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ELIMINATION OF DOMINATED STRATEGIES AND INESSENTIAL PLAYERS

We study the process, called the IEDI process, of iterated elimination of (strictly) dominated strategies and inessential players for finite strategic games. Such elimination may reduce the size of a game considerably, for example, from a game with a large number of players to one with a few players. We extend two existing results to our context; the preservation of Nash equilibria and order-independence. These give a way of computing the set of Nash equilibria for an initial situation from the endgame. Then, we reverse our perspective to ask the question of what initial situations end up at a given final game. We assess what situations underlie an endgame. We give conditions for the pattern of player sets required for a resulting sequence of the IEDI process to an endgame. We illustrate our development with a few extensions of the battle of the sexes.

Keywords: *dominated strategies, inessential players, iterated elimination, order-independence, estimation of initial games*

1. Introduction

Elimination of dominated strategies is a basic notion in game theory, and its relationships to other solution concepts, such as rationalizability, have been extensively discussed [5, 11]. Its nature, however, differs from other solution concepts; it suggests negatively what would/should not be played, while other concepts suggest/predict what would/should be chosen in games. In this paper, we also consider the elimination of inessential players whose unilateral changes of strategies do not affect any player's payoffs including his own. This concept is as basic as that of dominated strategies. We

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consider the process of iterated elimination of dominated strategies and of inessential players, which we call the IEDI process.

These two types of elimination interact with each other, and the situation differs from that of only elimination of dominated strategies. To see such interactions, as well as their negative nature, we consider three examples here. The first is described in a precise manner but the other two in an indicative manner.

Example 1.1. Battle of the sexes with a second boy. Consider a "battle of the sexes" situation consisting of boy 1, girl 2, and another boy 3. Each boy $i = 1, 3$ has two strategies, $\boldsymbol{s}_{i1}, \boldsymbol{s}_{i2}$ and girl 2 has four strategies, $\boldsymbol{s}_{21}, ..., \boldsymbol{s}_{24}$. Boy 1 and girl 2 can date at the boxing arena ($\boldsymbol{s}_{11} = \boldsymbol{s}_{21}$) or the cinema ($\boldsymbol{s}_{12} = \boldsymbol{s}_{22}$) but make decisions independently. Now, boy 3 enters this scene. Girl 2 can date boy 3 in a different arena $(\boldsymbol{s}_{23} = \boldsymbol{s}_{31})$ or cinema $(\boldsymbol{s}_{24} = \boldsymbol{s}_{32})$. When 1 and 2 consider their date, they would be happy even if they fail to meet; 3's choice does not affect their payoffs at all. Also, we assume that when 3 thinks about the case that 2 chooses to date boy 1, boy 3 is sadly indifferent between his arena and cinema. The same indifference is assumed for 1 when 2 chooses to date 3. Assuming this, their payoffs are described as Tables 1 and 2. The numbers in the parentheses in Table 1 are 3's payoffs. The dating situation for 3 and 2 is parallel and described in Table 2; but girl 2 is much less happy.

Table 1. Between 1 and 2

1\2 (3)	$\boldsymbol{s}_{21}$	$\boldsymbol{s}_{22}$
$\boldsymbol{s}_{11}$	15, 10 (–10)	5, 5 (–5)
$\boldsymbol{s}_{12}$	5, 5 (–5)	10, 15 (–10)

Table 2. Between 3 and 2

3\2 (1)	$\boldsymbol{s}_{23}$	$\boldsymbol{s}_{24}$
$\boldsymbol{s}_{31}$	15, 1 (–10)	5, 0 (–5)
$\boldsymbol{s}_{32}$	5, 0 (–5)	10, 2 (–10)

In this game, 2's strategies $\boldsymbol{s}_{23}$ and $\boldsymbol{s}_{24}$ are dominated by $\boldsymbol{s}_{21}$ and $\boldsymbol{s}_{22}$, since she wants to date boy 1. Eliminating these dominated strategies, we obtain a smaller game. Now, 3 is inessential in the sense that 3's choice does not affect any of the players. Thus, we eliminate 3 as an inessential player, and obtain the 2-person battle of the sexes.

In the game theory literature, it is standard to start with a given game, and analyze it with some solution concepts. Some abstraction takes place before reaching this given game. In the above case, eliminations of the dominated strategies for girl 2 and of boy 3 as an inessential player constitute this process to obtain the 2-person battle of the sexes.

In Example 1.1, elimination of dominated strategies generates inessential players. However, the possible interactions between elimination of dominated strategies and of inessential players are more complicated and can be summarized as follows: (a) elimination of dominated strategies may generate both new dominated strategies and new inessential players; (b) elimination of inessential players can only generate new inessential players. Hence, we obtain a process of iterated elimination of dominated strategies and of inessential players, which is our IEDI process. This is an extension of the process known as "iterated elimination of dominated strategies" in the literature [5, 11]. Elimination of both may reduce a large game into a small game in the sense of the sizes of the player set and strategy sets. Also, the following examples show very different social situations underlying the same battle of the sexes.

Example 1.2. A game with many players quickly reduced to a small game. We add 99 boys to Example 1.1, who are the same as boy 3 from the dating perspective. Now, the situation consists of 102 players but all could be essential unless 2 ignores these 100 boys. Her strategies to date any one of them are dominated by her dating strategies involving boy 1. Once these dominated strategies are eliminated, the boys from 3 to 102 all become inessential and can be eliminated. Again, we have the 2-person battle of the sexes.

In Example 1.2, we need only two steps if we allow simultaneous elimination of multiple dominated strategies and multiple inessential players. However, there are different situations where many steps are required to reach an endgame. In the next example, the resulting outcome is the same 2-person battle of the sexes but the process is intrinsically longer.

Example 1.3. Reduction takes many steps. Again, we add 99 boys to Example 1.1, where they are "onlookers". We assume that player $k+1$ is a friend of k and $k+1$'s opinion affects only k's payoffs ($k = 3, ..., 101$); $k+1$ has two actions: either to encourage k to tell his opinion to $k-1$ or not $(k \geq 4)$ and 4 can encourage 3 to cheer up. We assume that if 2 chooses to date 1, then 3 would be indifferent between his choices with or without 4's encouragement. The argument in Example 1.1 is applied to this; i.e., eliminating 2's dominated strategies $\boldsymbol{s}_{23}$ and $\boldsymbol{s}_{24}$, boy 3 becomes inessential and is eliminated. Then, boy 4 loses a friend to cheer up and becomes inessential. Similarly, if k disappears, then $k+1$ is inessential. After 100 iterative eliminations, we have the 2-person battle of the sexes. In this example, eliminations of inessential players only generate new inessential players.

The three examples above have different initial situations and show different elimination processes, while the endgame is the same. Such processes can have different possible combinations for the elimination of dominated strategies and inessential players. In an IEDI process, we take the order in which dominated strategies and then inessential players are eliminated into consideration. The sequence resulting from this process is called an IEDI sequence. Among such possible sequences, one type is representative, which we call the strict IEDI sequence; in each successive round, first all dominated strategies are eliminated and then all inessential players are eliminated.

Two existing results in the literature are converted to our context. One is the preservation theorem ([5], Theorem 4.35), stating that the Nash equilibria are preserved in the process of eliminating dominated strategies. This is extended to the IEDI process (Theorem 2.1). The other is the order-independence theorem [1, 3]: the process results in the same endgame regardless of the order in which dominated strategies are eliminated. This is also extended to our context (Theorem 3.1), and it is additionally shown that the strict IEDI sequence is the shortest and smallest among possible IEDI sequences.

These two results give a simple way of computing the set of Nash equilibria from the endgame to that of the initial game; the method is given explicitly as (8) in Section 3.1.

