \in M$ be a player s.t. $v(M \setminus \{i\}) = 1$. We say that player i is order $k+1$ critical for coalition *M*, via coalition $K \subseteq M \setminus \{i\}$, with $|K| = k$ iff

$$v(M \setminus K) - v(M \setminus (K \cup \{i\})) = 1 \qquad (3)$$

and *K* is the set of minimal cardinality satisfying (3), i.e.

$$v(M \setminus (T \cup \{i\})) = 1 \qquad (4)$$

for any $T \subset K$ with $|T| < k$.

The interpretation is similar to the previous one: Player i is not critical for *M* but there exists in M a coalition *K*, not including *i*, s.t. M becomes losing when all the players in $K \cup \{i\}$ leave.

It should also be noticed that the notion of minimal cardinality is crucial to unambiguously assign the order of criticality of a player in a coalition.

Example 2. Consider the weighted majority situation [31, 21, 5, 3, 3, 3, 3, 2]; player 7 becomes critical whenever either a coalition involving player 2 and any one of the players 3–6 are involved (they are the coalition *K* in the definition), or three of the players 3–6 are involved. According to Definition 2, in the grand coalition of all players, player 7 is third order critical.

Notice also that the above definition encompasses the definitions for lower orders. In particular, we obtain first order criticality when (3) is satisfied by $K = \varnothing$ of 0-cardinality, leading to $v(M) - v(M \setminus \{i\}) = 1$. For second order criticality, consider $K = \{j\}$, then, by (3), $v(M \setminus \{j\}) = 1$ and $v(M \setminus \{i, j\}) = 0$. Moreover, since $\{j\}$ is the set of minimal cardinality which makes i critical, then $v(M \setminus \{i\}) = 1$.

We can derive a more general result than Proposition 1.

Proposition 2. If player $i \in M$ is order $k+1$ critical for coalition *M*, via coalition $K \subset M$, then each player $j \in K$ is order $k+1$ critical for coalition *M*, via coalition $K \cup \{i\} \setminus \{j\}$.

Proof. Define $K' = K \setminus \{j\}$. Since i is critical for *M* via coalition $K = K' \cup \{j\}$, then

$$v(M \setminus (K' \cup \{j\})) - v(M \setminus (K' \cup \{i, j\})) = 1 \quad (5)$$

We want to show that j is critical for M via $K' \cup \{i\}$.

Now $v(M \setminus (K' \cup \{i, j\})) = 0$ by (5), and, since $|K'| < k$, $v(M \setminus (K' \cup \{i\})) = 1$ by (4). Therefore,

$$v(M \setminus (K' \cup \{i\})) - v(M \setminus (K' \cup \{i, j\})) = 1$$

We need to verify the minimality of $K' \cup \{i\}$. Consider $T \subset K' \cup \{i\}$. There are two cases:

A. $i \in T$, then

$$v(M \setminus (T \cup \{j\})) = v(M \setminus [T \cup \{j\} \setminus \{i\}] \cup \{i\}) = 1$$

since $|T \cup \{j\} \setminus \{i\}| < k$.

B. $i \notin T$, then $T \cup \{j\} \subset K$ and $v(M \setminus (T \cup \{j\})) \geq v(M \setminus K) = 1$ and Eq. (4) is always satisfied.

Remark 2. A null player is never critical.

Remark 3. Note that when there are critical players of order $k+1$, there are at least $k+1$, but they can be greater in number as in the following example.

Example 3. Consider the weighted majority situation [51; 44, 3, 3, 3, 3, 3, 3]; in this case, the first party is the unique critical one for the grand coalition, while the other six parties are fourth order critical; note that there are no critical parties of order 2 or 3.

In view of Propositions 1 and 2 and Remark 2, we obtain the following corollary.

Corollary 1. Let $M \subseteq N$ be a winning coalition, then the players in M may be partitioned according to their order of criticality, plus those who are never critical.

Proposition 3. Let $i \in M$ be a player critical of order $k+1$ for coalition M, via coalition $K \subset M$; if a player $j \in K$ leaves the coalition, then i is a critical player of order k for coalition $M \setminus \{j\}$, via coalition $K \setminus \{j\}$.

Proof. It is sufficient to note that $M \setminus (K \cup \{i\}) = (M \setminus \{j\}) \setminus ((K \setminus \{j\}) \cup \{i\})$, so both of them are losing and $|K \setminus \{j\}| = |K| - 1$.

After defining the various orders of criticality, we want to provide an index to measure how much a player may profit from being critical. The first step is to measure the power of a player w.r.t. a given coalition, taking into account his order; then we may aggregate the power of a player w.r.t. all the coalitions he may belong to.

