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SOLVING LINEAR FRACTIONAL MULTILEVEL PROGRAMS

The linear fractional multilevel programming (LFMP) problem has been studied and it has been proved that an optimal solution to this problem occurs at a boundary feasible extreme point. Hence the *K*th-best algorithm can be proposed to solve the problem. This property can be applied to quasiconcave multilevel problems provided that the first (n – 1) level objective functions are explicitly quasimonotonic, otherwise it cannot be proved that there exists a boundary feasible extreme point that solves the LFMP problem.

Keywords: *multilevel, linear fractional, quasiconcave, quasiconvex, Kth-best*

1. Introduction

Multilevel programming involves optimization problems where the constraint region of the first level problem is implicitly determined by the second level problem and the constrained region of the second level problem is determined by the third level problem, and so on. It has been applied to decentralized planning problems involving a decision process with a hierarchical structure. In terms of modeling, multilevel problems are programming problems which have a subset of their variables controlled by the optimal solution of another level problem parameterized by the remaining variables. The second level decision maker optimizes his objective function under the given parameters from the first level decision maker. This one, with complete information on the possible reactions of the second level decision maker, selects the parameters so as to optimize its own objective function.

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Multilevel programs can be formulated as:

$$\begin{aligned}
&\min_{x_1} f_1\left(x_1, x_2, ..., x_n\right)\\
&\text{where}\left(x_2, x_3\, ..., x_n\right)\text{solves}\\
&\min_{x_2} f_2\left(x_1, x_2\, ..., x_n\right)\\
&...\\
&\min_{x_2} f_2\left(x_1, x_2\, ..., x_n\right)\\
&\text{where }\left(x_n\right)\text{ solves}\\
&\min_{x_n} f_n\left(x_1, x_2\, ..., x_n\right)\\
&\text{s.t. }\left(x_1, x_2, ..., x_n\right)\in S
\end{aligned} \tag{1}$$

where $x_1 \in R^{n_1}, x_2 \in R^{n_2}, ..., x_n \in R^{n_n}$ are the variables controlled by the first, second and the nth level decision maker, respectively. $f_1, f_2, ..., f_n: R^n \to R,\ n = n_1 + n_2 + ... + n_n$. $S \subset R^n$ defines the common constraint region and

$$S_n\left(x_1, x_2, ..., x_{n-1}\right) = \left\{x_n \in R^{n_n}: \left(x_1, x_2, ..., x_n\right) \in S\right\}$$

Let S_{n-1} be the projection of S onto $R^{n_1} \times R^{n_2} \times ... \times R^{n_{n-1}}$. Then for each $(x_1, x_2, ..., x_{n-1}) \in S_{n-1}$, the nth level decision maker solves the problem

$$\begin{aligned}
&\min_{x_n} f_n\left(x_1, x_2, ..., x_n\right)\\
&\text{s.t. } x_n \in S_n\left(x_1, x_2, ..., x_{n-1}\right)
\end{aligned} \tag{2}$$

The feasible region of the first level decision maker, called the inducible region represented by $(IR)_1$, is implicitly determined by the second level optimization problem and the feasible region of the second level decision maker, represented by $(IR)_2$, is implicitly determined by the third level optimization problem, and so on. Likewise, the feasible region of the $(n-1)$th level decision maker, called the inducible region represented by $(IR)_{n-1}$, is implicitly determined by the nth level optimization problem

$$\left(IR\right)_{n-1} = \left\{\left(x_1, x_2, ..., x_n^*\right): x_1 \in S_1, x_2 \in S_2, ..., x_n^* \in M\left(x_{n-1}\right)\right\}$$

where $M(x_{n-1})$ denotes the set of all optimal solutions of problem (2).

Here we assume that S is not empty and that for any decision taken by the $(n-1)$th level decision maker, the nth level decision maker has some room to respond, i.e.

$M(x_{n-1}) \neq \phi$. Likewise, for any decision taken by the $(n-2)$th level decision maker, the $(n-1)$th level decision maker has some room to respond, i.e. $M(x_{n-2}) \neq \phi$, and so on. Likewise, for any decision taken by the first level decision maker, the second level decision maker has some room to respond, i.e. $M(x_1) \neq \phi$.

The above defined multilevel programming problem is a non-convex optimization problem and its main feature is that, unlike other general mathematical problems, the multilevel problem may not possess a solution, even when $f_1, f_2, ..., f_n$ are continuous and S is compact. In particular, difficulties may arise when the $M(x_i)$ are not single valued for all permissible $x_i,\ i = 1, 2, ..., n-1$.