The IEDI process can be regarded as an abstraction process from a social situation into a simple description by eliminating some "irrelevant" factors. The above examples show that there are very different underlying situations that end up at the same endgame. In Section 4, we ask the reverse question of what are possible underlying situations that end up at a given game. We focus on a sequence of pairs of sets of players, which specifies the player sets and the subsets of players with dominated strategies to be eliminated. Once such a sequence and an endgame are given, we reconstruct an IEDI sequence. The characterization theorem (Theorem 4.1) gives conditions to reconstruct a strict IEDI sequence. Using this, we can infer the possible underlying situations behind a given endgame.

The paper is organized as follows: Section 2 gives basic definitions of dominance, an inessential player, and presents our preservation theorem. Section 3 defines the IEDI process, and proves our order-independence theorem. Section 4 gives and proves the characterization theorem. In Section 5, we give a summary and discuss some remaining problems.

2. Elimination of dominated strategies and inessential players

We define three ways of reducing a game by elimination of dominated strategies and of inessential players but we show that one way is more effective than the others. We also show that the Nash equilibria are preserved in these reductions.

2.1. Basic definitions

Let $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ be a finite strategic game, where N is a set of players, S_i is a nonempty set of strategies, and $h_i : \Pi_{j\in N} S_j \to \mathbb{R}$ is the payoff function for player $i \in N$. We allow N to be empty, in which case the game is the empty game, denoted as $s \in S_N := \Pi_{j\in N} S_j$ as $(s_I;\, s_{N-I})$, where $s_I = \{s_j\}_{j\in I}$ and $s_{N-I} = \{s_j\}_{j\in N-I}$. When $I = \{i\}$, we write S_{-i} for $S_{N-\{i\}}$ and $(s_i;\, s_{-i})$ for $(s_{\{i\}};\, s_{N-\{i\}})$. Let G be given, and $s_i, s_i' \in S_i$. We say that s_i' dominates s_i in G iff $h_i(s_i';\, s_{-i}) > h_i(s_i;\, s_{-i})$ for all $s_{-i} \in S_{-i}$. When s_i is dominated by some s_i', we simply say that s_i is dominated in G.

We say that i is an inessential player in G iff for all $j \in N$,

$$h_j(s_i;\, s_{-i}) = h_j(s_i';\, s_{-i}) \text{ for all } s_i, s_i' \in S_i \text{ and } s_{-i} \in S_{-i} \tag{1}$$

A choice by player i does not affect any player's payoff including i's own, provided that the others' strategies are fixed. Note that when $|S_i| = 1$, player i is already inessential.

We find a weaker version of this concept in Moulin [7], who defined the concept of *d*-solvability by only requiring (1) for $j = i$. Once player i becomes inessential in this sense, he may stop thinking about his choice but it may still affect the others' payoffs; in this case, he is still relevant to them.

Although (1) is an attribute of a single player, we can treat a group of such players as inessential, which is stated in the following lemma[2].

Lemma 2.1. Let I be a set of inessential players in G. Then, for all $j \in N$,

$$h_j(s_I;\, s_{N-I}) = h_j(s_I';\, s_{N-I}) \text{ for all } s_I, s_I' \in S_I \text{ and } s_{N-I} \in S_{N-I} \tag{2}$$

Proof. Let $I = \{i_1, ..., i_k\}$, and $I_t = \{i_1, ..., i_t\}$ for $t = 1, ..., k$. Also, let $s, s' \in S_N$ be fixed. We prove $h_j(s_{I_t};\, s_{N-I_t}) = h_j(s_{I_t}';\, s_{N-I_t})$ by induction on $t = 1, ..., k$. The base case, i.e., $h_j(s_{i_1};\, s_{-i_1}) = h_j(s_{i_1}';\, s_{-i_1})$, is obtained from (1). Suppose that $h_j(s_{I_t};\, s_{N-I_t})$ $= h_j(s_{I_t}';\, s_{N-I_t})$. Since $s = (s_{I_t}; s_{N-I_t}) = (s_{I_{t+1}}; s_{N-I_{t+1}})$, we have $h_j(s_{I_{t+1}}; s_{N-I_{t+1}})$ $= h_j(s_{I_t}; s_{N-I_t})$. By (1), $h_j(s_{I_t}';\, s_{N-I_t}) = h_j(s_{I_{t+1}}';\, s_{N-I_{t+1}})$. By the supposition, we obtain

[2]The concept of an inessential player conceptually differs from that of a "dummy player" in cooperative game theory (cf., Osborne–Rubinstein [11], p. 280). Using the maxmin definition of a characteristic function game, we have examples to show the logical independence of those two concepts.

$h_j(s_{I_{t+1}}; s_{N-I_{t+1}}) = h_j(s_{I_t}; s_{N-I_t}) = h_j(s'_{I_t}; s_{N-I_t}) = h_j(s'_{I_{t+1}}; s_{N-I_{t+1}})$. Thus, the assertion holds for $t+1$. ■

Let I be a set of inessential players in G, $N' = N - I$, and i any player in N'. The restriction h'_i of h_i on $\Pi_{j\in N'} S'_i$ with $\varnothing \neq S'_j \subseteq S_j$ for $j \in N'$ is defined by

$$h'_i(s_{N'}) = h_i(s_I; s_{N'}) \text{ for all } s_{N'} \in S'_{N'} \text{ and } s_I \in S_I \tag{3}$$

The well-definedness of h'_i is guaranteed by Lemma 2.1. Thus, $(N', \{S'_i\}_{i\in N'}, \{h'_i\}_{i\in N'})$ is the strategic game obtained from $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ by eliminating a set of inessential players I and some strategies from S_i for $i \in N'$.

We first give a general definition: We say that $G' = (N', \{S'_i\}_{i\in N'}, \{h'_i\}_{i\in N'})$ is a *D*--reduction of $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ iff

DR1. $N' \subseteq N$ and any $i \in N - N'$ is an inessential player in G;

DR2. For all $i \in N'$, $S'_i \subseteq S_i$ and any $s_i \in S_i - S'_i$ is a dominated strategy in G;

DR3. h'_i is the restriction of h_i to $\Pi_{j\in N'} S'_j$.

Some dominated strategies and inessential players in G may not be eliminated during the reduction to G'. Such dominated strategies and inessential players remain dominated and inessential in G', which is stated in Lemma 2.2.1 and 2.2.2. Claim 2.2.3 implies that elimination of inessential players generates no new dominated strategies.

Lemma 2.2. Let $G' = (N', \{S'_i\}_{i\in N'}, \{h'_i\}_{i\in N'})$ be a *D*-reduction of G.

2.2.1. If $s_i \in S'_i$ $(i \in N')$ is dominated in G, then it is dominated in G'.

2.2.2. If $i \in N'$ is an inessential player in G, then it is inessential in G'.

2.2.3. Suppose that $S'_i = S_i$ for all $i \in N'$. Let $i \in N'$ and $s_i \in S_i$. Then, s_i is dominated in G if and only if it is dominated in G'.

Proof of 2.2.1. Suppose that s_i is dominated by s'_i in G. Then, $h_i(s'_i; s_{N-i}) > h_i(s_i; s_{N-i})$ for all $s_{N-i} \in S_{N-i}$. We can assume without loss of generality that s'_i is not a dominated strategy in G, so $s'_i \in S'_i$. We have, by (3), for all $s_{N-N'} \in S_{N-N'}$, $h'_i(s'_i; s_{N'-i}) = h_i(s'_i; s_{N'-i}; s_{N-N'}) > h_i(s_i; s_{N'-i}; s_{N-N'}) = h'_i(s_i; s_{N'-i})$ for all $s_{N'-i} \in S'_{N'-i}$. Thus, s_i is dominated by s'_i in G'. The proof of (2.2.2) is similar.