We want now to compute the probability of a player $i \in N$ being SOC in v for some coalitions via another player $j \in N$. First, consider a coalition $S \in 2^{N \setminus \{i, j\}}$ with $v(S \cup \{i, j\}) = 1$ and define $C_{ij}(S)$ as follows:

$$C_{ij}(S) = \min\{v(S \cup \{i\}), v(S \cup \{j\})\} - v(S)$$

By the monotonicity of v, we have four possible cases, as shown in Table 1.

Table 1. Possible cases for player i (j) to be SOC for coalition $S \cup \{i, j\}$ via player j (i)

No.	$v(S \cup \{i\})$	$v(S \cup \{j\})$	$v(S)$	$C_{ij}(S)$
1	0	1	0	0
2	1	0	0	0
3	1	1	1	0
4	1	1	0	1

The only case in which i is SOC for $S \cup \{i, j\}$ via j is the last one (4) and $C_{ij}(S) = 1.$ Note also that, in general, $C_{ij}(S) = C_{ji}(S).$

Let $\mathbf{p} = (p_0, ..., p_{n-1})$ be a probability vector as defined in Section 2. If we want to compute the probability that i is SOC for some coalitions via j, we should compute the following expression:

$$\Gamma_{ij}^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i, j\}}} p_{s+1} C_{ij}(S)$$

By Proposition 1, it immediately follows that $\Gamma_{ij}^{\mathbf{p}}(v) = \Gamma_{ji}^{\mathbf{p}}(v)$ for each $i, j \in N.$ Following the same approach used to define semivalues (see relation (1)), we can also compute the total probability that player i is SOC for some coalition via some other player using the following summation:

$$C_i^{\mathbf{p}}(v) = \sum_{j \in N \setminus \{i\}} \Gamma_{ij}^{\mathbf{p}}(v)$$

Now, consider the game (N, v^{ij}) such that for each $S \in 2^{N \setminus \{i, j\}}$:

$$v^{ij}(S \cup \{i, j\}) = v^{ij}(S \cup \{i\}) = v^{ij}(S \cup \{j\}) = \min\{v(S \cup \{i\}), v(S \cup \{j\})\}$$

and

$$v^{ij}(S) = v(S)$$

The game (N, v^{ij}) represents a coalitional situation where the role of i (j) is negatively influenced by j (i), that is the value of each coalition M containing either i or j is lowered to the worst value from the pair $v(M \cup \{j\} \setminus \{i\})$ and $v(M \cup \{i\} \setminus \{j\})$. It is easy to check that $\Gamma_{ij}^{\mathbf{p}}(v) = \pi_i^{\mathbf{p}}(v^{ij})$.

Example 4. Consider a simple game (N, v) with $N = \{1, 2, 3, 4\}$ whose minimal winning coalitions are $\{1, 2, 3\}$ and $\{1, 2, 4\}$. Note that 3 (4) is SOC for $\{1, 2, 3, 4\}$ via 4 (3), and no other player is SOC via another player for any coalition. Taking the vector $\mathbf{p}^{\beta}$ as the probability vector yielding the Banzhaf index $\pi^{\mathbf{p}^{\beta}}$ (see Section 2), we obtain $\Gamma_{34}^{\mathbf{p}^{\beta}}(v) = \Gamma_{43}^{\mathbf{p}^{\beta}}(v) = 1/8$ and $\Gamma_{ij}^{\mathbf{p}^{\beta}}(v) = 0$ for all the other i and j in N with $\{i, j\} \neq \{3, 4\}$. We have $v^{34}(S) = v^{43}(S) = v(S)$ for each $S \in 2^N$. So, $\Gamma_{34}^{p^{\beta}}(v) = \Gamma_{43}^{p^{\beta}}(v)$ $= \pi_3^{p^{\beta}}(v^{34}) = \pi_4^{p^{\beta}}(v^{43}) = 1/8$. In addition, we also have $v^{ij}(S) = 0$ for each $S \in 2^N$ and for all i and j with $\{i, j\} \neq \{3, 4\}$; hence, $\Gamma_{ij}^{\mathbf{p}^{\beta}}(v) = \pi_i^{\mathbf{p}^{\beta}}(v^{ij}) = 0$ for all i and j with $\{i, j\} \neq \{3, 4\}$.

In this example, the total probabilities of being SOC are $C_1^{\mathbf{p}^{\beta}}(v) = C_2^{\mathbf{p}^{\beta}}(v) = 0$ and $C_3^{\mathbf{p}^{\beta}}(v) = C_4^{\mathbf{p}^{\beta}}(v) = 1/8$.