1.1. Formulation of the problem

In this paper, the linear fractional multilevel programming (LFMP) problem is considered in which all the objective functions are linear fractional and S is a polyhedron, which is assumed to be nonempty and bounded. Using the common notation in multilevel programming, the LFMP problem can be written as follows:

$$
\begin{aligned}
&\min f_1(x_1, x_2, ..., x_n) = \frac{\alpha_1 + c_{11}x_1 + c_{12}x_2 + ... + c_{1n}x_n}{\beta_1 + d_{11}x_1 + d_{12}x_2 + ... + d_{1n}x_n} \\
&\text{where } x_2 \text{ solves} \\
&\min f_2(x_1, x_2, ..., x_n) = \frac{\alpha_2 + c_{21}x_1 + c_{22}x_2 + ... + c_{2n}x_n}{\beta_2 + d_{21}x_1 + d_{22}x_2 + ... + d_{2n}x_n} \\
&... \\
&\text{where } x_{n-1} \text{ solves} \\
&\min f_n(x_1, x_2, ..., x_n) = \frac{\alpha_n + c_{n1}x_1 + c_{n2}x_2 + ... + c_{nn}x_n}{\beta_n + d_{n1}x_1 + d_{n2}x_2 + ... + d_{nn}x_n} \\
&\text{s.t.} \left(x_1, x_2, ..., x_n\right) \in S
\end{aligned}
\tag{3}
$$

where for $i, j \in \{1, 2, ..., n\}$ c_{ij} and d_{ij} are vectors of conformable dimensions and α_i and β_i are scalars, $i \in \{1, 2, ..., n\}$. We require that

$$
\beta_i + d_{i1}x_1 + d_{i2}x_2 + ... + d_{in}x_n \neq 0, \qquad i \in \{1, 2, ..., n\}, \qquad \forall \left(x_1, x_2, ..., x_n\right) \in S.
$$

We assume that

$$
\beta_i + d_{i1}x_1 + d_{i2}x_2 + ... + d_{in}x_n > 0, \qquad i \in \{1, 2, ..., n\}, \qquad \forall \left(x_1, x_2, ..., x_n\right) \in S
$$

If this assumption is not valid, it suffices to consider the linear fractional objective function

$$-\frac{\alpha_i + c_{i1}x_1 + c_{i2}x_2 + \ldots + c_{in}x_n}{\beta_i + d_{i1}x_1 + d_{i2}x_2 + \ldots + d_{in}x_n}$$

Moreover, it is also assumed that $M(x_i)$ is a singleton set $\forall(x_1, x_2, ..., x_i) \in S_i$.

Fractional programming with only one level of decision making has received remarkable attention in the literature [5]. It is worth mentioning that objective functions which are ratios are frequently used in stochastic programming problems.

1.2. Approaches to solving linear fractional multilevel programs

Various approaches have been proposed in the literature to make sure that the multilevel problem is well posed. The most common one is to assume that for each value of the first level variable x_1 there is a unique solution to the second level problem, i.e. the set $M(x_1)$ is a singleton set $\forall x_1 \in S_1$. Likewise for each value of the $(n-1)$th variable x_{n-1}, there is a unique solution to the nth level problem, i.e. the set $M(x_{n-1})$ is a singleton set $\forall(x_1, x_2, ..., x_{n-1}) \in S_{n-1}$. Other approaches focus on the way of selecting $x_i^* \in M(x_{i-1})$, in order to evaluate $f_{i-1}(x_1, x_2, ..., x_i)$ when $M(x_{i-1})$ is not a singleton. Among the rules that have been proposed, it is worth mentioning the optimistic or weak approach and the pessimistic or strong approach. The first one assumes that the $(i-1)$th level decision maker is able to influence the ith level decision maker so that the latter always selects the variables x_i to provide the best value of f_{i-1}. In the pessimistic approach the $(i-1)$th level decision maker behaves as though the ith level decision maker always selects the optimal decision which gives the worst value of f_{i-1}. Finally, other approaches consider a local reduction of the problem.

Several approaches have been given for solving bilevel and multilevel linear fractional programming problems. In the weighted approach to the bilevel linear fractional programming problem, a non-dominated solution set is obtained and the objective functions for both levels are combined into one objective function by finding appropriate weights and the relative weights represent the relative importance of the objective functions [12]. A globally convergent algorithm has also been proposed to solve bilevel linear fractional programming problems [13]. Necessary and sufficient optimality conditions have been already given for bilevel multiobjective programming problems [9].

In this paper, we give a geometrical characterization of the optimal solution to the LFMP problem in terms of what is called a boundary feasible extreme point. This result extends the characterization proved by Liu and Hart for the linear bi-level programming problem [10, 11]. This property is the key to concluding that the *K*th-best algorithm can be used to solve the linear fractional bilevel programming problem [4]. This paper extends this work and uses the *K*th-best algorithm to solve linear fractional multilevel programming problems. Also, we give a brief note on the quasiconcave multilevel problem.