2.2.3. The only-if part is immediate. Consider the if part. Suppose that s_i is dominated by s_i' in G'. Then, $h_i'(s_i';s_{N'-i}') > h_i'(s_i;s_{N'-i}')$ for all $s_{N'-i}' \in S_{N'-i}'$. Let $s_{N'-i}'$ be any element in $S_{N'-i}' = S_{N'-i}$. By (3), for all $s_{N-N'} \in S_{N-N'}$, $h_i(\, s_i';\; s_{N'-i}';\; s_{N-N'}) = h_i'(\, s_i';\; s_{N'-i}')$ $> h_i'(s_i;\; s_{N'-i}') = h_i(s_i; s_{N'-i}'; s_{N-N'})$. Thus, s_i is dominated by s_i' in G. ■

A *D*-reduction allows simultaneous elimination of both dominated strategies and inessential players. However, it would be easier to separate these types of elimination. First, let $N' = N$ hold in DR1, i.e., G' results from G by eliminating some dominated strategies; in this case, G' is called a *ds*-reduction of G, denoted as $G \to_{ds} G'$. Second, let $S_i' = S_i$ for all $i \in N'$ in DR2, i.e., G' results from G by eliminating some inessential players; in this case, G' is called an *ip*-reduction of G, denoted by $G \to_{ip} G'$. When all dominated strategies are eliminated in $G \to_{ds} G'$, it is called the *strict ds*-reduction, and similarly, when all inessential players are eliminated in $G \to_{ip} G'$ it is called the strict *ip*-reduction.

We focus on the order in which *ds*-reduction is applied and then *ip*-reduction is done. We say that G' is a *DI*-(compound) reduction of G iff there is an interpolating game $\underline{G}$ such that $G \to_{ds} \underline{G}$ and $\underline{G} \to_{ip} G'$. We say that G' is the strict *DI*-reduction of G iff both $G \to_{ds} \underline{G}$ and $\underline{G} \to_{ip} G'$ are strict. Even if $G \neq G'$, it is possible that $G = \underline{G}$ or $\underline{G} = G'$.

For comparison, we consider another compound reduction; G' is an *ID*-reduction of G iff $G \to_{ip} \underline{G} \to_{ds} G'$ for some $\underline{G}$. Lemma 2.3.1 states that *ID*-reductions are equivalent to *D*-reductions but 2.3.2 that a *DI*-reduction allows more possibilities. The converse of 2.3.2 does not hold; in Example 1.1, 3 becomes inessential after elimination of 2's dominated strategies.

Lemma 2.3.1. G' is a *D*-reduction of G if and only if G' is an *ID*-reduction of G.
2.3.2. If G' is a *D*-reduction of G, then G' is a *DI*-reduction of G.

Proof of 2.3.1. Only-If. Let G' be a *D*-reduction of G. It follows from Lemma 2.2.1 that we can postpone elimination of dominated strategies until the elimination of inessential players has been carried out. Hence, G' can be an *ID*-reduction. (If): Let G' be an *ID*-reduction of G, i.e., $G \to_{ip} \underline{G} \to_{ds} G'$ for some $\underline{G}$. Lemma 2.2.3 states that G has the same set of dominated strategies as G. Hence, we can combine these two reductions into one, which yields the *D*-reduction of G'.

2.3.2. This can be proved by a similar argument to the only-if part of 2.3.1 postponing *ip*-reductions, instead of *ds*-reductions. ■

2.2. Preservation of Nash equilibria

The concept of a *D*-reduction reduces a game by eliminating irrelevant players as well as irrelevant actions for some players. It is desirable that such a reduction loses no essential features of the social situation being modeled. This is what Merterns' [6] "small world axiom" requires for a solution concept. Here, we show that this holds for the concept of Nash equilibrium with respect to a *D*-reduction. In addition, the converse holds in our case.

We say that $s \in S$ is a Nash equilibrium in a nonempty game G iff for all $i \in N$, $h_i(s) \geq h_i(s_i'; s_{-i})$ for all $s_i' \in S_i$. Let θ be the null symbol, i.e., for any $s \in S$ we set $(\theta; s) = s$, and stipulate that the restriction of s to the empty game $G_\varnothing$ is the null symbol θ. Also, we stipulate that θ is the Nash equilibrium in $G_\varnothing$.

We have the following theorem. The first claim corresponds to Mertens' [6], p. 733, "small world axiom", for the case of Nash equilibrium. Both claims are presented in [5], Theorem 4.35, p. 109, for the case of elimination of dominated strategies only.

Theorem 2.1. Preservation of Nash equilibria. Let G' be a *D*-reduction of G.

A. If s_N is an NE in G, then its restriction $s_{N'}$ on G' is an NE in G'.

B. If $s_{N'}$ is an NE in G', then $(s_{N'}; s_{N-N'})$ is an NE in G for any $s_{N-N'}$ in $\Pi_{j \in N-N'} S_j$.

Proof of A. Let s be an NE in G. If $i \in N$, $h_i(s_i; s_{-i}) \geq h_i(s_i'; s_{-i})$ for any $s_i' \in S_i$. Let $i \in N'$. Then $s_i \in S_i'$, since s_i is not dominated in G. Let $s_i' \in S_i'$. Since G' is a *D*-reduction, we have $h_i'(s_i; s_{N'-i}) = h_i(s_i; s_{N-i}) \geq h_i(s_i'; s_{N-i}) = h_i'(s_i'; s_{N'-i})$; so $s_{N'}$ is an NE in G'.

B. Let $s_{N'}$ be an NE in G'. We choose any $s_{N-N'} \in S_{N-N'}$. We let $G^o = (N, \{S_i^o\}_{i\in N}, \{h_i\}_{i \in N})$ where $S_j^o = S_j'$ if $j \in N'$; and $S_j^o = S_j$ if $j \in N - N'$. The restriction of h_i on $\Pi_{j \in N} S_j^o$ is denoted by h_i itself. First, we show that $(s_{N'}; s_{N-N'})$ is an NE in G^o.

Let $i \in N'$. Then, $h_i'(s_{N'}') = h_i(s_{N'}'; s_{N-N'})$ for any $s_{N'}' \in S_{N'}'$ by Lemma 2.1, since the players in $N - N'$ are inessential in G. Since $s_{N'}$ is an NE in G', we have $h_i(s_i; s_{N'-i}; s_{N-N'}) = h_i'(s_i; s_{N'-i}) \geq h_i'(s_i'; s_{N'-i}) = h_i(s_i'; s_{N'-i}; s_{N-N'})$ for all $s_i' \in S_i'$. Let

$i \in N - N'$. Then, $h_i(s_i; s_{N'-i}; s_{N-N'}) = h_i(s_i'; s_{N'-i}; s_{N-N'})$ for all $s_i' \in S_i^o$. Hence, $(s_{N'}; s_{N-N'})$ is an NE in G^o.

Now we show that $(s_{N'}; s_{N-N'})$ is an NE in G. Let $i \in N'$. Suppose that $i \in N'$ has a strategy s_i'' in G such that $h_i(s_i''; s_{N-i}) > h_i(s_i; s_{N-i})$. We can choose such an s_i'' giving the maximum $h_i(s_i''; s_{N-i})$. This s_i'' is not dominated in G. Hence, s_i'' remains in G', which contradicts the statement that $s_{N'}$ is an NE in G'. ■

Let $NE(G)$ and $NE(G')$ be the sets of Nash equilibria for a game G and its D-reduction G'. It follows from Theorem 2.1 that $NE(G)$ and $NE(G')$ are connected by:

$$NE(G) = \Pi_{j \in N-N'} S_j \times NE(G') \tag{4}$$

Here, we stipulate that when $N - N' = \varnothing$, $\Pi_{j \in N-N'} S_j$ is the unit set with respect to the set multiplication $\times$, i.e., $NE(G) = NE(G')$. When G' is an empty game $G_\varnothing$, the Nash equilibrium of $G_\varnothing$ is the null symbol θ, and Theorem 2.1.B states that any strategy profile $s = (\theta; s)$ is a Nash equilibrium in G. It follows from Lemma 2.3.1 that (4) holds when $G_\varnothing$ is an ID-reduction of G.