Now, consider a game (N, v) and take a coalition M such that $i \in M$ and $K \subseteq M \setminus \{i\}$ with $|K| = k$, $k \geq 2$. Similarly to the above, we can compute the value

$$C_{iK}(M) = \min\{v(M \setminus T) : T \subset K \cup \{i\}\} - v(M \setminus (K \cup \{i\}))$$

that is equal to 1 iff i is order $k+1$ critical for coalition M via coalition $K \subset M$. Consequently, if we want to compute the probability that i is order $k+1$ critical for some coalitions via coalition K, we should compute the following expression:

$$\Gamma_{iK}^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i\}} \text{ with } K \subseteq S} p_s C_{iK}(S)$$

We conclude this section with an example of a possible application of the notions of criticality of first and second order to the analysis of the power of political parties in a realistic scenario.

Example 5. Consider the political situation described in Section 1 concerning the Italian Senate during the eighties. More precisely, the distribution of seats among the political parties of the largest alliance in the Italian Senate during the IX Legislature (1979–1983) was the one shown in Table 2.

Table 2. The distribution of seats in the largest alliance in the Italian Senate during the IX Legislature (1979–1983)

Party	Seats
Democrazia Cristiana (DC)	145
Partito Socialista Italiano (PSI)	32
Partito Socialdemocratico Italiano (PSDI)	9
Partito Repubblicano Italiano (PRI)	6
Partito Liberale Italiano (PLI)	2

At that time, the quota needed to have a majority within the Senate was 162. This leads to the weighted majority situation $[162; 145, 32, 9, 6, 2]$ on the player set $\{\text{DC, PSI, PSDI, PRI, PLI}\}$. Notice that a coalition is winning if it contains one of the following minimal winning coalitions $\{\{\text{DC, PSI}\}, \{\text{DC, PSDI, PRI, PLI}\}\}$. The symmetric relation of SOC exists between several pairs of players for the coalition $\{\text{DC, PSI, PSDI, PRI, PLI}\}$, specifically: PSI vs. PSDI, PSI vs. PRI and PSI vs. PLI.

Using the probability vector $\mathbf{p}^{\beta}$, we can compute the probability of being first order critical (i.e., the Banzhaf index) and of being SOC (using the index $C^{\mathbf{p}^{\beta}}$), as shown in Table 3.

Table 3. The Banzhaf index and the total probability of being SOC in the Italian Senate

Party	DC	PSI	PSDI	PRI	PLI
$\pi^{\mathbf{p}^{\beta}}$	$\frac{9}{16}$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$
$C^{\mathbf{p}^{\beta}}$	0	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

For the sake of completeness, we recall that in 1983 the PSI threatened to leave the five-party alliance unless Bettino Craxi, the PSI party's leader, was made Prime Minister. The DC party accepted this compromise, in order to avoid a new election.

Maybe, the DC party had evaluated the threat of the PSI as credible in view of the high $C^{\mathbf{P}^{\beta}}$ index for the PSI party.

4. Credible criticality

We now consider an alternative notion of first order criticality where the fact that player i can threaten coalition M in a credible way is made possible by the fact that there exists another opportunity for i to be winning without the help of the players in M. We want to remark that this hypothesis is different from that in [3], where i could ask for the help of some players in M, but not all.

Definition 3. Let $M \subseteq N$ with $v(M) = 1$. A player $i \in M$ is said to be first order credibly critical (or simply credibly critical) for coalition M iff it satisfies the following two conditions: (i) i is critical for M (i.e. $v(M \setminus \{i\}) = 0$) and (ii) there exists another coalition $S \subseteq N \setminus M$ that becomes winning with the addition of i, i.e. $v(S \cup \{i\}) = 1$.

Example 6. Consider a simple game (N, v) with $N = \{1, 2, 3\}$ whose minimal winning coalitions are $\{1, 2\}$ and $\{2, 3\}$. Player 2 is credibly critical for coalition $\{1, 2\}$ (in fact $v\,(1, 2) = 1$, $v\,(1) = 0$ and $v\,(2, 3) = 1$) but players 1 and 3 are never credibly critical.

Suppose $M \cup \{i\}$ is a winning coalition, where $M \subseteq N \setminus \{i\}$. When player i is in negotiations with the coalition M, the property of the credible criticality of player $i \in N$ can affect the ability of player i to gain power over $M \cup \{i\}$ by defeating the resistance of the other members of M to assigning the marginal contribution $v(M \cup \{i\}) - v(M)$ to i.