The paper is organized as follows. The next section provides the main theoretical result on optimality. Then the *K*th-best algorithm is proposed to solve the problem and a formal proof of its correctness is given. Finally, the last section concludes the paper with final remarks on more general multilevel problems for which the characterization of the optimal solution is still valid and the *K*th-best algorithm can be applied to solve them.

2. Theoretical properties

Before proving the main result on the optimal solution of problem (3), we list some preliminary definitions and results.

Definition 1 [6]. Let *f* be a real-valued function defined on a convex subset *D* of R^n, then

1. *f* is quasiconcave on *D* iff $d^1,\ d^2 \in D, \lambda \in [0,1]$, and $f\left(d^1\right) \le f\left(d^2\right) \Rightarrow f\left(d^1\right) \le f\left[\left(1-\lambda\right)d^1 + \lambda d^2\right]$. The function *f* is quasiconvex iff –*f* is quasiconcave.

2. *f* is semistrictly quasiconcave on *D* iff $d^1, d^2 \in D,\ d^1 \ne d^2,\ \lambda \in \left(0,|\ |1\right)$ and $f\left(d^1\right) < f\left(d^2\right) \Rightarrow f\left(d^1\right) < f\left[\left(1-\lambda\right)d^1 + \lambda d^2\right]$. The function *f* is semistrictly quasiconvex iff –*f* is semistrictly quasiconcave.

3. *f* is explicitly quasiconcave on *D* iff it is quasiconcave and semistrictly quasiconcave on *D*. The function *f* is explicitly quasiconvex iff –*f* is semistrictly quasiconcave.

4. *f* is explicitly quasimonotonic on *D* iff it is explicitly quasiconcave and explicitly quasiconvex on *D*.

Note that the linear fractional functions

$$f_i\left(x_1, x_2, ..., x_n\right) = \frac{\alpha_i + c_{i1}x_1 + c_{i2}x_2 + ... + c_{in}x_n}{\beta_i + d_{i1}x_1 + d_{i2}x_2 + ... + d_{in}x_n}, \qquad i \in \{1, 2, ..., n\}$$

are explicitly quasimonotonic on *S* if $\beta_i + d_{i1}x_1 + d_{i2}x_2 + ... + d_{in}x_n \ne 0$ in *S*.

On the other hand, since $f_1, f_2, ..., f_n$ are quasiconcave and S is a nonempty and compact polyhedron, the LFMP problem is a particular case of the quasiconcave multilevel problem. Hence,

• The feasible region of the LFMP consists of the union of connected faces of the polyhedron S. As a consequence, in general, the inducible region is a nonconvex set.

• There exists an extreme point of IR, thus an extreme point of the polyhedron S, which is an optimal solution of the LFMP problem.

Definition 2 [9]. A point $\left(x_1, x_2, ..., x_n\right) \in \left(IR\right)_{n-1}$ is a boundary feasible point if there exists an edge E of S such that $\left(x_1, x_2, ..., x_n\right)$ is an extreme point of E and the other extreme point of E is not an element of $\left(IR\right)_{n-1}$.

Now we characterize the optimal solution to the LFMP problem. To begin, let us consider the relaxed problem

$$\begin{aligned} &\min f_1\left(x_1, x_2, ..., x_n\right) = \frac{\alpha_1 + c_{11}x_1 + c_{12}x_2 + ... + c_{1n}x_n}{\beta_1 + d_{11}x_1 + d_{12}x_2 + ... + d_{1n}x_n} \\ &\text{s. t. } \left(x_1, x_2, ..., x_n\right) \in S \end{aligned} \tag{4}$$

Note that f_1 is a quasiconcave function and S is a nonempty and compact polyhedron, so that there is an extreme point of S which solves the above problem. If this is a point of the induced region $\left(IR\right)_{n-1}$, then it is an optimal solution to the LFMP problem.

In general, by solving the relaxed problem we will not obtain an optimal solution of the multilevel problem, since decision makers usually have conflicting objectives. In this case, to characterize in a more precise way the geometry of the optimal solution to the LFMP problem, we will prove in the next theorem that it occurs at a boundary feasible extreme point.

Theorem 1. If there exists an extreme point of S not in the induced region $\left(IR\right)_{n-1}$ which is an optimal solution of the relaxed problem (4), then there exists a boundary feasible extreme point that solves the LFMP problem.