For a DI-reduction G' of G, (4) should be modified slightly. Let $G \to_{ds} \underline{G}$ and $\underline{G} \to_{ip} G'$, where $\underline{G} = (\underline{N}, \{\underline{S}_i\}_{i \in \underline{N}}, \{\underline{h}_i\}_{i \in \underline{N}})$ is the interpolating game. Then,

$$NE(G) = \prod_{j \in N-N'} \underline{S}_j \times NE(G') \tag{5}$$

Since $\underline{N} = N$ and $\underline{S}_i = S_i'$ for all $i \in N'$, we first have $NE(\underline{G}) = \Pi_{j \in N-N'} \underline{S}_j \times NE(G')$ by (4), and then we obtain $NE(G) = NE(\underline{G}) = \Pi_{j \in N-N'} \underline{S}_j \times NE(G')$. Note $NE(G) = NE(\underline{G})$, since the dominated strategies in G are not in $NE(G)$. The formula (5) will be used to give a way of computing the set of NE's of an initial game from the endgame in the IEDI process.

Theorem 2.1 holds with respect to mixed strategy Nash equilibria, as well as rationalizability, correlated equilibria and Nash's [8] non-cooperative solution. So far, we only have positive results as long as the concepts of purely non-cooperative solutions are concerned[3].

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[3]The solution concept called the intrapersonal coordination equilibrium in Kaneko–Kline [4] is regarded as a concept of a non-cooperative solution but it is incompatible with the elimination of dominated strategies. It captures some cooperative aspects through an individual's intrapersonal thinking about

3. The IEDI process and generated sequences

Here, we consider the process of iterated elimination of dominated strategies and inessential players (the IEDI process). In Section 3.1, we present an extension of the order-independence theorem, and in Section 3.2, we give a theorem dividing elimination of inessential players from that of dominated strategies.

3.1. IEDI sequences and order-independence

We say that $\Gamma(G^0) = \langle G^0, G^1, ..., G^\ell \rangle$ is an IEDI sequence from a game G^0 iff

$$G^{t+1} \text{ is a } DI\text{-reduction of } G^t \text{ and } G^{t+1} \neq G^t \text{ for each } t = 0, ..., \ell - 1; \qquad (6)$$

$$G^\ell \text{ has no dominated strategies and no inessential players} \qquad (7)$$

We say that $\Gamma(G^0) = \langle G^0, ..., G^\ell \rangle$ is the strict IEDI sequence iff G^{t+1} is the strict *DI*-reduction of G^t for $t = 0, ..., \ell - 1$. The strict IEDI sequence is uniquely determined by G^0.

Example 3.1. Consider Example 1.1. The strict IEDI sequence is given in Fig. 1. Player 2's strategies $\boldsymbol{s}_{23}$ and $\boldsymbol{s}_{24}$ are dominated by $\boldsymbol{s}_{21}$ and $\boldsymbol{s}_{22}$. Then, by eliminating $\boldsymbol{s}_{23}$ and $\boldsymbol{s}_{24}$, we get the second interpolating 3-person game. Now, 1 and 2 focus on their dating, ignoring player 3 as inessential. Eliminating him, we obtain a 2-person battle of the sexes. This is a *DI*-reduction of $G^0 = G$. This IEDI has length 1. There are two other IEDI's; $\boldsymbol{s}_{23}$ and $\boldsymbol{s}_{24}$ are eliminated sequentially, then player 3 is eliminated as inessential. Each has length 2.

$G^0 \rightarrow_{ds}$

1\2\3	$\boldsymbol{s}_{21}$	$\boldsymbol{s}_{22}$
$\boldsymbol{s}_{11}$	15, 10, –10	5, –5, 5
$\boldsymbol{s}_{12}$	5, 5, –5	10, –10, 15

$\boldsymbol{s}_{31}$ or $\boldsymbol{s}_{32}$

$\rightarrow_{ip}$

1\2	$\boldsymbol{s}_{21}$	$\boldsymbol{s}_{22}$
$\boldsymbol{s}_{11}$	15, 10	5, 5
$\boldsymbol{s}_{12}$	5 , 5	10, 15

Fig. 1. The strict IEDI from Example 1.1

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others' thinking. An example of the non-preservation of such equilibrium occurs in the Prisoner's Dilemma.

It is known as the order-independence theorem [3, 1] that with iterated elimination of only dominated strategies, the order in which strategies are eliminated does not affect the endgame. Here, we extend this result to the above definition including elimination of inessential players. We focus not only on the endgames of IEDI sequences but also on comparisons between these sequences.

We say that $G' = (N', \{S_i'\}_{i\in N'}, \{h_i'\}_{i\in N'})$ is a subgame of $G = (N, \{S_i\}_{i\in N}, \{h_i\}_{i\in N})$ iff $N' \subseteq N$ and $S_i' \subseteq S_i$ for all $i \in N'$. If G' is a *D*-reduction of *G*, then G' is a subgame of *G*. For an IEDI sequence $\Gamma(G^0) = \langle G^0, \ ..., G^\ell \rangle$, if $t < k$, G^k is a subgame of G^t.

We have the following theorem, which will be proved at the end of this section.

Theorem 3.1. Order-independence, shortest, and smallest. Let G^0 be a game, and $\Gamma^*(G^0) = \langle G^{*0}, \ \ldots, G^{*\ell^*} \rangle$ the strict IEDI sequence from $G^0 = G^{*0}$. Then for any IEDI sequence $\Gamma(G^0) = \langle G^0, \ \ldots, G^\ell \rangle$ from G^0, **(A)** $G^{*\ell^*} = G^\ell$; **(B)** $\ell^* \le \ell$; **(C)** for each $t \le \ell^*$, G^{*t} is a subgame of G^t.

Claim (A) is order-independence[4]. Claims (B) and (C) mean that the strict IEDI sequence is the shortest and smallest with respect to the length of IEDI sequences and the size of their component games, respectively.

In Example 1.2, the strict IEDI sequence has length 1. However, there are many non-strict IEDI sequences with much longer lengths. In this example, girl 2 has many dating choices, e.g., 2 (choices) $\times$ 101 (boys) $= 202$ choices. Hence, the longest IEDI sequence consists of the sequential elimination of 200 dominated strategies and 100 inessential players; the length is thus 300. There are also many possible orders of these eliminations.

Example 1.3 does not require player 2 to have more strategies. Here, the strict IEDI has length 100, and the longest IEDI sequence has length 102, since it takes two steps to eliminate $\boldsymbol{s}_{23}$ and $\boldsymbol{s}_{24}$, and then players from 3 to 102 are eliminated sequentially.

If we focus initially only on elimination of dominated strategies, the 100 players remain in these games. Eliminating them, the games are reduced to the 2-person battle of the sexes.

We have other elimination processes adopting different reductions such as *D*- and *ID*-reductions. From Lemma 2.3, the strict IEDI $\Gamma^*(G^0)$ is shorter and smaller than the sequences based on *D*- or *ID*-reductions. It would also be possible to apply only *ds*-reductions until all dominated strategies are eliminated and then to apply *ip*--reductions. The strict IEDI sequence is shorter than or equal to this sequence, as long

[4]The order-independence theorem does not hold for weak dominance (cf. [10], p. 60). See [1] for comprehensive discussions on order-independence theorems for various types of dominance relations.

as we count each of the *DI*-reductions in the strict IEDI as one step. However, some IEDI might be shorter than the strict IEDI if we count each *DI*-reduction consisting of nontrivial sub-reductions as 2 steps.