More in general, we can think of a situation where the marginal contribution $v(M \cup \{i\}) - v(M)$ is assigned to player i only if i could potentially take part in another coalition $S \subseteq N \setminus M$ at least as powerful as $M \cup \{i\}$, such that $M \cap S = \varnothing$. For a simple game, this leads us to the following definition of the *credible* marginal contribution of player i to coalition $M \subseteq N \setminus \{i\}$:

$$\hat{m}_i^v(M) = \begin{cases} v(M \cup \{i\}) - v(M) & \text{iff} \quad v(N \setminus M) = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Remark 4. A credible marginal contribution exists and can be computed for each player i and each coalition $M \subseteq N \setminus \{i\}$ on each possible game (N, v). However, in the remainder of the paper we will focus on the computation of the quantity $\hat{m}_i^v(M)$ for simple games.

Remark 5. A more concise way to define the quantity $\hat{m}_i^v(S)$ for each $i \in N$ and $M \in 2^{N \setminus \{i\}}$ is as follows*:*

$$\hat{m}_i^v(M) = \min\{v(M \cup \{i\}) - v(M), v(N \setminus M)\} \tag{7}$$

Now, consider again a probability vector $\mathbf{p} = (p_0, \ldots, p_{n-1})$ as in the previous section. As a measure of the power that players may credibly claim in a simple game, we define the following *credible semivalue* engendered by the vector $\mathbf{p}$. Hence,

$$\hat{\pi}_i^{\mathbf{p}}(v) = \sum_{S \in 2^{N \setminus \{i\}}} p_s \hat{m}_i^v(S) = \sum_{S \in W_i(v)} p_s \hat{m}_i^v(S) \tag{8}$$

For simple games, $\hat{\pi}_i^{\mathbf{p}}(v)$ can be interpreted as the probability of player i being credibly critical (under the probability vector $\mathbf{p}$). The next examples show that the vector of indices given by a semivalue $\pi^{\mathbf{p}}$ engendered by a probability vector $\mathbf{p}$ can be drastically different from the one given by the credible semivalue $\hat{\pi}^{\mathbf{p}}$ engendered by the same probability vector.

Example 7. Consider the simple game (N, v) with $N = \{1, 2, 3\}$ whose unique minimal winning coalition is $\{1, 2\}$. By the null player property, for any semivalue $\pi^{\mathbf{p}}$, $\pi_3^{\mathbf{p}}(v) = 0$ and by symmetry $\pi_1^{\mathbf{p}}(v) = \pi_2^{\mathbf{p}}(v)$. Notice also that no player $i \in \{1, 2, 3\}$ is credibly critical. So each credible semivalue yields $\hat{\pi}_i^{\mathbf{p}}(v) = 0$ for each $i \in \{1, 2, 3\}$.

The next example shows that not only can the values $\pi^{\mathbf{p}}$ and $\hat{\pi}^{\mathbf{p}}$ be very different, but even the ranking of players according to $\pi^{\mathbf{p}}$ and $\hat{\pi}^{\mathbf{p}}$ need not be preserved.

Example 8. Consider the simple game (N, v) with $N = \{1, 2, 3, 4, 5\}$ whose minimal winning coalitions are $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$ and $\{3, 4, 5\}$. Consider the semivalue $\pi^{\mathbf{p}^\beta}$ corresponding to the Banzhaf value. Thus, the Banzhaf value gives $\pi_1^{\mathbf{p}^\beta} = \pi_2^{\mathbf{p}^\beta} = 6/16$ and $\pi_3^{\mathbf{p}^\beta} = \pi_4^{\mathbf{p}^\beta} = \pi_5^{\mathbf{p}^\beta} = 4/16$. Now consider the notion of credible criticality. Note that 1 and 2 are never credibly critical, whereas 3, 4 and 5 are credibly critical whenever they are critical. Consequently, $\hat{\pi}_1^{\mathbf{p}^\beta} = \hat{\pi}_2^{\mathbf{p}^\beta} = 0$ and $\hat{\pi}_3^{\mathbf{p}^\beta} = \hat{\pi}_4^{\mathbf{p}^\beta} = \hat{\pi}_5^{\mathbf{p}^\beta} = 4/16$.

Proposition 4. A credible semivalue $\hat{\pi}^{\mathbf{p}}$ satisfies the null player property and the symmetry property.

Proof. Both properties follow from the definition of credible marginal contribution as for the case of classical semivalues.