We can see Theorem 3.1 from the viewpoint of the preservation of Nash equilibria. By applying (4) to $\Gamma^*(G^0) = \langle G^{*0}, ..., G^{*\ell^*} \rangle$ repeatedly, we obtain the result that if $G^{*\ell^*}$ has a Nash equilibrium, then so does $G^{*0} = G^0$. This holds even if $G^{*\ell^*}$ is the empty game. If $G^{*\ell^*}$ has no Nash equilibria, the initial game $G^{*0} = G^0$ has no Nash equilibria either.

This gives a method for computing the NE set, $NE(G^0)$, for any given game G^0. Let $\Gamma(G^0) = \langle G^0, \ldots, G^\ell \rangle$ be an IEDI, and $\underline{G}^t = (N^t, \{\underline{S}_i^t\}_{i \in N^t}, \{\underline{h}_i^t\}_{i \in N^t})$ the interpolating game between G^t and G^{t+1} for $t = 0, ..., \ell - 1$. The set $NE(G^0)$ is written as:

$$NE(G^0) = \Pi_{j \in N^0 - N^1} \underline{S}_j^0 \times ... \times \Pi_{j \in N^{\ell-1} - N^\ell} \underline{S}_j^{\ell-1} \times NE(G^\ell) \tag{8}$$

It follows from (5) that $NE(G^t) = NE(\underline{G}^t)$ and $NE(\underline{G}^t) = \Pi_{j \in N^t - N^{t+1}} \underline{S}_j^t \times NE(G^{t+1})$ for $t =$ $0, ..., \ell - 1$. Repeating this process from $\ell - 1$, we obtain (8). Thus, we have an algorithm for computing $NE(G^0)$ along the IEDI process. Formula (8) gives the set $NE(G^0)$ regardless of an IEDI sequence used but the strict IEDI gives the shortest computation.

In Examples 1.1, (8) gives $NE(G^0) = \{(\boldsymbol{s}_{11}, \boldsymbol{s}_{21}), (\boldsymbol{s}_{12}, \boldsymbol{s}_{22})\} \times \{\boldsymbol{s}_{31}, \boldsymbol{s}_{32}\}$. Similarly, we obtain $NE(G^0) = \{(\boldsymbol{s}_{11}, \boldsymbol{s}_{21}), (\boldsymbol{s}_{12}, \boldsymbol{s}_{22})\} \times S_3 \times \ldots \times S_{102}$ for Examples 1.2 and 1.3.

Table 3. *d*-solvable but nonempty

1\2	$\boldsymbol{s}_{21}$	$\boldsymbol{s}_{22}$
$\boldsymbol{s}_{11}$	1, 1	0, 1
$\boldsymbol{s}_{12}$	1, 0	0, 0

Finally, we look at Moulin's [7] concept of *d*-solvability; a game G^0 is *d*-solvable iff a sequence $\langle G^0, ..., G^\ell \rangle$ with $G^{t-1} \to_{ds} G^t$ for $t = 1, ..., \ell$, such that in G^ℓ, each $i \in N^\ell$ has constant payoffs when the others' strategies are fixed. If G^0 has an IEDI $\langle G^0, ..., G^\ell \rangle$ with $G^\ell = G_\varnothing$, then G^0 is *d*-solvable. The converse does not hold; Table 3, given in [7], has no dominated strategies and no inessential players but is *d*-solvable.

Now let us prove Theorem 3.1. First, we refer to Newman's lemma (see also [1]). An abstract reduction system is a pair $(X, \to)$, where X is an arbitrary nonempty set and $\to$ is a binary relation on X. We say that $\{x_\nu; \nu = 0, ...\}$ is a $\to$ sequence in $(X, \to)$ iff for all $\nu \geq 0$, $x_\nu \in X$ and $x_\nu \to x_{\nu+1}$ (as long as $x_{\nu+1}$ is defined). We use $\to^*$ to denote the transitive reflexive closure of $\to$. We say that $(X, \to)$ is weakly confluent iff for each $x, y, z \in X$ with $x \to y$ and $x \to z$, there is some $x' \in X$ such that $y \to^* x'$ and $z \to^* x'$.

Lemma 3.1 ([9]). Let $(X, \to)$ be an abstract reduction system satisfying N1: each $\to$ sequence in X is finite; and N2: $(X, \to)$ is weakly confluent. Then, for any $x \in X$, there is a unique endpoint *y* with $x \to^* y$.

Proof of Theorem 3.1 (A). Let $\mathcal{G}$ be the set of all finite strategic games. Then $(\mathcal{G}, \to_{DI})$ is an abstract reduction system, where we write $G \to_{DI} G'$ for $G \to_{ds} \underline{G}$ and $\underline{G} \to_{ip} G'$ for some interpolating $\underline{G}$ and $G \neq G'$. The relation $\to_{ID}$ is reflexive. Each $\to_{DI}$ sequence is finite, i.e., N1 holds. Let us show N2. Let $G, G', G'' \in \mathcal{G}$ with $G \to_{DI} G'$ and $G \to_{DI} G''$. Let G^* be the strict *DI*-reduction of *G*. Then, G^* is a *DI*--reduction of both G' and G''. Hence, $G' \to_{DI}^* G^*$ and $G'' \to_{DI}^* G^*$. Thus, it follows from Lemma 3.1 that for any $G^0 \in \mathcal{G}$, there is a unique endpoint G^*. Hence, the strict IEDI sequence $\Gamma^*(G^0) = \langle G^{*0}, \ldots, G^{*\ell^*} \rangle$ has the same endgame, i.e., $G^* = G^{*\ell^*} = G^\ell$.

Now, we prove (C) in a weaker form. Then, we prove (B), from which (C) follows.

C*. We prove by induction on t that G^{*t} is a subgame of G^t for each $t \leq \min(\ell, \ell^*)$. This holds by definition for $t = 0$. Suppose that this holds for $t < \min(\ell, \ell^*)$. Let $G^{*t} \to_{ds} \underline{G}^{*t} \to_{ip} G^{*t+1}$ and $G^t \to_{ds} \underline{G}^t \to_{ip} G^{t+1}$. From Lemma 2.2.1, if a strategy s_i in G^{*t} is dominated in G^t, it is also dominated in G^{*t}. From Lemma 2.2.2, if a player i in G^{*t} is inessential in $\underline{G}^t$ he is also inessential in G^{*t}. We obtain G^{*t+1} by eliminating all dominated strategies in G^{*t} and all inessential players in $\underline{G}^{*t}$; so G^{*t+1} is a subgame of G^{t+1}.

B. Let $\Gamma(G^0) = \langle G^0, \ldots, G^\ell \rangle$ be any IEDI sequence. From (A), $G^{*\ell^*} = G^\ell$. If $\ell < \ell^*$, then, from (6), $G^{*\ell^*} = G^\ell$ is a strict subgame of $G^{*\ell}$. From (C^*), $G^{*\ell}$ is a subgame of G^ℓ, which is a contradiction. Thus, $\ell^* \leq \ell$, i.e., (B) holds. This implies (C). ■

3.2. The elimination divide

An IEDI sequence is partitioned into two segments, $G^1, \ldots, G^{m_0-1}$ and $G^{m_0}, \ldots, G^{\ell}$, so that in the first segment both dominated strategies and inessential players can be eliminated, and in the second only inessential players are eliminated.

Theorem 3.2. Partition of an IEDI sequence. Let $\Gamma(G^0) = \langle G^0, \ \ldots, G^{\ell} \rangle$ be an IEDI sequence from G^0. There is exactly one m_0 $(0 \le m_0 \le \ell)$ such that (i): some dominated strategy is eliminated going from G^{m_0-1} to G^{m_0}; (ii): for each $t\,(m_0 \le t \le \ell - 1)$, no dominated strategies are eliminated but some inessential player is eliminated going from G^{m_0-1} to G^{m_0}.

Proof. Suppose that G^t has no dominated strategies. Then, G^{t+1} is obtained from G^t by eliminating inessential players. It follows from Lemma 2.2.3 that G^{t+1} has no dominated strategies. Thus, for any $t' > t$, $G^{t'}$ has no dominated strategies. Hence, we choose m_0 to be the smallest value among such t's. ■

We call the m_0 given by Theorem 3.2 the elimination divide. In Example 2.1, $m_0 = 0$, and the segment after m_0 may have the length greater than 1. The elimination divide m_0 plays an important role in Section 4.

4. Characterization of initial situations

We have studied IEDI sequences generated from a given initial game G^0, and have seen that there are many different initial situations, as well as many IEDI sequences that lead to the same endgame G.

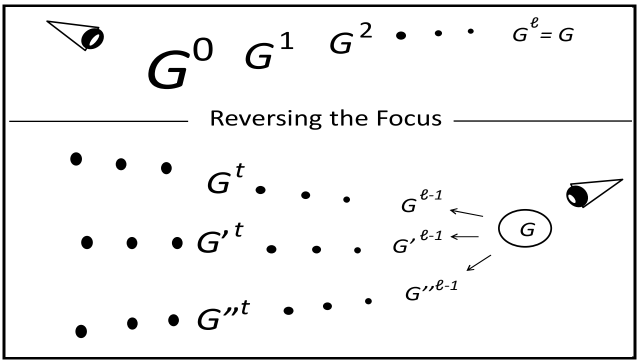

Fig. 2. Starting from the final game

Here, we study the class of those initial situations that lead to a given endgame G; i.e., we reverse our point of view from the top of Fig. 2 to the bottom. We characterize what underlying social situations can lie behind the same G. We give conditions for a given pattern of player sets corresponding to a sequence of the IEDI process that leads to a given game.

4.1. Evolving player configurations and the corresponding strict IEDI sequences

We start with a sequence $\eta = [(N^0, T^0), ..., (N^\ell, T^\ell)]$ of pairs of sets of players, which we call a sequence of evolving player configurations (EPC). Here, $N^0, ..., N^\ell$ are the player sets and $T^0, ..., T^\ell$ are the subsets of players with dominated strategies corresponding to some IEDI sequence $\Gamma(G^0) = \langle G^0, ..., G^\ell \rangle$. We wish to determine what conditions on η guarantee the existence of some strict IEDI sequence $\Gamma(G^0)$ corresponding to η.

We give four conditions on η, and the first three are as follows:

PC0. $T^t \subseteq N^t$ for $t = 0, \dots, \ell$; and $N^0 \supseteq \dots \supseteq N^\ell$ with $|N^\ell| \neq 1$;

PC1. For any $t < \ell$, if $T^t = \emptyset$, then $N^t \supsetneq N^{t+1}$;

PC2. For some m_0 $(0 \le m_0 \le \ell)$, $T^{m_0} \neq \emptyset$ and $T^t = \emptyset$ if $t \ge m_0$.

PC0 is basic. It intends to mean that the player sets are decreasing with the eliminations of inessential players. $N^t - N^{t+1}$ is the set of inessential players to be eliminated and T^t is a set of players in N^t with dominated strategies to be eliminated. It also requires that the changes do not stop with a single player. PC1 corresponds to the requirement $G^t \neq G^{t+1}$ in (6). The number m_0 in PC2 is the elimination divide discussed in Section 3.2.

The fourth condition is for a strict IEDI sequence. We say that an EPC sequence $\eta = [(N^0, T^0), \dots, (N^\ell, T^\ell)]$ is strict iff

PC3. For $t = 1, ..., m_0$, if $|T^{t-1}| = 1$, then $T^{t-1} \cap T^t = \emptyset$.

This states that if a single player's dominated strategies are to be eliminated, this elimination should not generate any new dominated strategies for him. Actually, PC0–PC3 are sufficient to guarantee the existence of a strict IEDI sequence.

To connect the EPC and IEDI sequences, we define the concept of a D-group. Let G' be a DI-reduction of G with $G \to_{ds} \underline{G} \to_{ip} G'$. We say that $T = \{i \in N : S_i \neq \underline{S}_i\}$ is the D-group from G *to* G'. When G' is the strict DI-reduction of G, T is the set of all players with dominated strategies in G. We have the following lemma.

Lemma 4.1. Necessary conditions for an EPC sequence. Let $\Gamma(G^0) = \langle G^0, \dots, G^\ell \rangle$ be an IEDI sequence with elimination divide m_0, N^t the player set of G^t for $t = 0, \dots, \ell,$

T^t the D-group from G^t to G^{t+1} for $t = 0, \ldots, \ell - 1$, and $T^\ell = \emptyset$. Then $\eta = [(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ satisfies PC0–PC2. If $\Gamma(G^0)$ is a strict IEDI sequence, then PC3 holds, too.

Proof. Let $G^t = (N^t, \{S_i^t\}_{i \in N^t}, \{h_i^t\}_{i \in N^t})$ for $t = 0, \ldots, \ell$. PC0 follows from (6) and (7), and PC1 corresponds to $G^t \neq G^{t+1}$ in (6). PC2 follows from the definition of the elimination divide m_0. Consider PC3: Let $\Gamma(G^0)$ be the strict IEDI from G. Let $T^{t-1} = \{i\}$. If $i \notin N^t$, then $i \notin T^t$, so $T^{t-1} \cap T^t = \emptyset$. Suppose $i \in N^t$. Let $G^{t-1} \to_{ds} \underline{G}^{t-1} \to_{ip} G^t$. Then, all of the dominated strategies for player i in G^{t-1} are eliminated in forming $\underline{G}^{t-1}$. From Lemma 2.2.3, i has no dominated strategies in G^t. Hence, $T^{t-1} \cap T^t = \emptyset$. ■

We say that $\eta = [(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ given in this lemma is the EPC sequence associated with $\Gamma(G^0) = \langle G^0, \ldots, G^\ell \rangle$. The converse of Lemma 4.1 is our present concern. Here, we confine ourselves to recoverability by strict IEDI sequences.

We have the following theorem, which is proved in Section 4.2.

Theorem 4.1. Characterization. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a game with $|S_i| \geq 2$ for all $i \in N$, which has no dominated strategies and no inessential players. Let $\eta = [(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ be a strict EPC sequence with $N^\ell = N$. Then, there exists a game G^0 and a strict IEDI sequence $\Gamma(G^0) = \langle G^0, \ldots, G^\ell \rangle$ such that (A) $G^\ell = G$; (B) $|S_i^t| \geq 2$ for all $i \in N^t$, $t = 0, \ldots, \ell - 1$; and (C) η is the EPC sequence associated with $\Gamma(G^0)$.

This theorem implies that there are a great multitude of possible underlying situations behind a given game G. Let us look at the EPC sequences associated with Examples 1.1–1.3. Example 1.1 has the strict IEDI sequence $\langle G^0, G^1 \rangle$ with its associated EPC sequence: $[(N^0, T^0), (N^1, T^1)] = [(\{1, 2, 3\}, \{2\}), (\{1, 2\}, \emptyset)]$. In Example 1.2, we have $[(N^0, T^0), (N^1, T^1)] = [(\{1, 2, ..., 102\}, \{2\}), (\{1, 2\}, \emptyset)]$. In Example 1.3, the strict IEDI sequence has length 100. The associated EPC sequence is given as $\eta = [(N^0, T^0), ..., (N^{100}, T^{100})]$ so that $N^t = \{1, 2\} \cup \{3 + t, ..., 102\}$ for $t = 0, \ldots, 100$, and $T^0 = \{2\}$, $T^t = \emptyset$ for $t = 1, \ldots, 100$.