Proposition 5. Consider a weighted majority game (N, v) with quota and weights $[q; w_1, ..., w_n]$. Let $\hat{\pi}^{\mathbf{p}}$ be defined according to relation (8) on any probability vector **p**. It follows that

$$w_i \geq w_j \Rightarrow \hat{\pi}_i^{\mathbf{p}} \geq \hat{\pi}_j^{\mathbf{p}}$$

for each $i, j \in N$.

Proof Let $i, j \in N$ with $w_i \geq w_j$ and $S \in 2^{N \setminus \{i, j\}}$. It follows from the definition of a weighted majority game that $v(S \cup \{i\}) \geq v(S \cup \{j\})$. So, by Eq. (7), it immediately follows that

$$\begin{aligned} \hat{m}_i^v(S) &= \min\{v(S \cup \{i\}) - v(S), v(N \setminus S)\} \\ &\geq \min\{v(S \cup \{j\}) - v(S), v(N \setminus S)\} = \hat{m}_j^v(S) \end{aligned} \tag{9}$$

Now take $S \in 2^{N \setminus \{i, j\}}$ and consider the credible marginal contribution $\hat{m}_i^v(S \cup \{j\})$ and $\hat{m}_j^v(S \cup \{i\})$. We obtain $v(S \cup \{i\}) \geq v(S \cup \{j\})$. Again by Eq. (7) it follows that

$$\begin{aligned} \hat{m}_i^v(S \cup \{j\}) &= \min\{v(S \cup \{i, j\}) - v(S \cup \{j\}), v(N \setminus (S \cup \{j\}))\} \\ &\geq \min\{v(S \cup \{i, j\}) - v(S \cup \{i\}), v(N \setminus (S \cup \{i\}))\} = \hat{m}_j^v(S \cup \{i\}) \end{aligned} \tag{10}$$

where the inequality follows from the fact that by the definition of a weighted majority game,

$$v(S \cup \{i, j\}) - v(S \cup \{j\}) \geq v(S \cup \{i, j\}) - v(S \cup \{i\})$$

and $v(N \setminus (S \cup \{j\})) \geq v(N \setminus (S \cup \{i\}))$, since $N \setminus (S \cup \{j\})$ is obtained by substituting *j* by *i* in $N \setminus (S \cup \{i\})$.

The proof follows by relations (9) and (10) and the fact that by relation (8)

$$\hat{\pi}_i^{\mathbf{p}}(v) = \sum_{S \in 2^{N\setminus\{i,j\}}} (\hat{m}_i^v(S) + \hat{m}_i^v(S \cup \{j\}))$$

for each $i \in N$.

The converse of Proposition 5 is not true, as shown by Example 7, where the game (N, v) can be generated by the weighted majority situation [3; 2, 2, 0] and $\hat{\pi}^{\mathbf{p}}(v) = (0, 0, 0)$ for any credible semivalue $\hat{\pi}^{\mathbf{p}}$.

5. Concluding remarks

We have introduced the concept of order of criticality for a party in a winning coalition, based on the minimal number of parties who must leave the coalition to make it losing such that the party of interest is the last to leave. Then we defined a measure of criticality that describes the relevance of a party in a majority coalition; this measure is given by the probability that a party is of a given order of criticality. Finally, we studied the credibility of a threat made by a party considering the possibility of forming an alternative majority by joining some of the parties in the opposition. Possible developments of this concept of credibility may account for the effectiveness of possible alternative majorities. For instance, it is possible to take into account the ideological contiguity of parties on a left-right axis [5], or majorities that had been formed in the past.

Another possible direction of investigation might be an analysis of credible criticality of higher orders; in this case, when player $i \in N$ is second (or higher) order critical for a winning coalition $M \ni i$, we cannot use the same approach as for first order criticality because this implies that the game is not proper, as the two disjoint coalitions $M \setminus \{i\}$ and $(N \setminus M) \cup \{i\}$ are both winning. Consequently, such a definition should allow some players in $M \setminus \{i\}$ to join an alternative majority.

Acknowledgement

The authors are happy to dedicate this paper to Gianfranco Gambarelli on the occasion of his retirement. For many years he has been providing hints for interesting topics to investigate, as well as his good humor to the whole community. We are sure that his retirement will not stop his positive influence on all of us. The authors are grateful to Guido Ortona for some fruitful discussions on blackmailing behavior and gratefully acknowledge two anonymous reviewers for their careful reading and useful comments. One of the authors (S.M.) benefited from the support of the French National Research Agency (ANR) project NETLEARN (grant no. ANR-13-INFR-004).

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Received 24 April 2016
Accepted 7 July 2016