We have many other EPC sequences. For example, for $t = 0, \ldots, 10$, let $N^t = \{1, 2\} \cup (\cup_{k=0}^{9} \{10k + (3 + t), \ldots, 10k + 12\})$, $T^0 = \{2\}$, and $T^t = \emptyset$ for $t = 1, \ldots, 10$. Players from 3 to 102 are divided into 10 groups $\{3, 4, \ldots, 12\}$, $\{13, 14, \ldots, 22\}$, …,

$\{93, 94, \ldots, 102\}$. Each has the same structure as the "onlookers" in Example 1.3 but each of 3, 13, 23, …, 93 wants to date girl 2, and 4 is a friend of 3, 14 is a friend of 13, …, and 94 is a friend of 93, and so on. In the strict IEDI associated with this EPC sequence, players 3, 13, 23, …, 93 become inessential and are eliminated in the first round, and then players 4, 14, …, 94 become inessential and are eliminated, and so on. The resulting game after 10 rounds is the same as the 2-person battle of the sexes.

The initial game of this IEDI sequence is very different from those in Examples 1.2 and 1.3. We can think about more complicated networks. As long as PC0–PC3 are satisfied by a given EPC sequence, Theorem 4.1 suggests a game situation with such a network. In this sense, we regard typical examples in game theory as being abstracted from many different situations.

Condition PC3 is not used in these examples. We can extend Example 4.2 with $[(N^0, T^0), (N^1, T^1)]$ to a situation including more steps. Suppose that after eliminating all the boys from 3 to 102, 1 and 2 find more strategies relevant to themselves. Then, there is a longer EPC sequence $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ with $N^t = \{1, 2\}$ and $T^t \neq \emptyset$ for all $t = 1, \ldots, \ell$. When $\langle G^0, \ldots, G^\ell \rangle$ is a strict IEDI sequence, PC3 implies that for some k_0 $(2 \le k_0 \le \ell)$, $T^t = \{1, 2\}$ for t $(2 \le t \le k_0)$, and $|T^t| = 1$ for t $(k_0 < t \le \ell)$. Up to step k_0, they agree to eliminate their dominated strategies together but after k_0, $T^{t+1} \cap T^t = \emptyset$, i.e., they alternatingly eliminate dominated strategies.

In Theorem 4.1, we have not considered the strategy sets in $\Gamma^*(G^0)$. However, it is possible to start with a given sequence of game forms (without specifying payoffs) rather than an EPC sequence. A detailed analysis remains open.

4.2. Proof of Theorem 4.1

Consider an EPC sequence $\eta = [(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ and $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ satisfying the conditions of the theorem with $N = N^\ell$. By induction from (N^ℓ, T^ℓ) to (N^0, T^0), we construct a sequence $G^\ell, G^{\ell-1}, \ldots, G^0$ from $G^\ell = G$, and show that for each $t = \ell - 1, \ldots, 0$, G^{t+1} is a strict *DI*-reduction of G^t; thus, $\langle G^0, \ldots, G^\ell \rangle$ is a strict IEDI generated from G^0.

Lemma 4.2 describes the construction of the interpolating $\underline{G}^t$ from G^{t+1}, i.e., $\underline{G}^t \rightarrow_{ip} G^{t+1}$. Since $G^\ell = G$ has no inessential players, we can assume that $|S_i| \ge 2$ for all $i \in N$. In the following lemmas, we use the same notation $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$

for a generic game, which should not be confused with the given game G in Theorem 4.1. Also, we consider the reverse direction from $G = G^{t+1}$ to $G' = \underline{G}^t$.

$$G^t \xleftarrow[\text{Lemma 4.3}]{} \underline{G}^t \xleftarrow[\text{Lemma 4.2}]{} G^{t+1} \tag{9}$$

Lemma 4.2. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a game with $|S_i| \geq 2$ for all $i \in N$, and let I' be a nonempty set of new players. Then, there is a game $G' = (N', \{S_i'\}_{i \in N'}, \{h_i'\}_{i \in N'})$ such that (i): $N' = N \cup I'$; (ii): $|S_i'| \geq 2$ for all $i \in N'$; and (iii): G is the strict *ip*-reduction of G'.

Proof. We choose strategy sets S_i, $i \in N'$ so that $S_i' = S_i$ for all $i \in N$ and $S_i' = \{\alpha, \beta\}$ for all $i \in I'$, where α, β are new strategies not in G. Then we define the payoff functions $\{h_i'\}_{i \in N'}$ so that the players in I' are inessential in G', but no players in N are inessential in G'. Let I be the set of inessential players in G. For each $i \in I$, we choose an arbitrary strategy, say $\mathbf{s}_{i1}$ from S_i. Then we define $\{h_i'\}_{i \in N'}$ as follows: (a): if $j \in I'$, $h_j'(s_{N'}) = |\{i \in I : s_i = \mathbf{s}_{i1}\}|$ for $s_{N'} \in S_{N'}$; (b): if $j \in N$, $h_j'(s_{N'}) = h_j(s_N)$ for $s_{N'} \in S_{N'}$, where s_N is the restriction of $s_{N'}$ to N. For any $j \in I'$, j's strategy s_j does not appear substantively in h_i' for any $i \in N \cup I'$. Thus, the players in I' are all inessential in G'. On the other hand, each player $i \in I$, as far as such a player exists in G, affects j's payoffs for $j \in I'$ because of (a) and $|S_i| \geq 2$. This means that no $i \in I$ is inessential in G'. Also, no $i \in N - I$ is inessential in G' by (b). Thus, only the players in I' are inessential. In sum, G is the strict *ip*-reduction of G'. ∎

Now, consider the construction from $\underline{G}^t$ to G^t in (9). For this, first we show the following fact: Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be an n-person game, and $j \in N$ a fixed player. Then, there are real numbers $\{\pi_j(s_j)\}_{s_j \in S_j}$ such that

$$\text{if } s_j \text{ dominates } s_j', \text{ then } \pi_j(s_j) < \pi_j(s_j') \tag{10}$$

Such $\{\pi_j(s_j)\}_{s_j \in S_j}$ are defined by induction as follows: First, we let $H^1 = G$. Let k be a natural number with $1 \leq k \leq |S_i| - 1$. Suppose that a game H^k is given. Take an arbitrary strategy s_j^k for player j in H^k so that it is not dominated at all in H^k. Then, $\pi_j(s_j^k) = k$. Then, H^{k+1} is obtained from H^k by eliminating s_j^k from the strategy set for player j in H^k. Additionally, we let $\pi_j(s_j^{|S_i|}) = |S_i|$. Thus, we have $\{\pi_j(s_j)\}_{s_j \in S_j}$. It remains to show that (10) holds. Suppose that s_j dominates s'_j in $G = H^1$. Then, s_j occurs before s'_j in the sequence $s_j^1, \ldots, s_j^{|S_i|}$ above constructed. Hence, $\pi_j(s_j) < \pi_j(s'_j)$.

Now, consider the step from $\underline{G}^t$ to G^t in (9). In the next lemma, G and G' are supposed to be $\underline{G}^t$ and G^t, respectively.

Lemma 4.3. Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$ be a game, and T a nonempty subset of N with $|S_i| \geq 2$ for all $i \in N - T$.

4.3.1. There is a game $G' = (N', \{S'_i\}_{i \in N'}, \{h_i\}_{i \in N'})$ such that G is a *ds*-reduction of G', T is the *D*-group from G' to G, and $|S'_i| \geq 2$ for all $i \in N$.

4.3.2. If the following condition holds for T,

$$\text{if } T = \{i\}, \text{ no pair of strategies } s_i, s'_i \in S_i \text{ exists such that } s_i \text{ dominates } s' \tag{11}$$

then G is the strict *ds*-reduction of the game G' given by 4.3.1.

Proof of 4.3.1. Let β_j be a new strategy for each $j \in T$. We define $\{S'_j\}_{j \in N}$ as follows:

$$S'_j = \begin{cases} S_j \cup \{\beta_j\} & \text{if } j \in T \\ S_j & \text{if } j \in N - T \end{cases} \tag{12}$$

For each $j \in N$, we extend h_j to $h'_j : \Pi_{i \in N} S'_i \to \mathbb{R}$ so that the restriction of h'_j to $\Pi_{i \in N} S_i$ is h_j itself and G is the strict *ds*-reduction of G', as follows: Let $j \in N$. First, h'_j is the same as h_j over $\Pi_{i \in N} S_i$, i.e., $h'_j(s) = h_j(s)$ if $s \in \Pi_{i \in N} S_i$. Let $s \in S' - S$. If $j \in N - T$,

$$h'_j(s) = \pi_j(s_j), \qquad \text{where } \pi_j(s_j) \text{ is above defined for } G \tag{13}$$

and if $j \in T$,

$$h'_j(s) = \begin{cases} \pi_j(s_j) & \text{if } s_j \neq \beta_j \\ \min\{\pi_j(t_j) : t_j \in S_j\} - 1 & \text{if } s_j = \beta_j \end{cases} \tag{14}$$

Now let $j \in N - T$, and let $s_j, s'_j \in S_j = S'_j$. Suppose that s_j dominates s'_j in G. Consider $s, s' \in S' - S$ such that the j-th components of s and s' are s_j and s'_j. From (13), we get $h'_j(s) = \pi_j(s_j) < \pi_j(s'_j) = h'_j(s')$. Hence, s_j does not dominate s'_j in G', which implies that j has no dominated strategies in G'. Second, let $j \in T$. We choose an $s^*_j \in S_j$ with $s^*_j \neq \beta_j$. From (14), we have, for any $s_{-j} \in S_{-j}$, $h'_j(\beta_j; s_{-j})$ $= \min\{\pi_j(t_j) : t_j \in S_j\} - 1 < \pi_j(s^*_j) = h'_j(s^*_j; s_{-j})$. This does not depend upon s_{-j}. Thus, s^*_j dominates β_j in G'. From the analysis of these two cases, we conclude that T is the D-group from G' to G.

4.3.2. Finally, we show that under condition (11), s_j does not dominate s'_j in G' for any $s_j, s'_j \in S_j = S'_j - \{\beta_j\}$ and $j \in T$. If s_j does not dominate s'_j in G, then this does not hold in G' either. Now let s_j dominate s'_j in G. From (11), we have $|T| > 1$. This guarantees the existence of $s, s' \in S' - S$ such that their j-th components are s_j and s'_j. From (14), $h'_j(s) = \pi_j(s_j) < \pi_j(s'_j) = h'_j(s')$. Hence, s_j does not dominate s'_j. It follows that G is the strict *ds*-reduction of G'. ■

Proof of Theorem 4.1. We construct a strict IEDI sequence $\Gamma(G^0) = \langle G^0, ..., G^\ell \rangle$ along $[(N^0, T^0), ..., (N^\ell, T^\ell)]$ from the endgame $G^\ell = G$ by backward induction. Let $G^\ell = G$. By assumption, condition (7) holds. Also, $|S^\ell_i| \geq 2$ for all $i \in N$.

Suppose that G^{t+1} is defined with $|S^{t+1}_i| \geq 2$ for all $i \in N^{t+1}$. From Lemma 4.2, we find an interpolating game $\underline{G}^t$ such that G^{t+1} is the strict *ip*-reduction of $\underline{G}^t$ with player set N^t and $|\underline{S}^t_i| \geq 2$ for all $i \in N^t$. From Lemma 4.3.1, we find another game G^t such that $\underline{G}^t$ is a *ds*-reduction of G^t with D-group T^t satisfying $|S^t_i| \geq 2$ for all $i \in N^t$.

Now we obtained an IEDI $\Gamma(G^0) = \langle G^0, ..., G^\ell \rangle$ with associated EPC sequence $[(N^0, T^0), ..., (N^\ell, T^\ell)]$. When PC3 holds, condition (11) in Lemma 4.3 is satisfied. Thus, from Lemma 4.3.2 $\underline{G}^t$ is the strict *ds*-reduction of G^t. ■

5. Conclusions

We have considered the process of iterated elimination of dominated strategies and of inessential players. The latter is newly introduced, and is interactive with the former. This introduction changes the analysis considerably. We have given modifications of existing results; Theorem 2.1 (preservation) and Theorem 3.1 (order--independence). Then, we presented Theorem 4.1 (characterization).

Result (4) on the preservation of Nash equilibria follows Theorem 2.1. Theorem 3.1 states that any sequence generated from a given game by the IEDI process ends up at the same game and that the strict IEDI sequence is the smallest and shortest among the IEDI sequences. Combining these results, we obtain a simple way to compute the set of Nash equilibria for the initial game from the equilibria for the endgame, which is expressed as (8).

Then we argued in Section 4 that typical examples considered in game theory are representatives of games abstracted from many different situations. Theorem 4.1 gives conditions for the form of IEDI sequences from possible situations that end up at a given game. These conditions imply that there are many underlying situations behind a given game. Examples 1.1–1.3, together with this theorem, show that the introduction of inessential players gives new perspectives about possible underlying social situations behind a game. Also, Theorems 2.1 and 3.1 give a way of computing the set of Nash equilibria. (8) associates the appropriate sequence of sets of Nash equilibria with the strict IEDI described by Theorem 4.1.

We have not touched upon some important problems. One problem is to relax the concept of "inessential players". The definition of an inessential player here is too stringent in that unilateral changes in his strategies have no effect on any player's payoffs. One possibility is to introduce ε-inessential players or ε-influences for an $\varepsilon > 0$. Player j is defined to be ε-inessential with respect to player i iff unilateral changes in j's strategies only affect i's payoffs within an ε-magnitude. Using this definition, we may allow boy 3 in Example 1.1 to be ε-indifferent between the arena and cinema when girl 2 chooses to date boy 1.

Our three examples suggest different problems. The payoffs for players depend only upon a set of neighbors. This is compatible with ε-inessential players or ε-influence. This is also along the research line of the present paper.

Another problem is the complexity of assessing preferences for an IEDI sequence. The results in this paper facilitate such considerations, since, in general, the strict IEDI sequence requires less analysis than any other IEDI sequence. However, based on the straightforward definition of complexity for preference comparisons, we have an example of a game where some IEDI sequence needs a smaller number of preference comparisons than the strict IEDI sequence. A detailed study is an open problem.

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References

[1] APT K.R., *Direct proofs of order independence*, Economics Bulletin, 2011, 31, 106–115.
[2] BÖRGERS T., *Pure strategy dominance*, Econometrica, 1993, 61, 423–430.
[3] GILBOA I., KALAI E., ZEMEL E., *On the order of eliminating dominated strategies*, Operations Research Letters, 1990, 9, 85–89.
[4] KANEKO M., KLINE J.J., *Understanding the other through social roles*, to be piblished in International Game Theory Review, 2015.
[5] MASCHLER M., SOLAN E., ZAMIR S., *Game Theory*, Cambridge University Press, Cambridge 2013.
[6] MERTENS J.F., *Stable equilibria – a reformulation II. The geometry, and further results*, Mathematics of Operations Research, 1991, 16, 694–753.
[7] MOULIN H., *Game Theory for the Social Sciences*, 2nd revised Ed., New York University Press, New York 1986.
[8] NASH J.F., *Non-cooperative games*, Annals of Mathematics, 1951, 54, 286–295.
[9] NEWMAN M.H.A., *On theories with a combinatorial definition of equivalence*, Annals of Mathematics, 1942, 43, 223–243.
[10] MYERSON R.B., *Game Theory*, Harvard University Press, Cambridge 1991.
[11] OSBORNE M., RUBINSTEIN A., *A Course in Game Theory*, The MIT Press, Cambridge 1994.

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