Proof: As previously mentioned, there exists an extreme point of S which is an optimal solution of the LFMP problem. Let this point be $\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)$. If it is a boundary feasible point, the proof is complete. If this is not so, every extreme point adjacent to $\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)$ is in $\left(IR\right)_1$ and

$$f_1\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right) \le f_1\left(x_1, x_2, ..., x_n\right) \tag{5}$$

for all points $(x_1, x_2, ..., x_n)$ adjacent to $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$. Firstly, we prove that there must be an extreme point $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ adjacent to $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ such that

$$f_1(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n) = f_1(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \tag{6}$$

For this purpose, let us consider the relaxed problem (4). Taking into account (5), $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ is a local extreme-minimum point of f_1 in *S*. Since f_1 is quasiconcave and explicitly quasiconvex on *S*, we can conclude that $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ is a global minimum of the relaxed problem (4), i.e.

$$f_1(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) \le f_1(x_1, x_2, ..., x_n), \qquad \forall (x_1, x_2, ..., x_n) \in S \tag{7}$$

By hypothesis there exists an extreme point $(\bar{y}_1, \bar{y}_2, ..., \bar{y}_n) \in S$ but not in the induced region $(IR)_1$ being an optimal solution of problem (4). Thus

$$f_1(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n) = f_1(\bar{y}_1, \bar{y}_2, ..., \bar{y}_n)$$

Notice that $(\bar{y}_1, \bar{y}_2, ..., \bar{y}_n)$ cannot be adjacent to $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ as is not a boundary feasible extreme point.

Since f_1 is continuous, quasiconvex and explicitly quasiconcave on *S*, the optimum set of problem (4) is the convex hull of some extreme points of *S*, thus itself a polyhedron [7]. It follows that there exists an edge path in the optimum set of problem (4) from $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ to $(\bar{y}_1, \bar{y}_2, ..., \bar{y}_n)$. Hence, there must be an extreme point $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ adjacent to $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ belonging to the optimum set of problem (4), thus verifying (6).

If $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ is a boundary feasible extreme point, then we can say that if there exists an extreme point not in $(IR)_1$ which is an optimal solution of the relaxed problem (4), then there exists a boundary feasible extreme point that solves the problem for the first two levels. If this is not so, we consider the extreme point $(\hat{x}_1, \hat{x}_2, ...\hat{x}_n)$ instead of $(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ and repeat the same argument. Thus we get an extreme point $(\breve{x}_1, \breve{x}_2, ..., \breve{x}_n)$ adjacent to $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ verifying (6).

If this new point is a boundary feasible extreme point, then the proof is complete. Otherwise, by repeating the process, because the number of extreme points of *S* is finite, eventually a boundary feasible extreme point will be reached in a finite number of steps, which solves the two level problem.

Again, we formulate the second relaxed problem

$$\min\ f_2\left(x_1, x_2 \ldots, x_n\right) = \frac{\alpha_2 + c_{21}x_1 + c_{22}x_2 + \ldots + c_{2n}x_n}{\beta_2 + d_{21}x_1 + d_{22}x_2 + \ldots + d_{2n}x_n} \tag{8}$$

$$\text{s.t.}\left(x_1, x_2 \ldots, x_n\right) \in S$$

By repeating this process, we can prove that if there exists an extreme point of S not in the induced region $(IR)_2$ which is an optimal solution of the relaxed problem (8), then there exists a boundary feasible extreme point that solves the three level problem.

By repeating the same process and after formulating the $(n-1)$th relaxed problem, we can prove that if there exists an extreme point of S not in the induced region $(IR)_{n-1}$ which is an optimal solution of the $(n-1)$th relaxed problem, then there exists a boundary feasible extreme point that solves the LFMP problem.

Remark 1. As here we use the assumption that for each value of the first level problem there exists a unique solution to the second level problem, i.e. the set $M(x_1)$ is a singleton $\forall x_1 \in S_1$. Likewise, $M(x_{n-1})$ is a singleton set $\forall (x_1, x_2, \ldots, x_{n-1}) \in S_{n-1}$. But if $M(x_1)$ is not a singleton $\forall x_1 \in S_1$, then problems are caused by the existence of multiple optima when solving the second level problem for a given $x_1 \in S_1$. This means that the inducible region is no longer formed by the union of the faces of the polyhedron S. Moreover, the first level decision maker may not be able to reach his optimal decision while forcing the optimal decision of the second level decision maker to be unique.

Let us discuss the following example of a three level LFMP problem:

$$\min f_1\left(x_1,\ x_2,\ x_3\right) = \frac{x_1 + 3x_2 + 3x_3 + 3}{x_1 + x_2 + x_3 + 5}$$

where x_2 solves

$$\min f_2\left(x_1,\ x_2,\ x_3\right) = \frac{-x_1 + 2x_2 + x_3 + 7}{x_1 + x_2 + x_3 + 2}$$

where x_3 solves

$$\min f_3\left(x_1,\ x_2,\ x_3\right) = \frac{x_1 + x_2 + 2x_3 + 5}{x_1 + x_2 + x_3 + 1}$$

where *S* is $\{(x_1, x_2, x_3) \in R_3\}$ with the following constraints:

$$x_1 + 2x_2 + x_3 \le 20, \qquad x_1 + x_2 + x_3 \le 12, \qquad 3x_1 - 4x_2 + x_3 \le 19$$
$$x_1 - 4x_2 + x_3 \le 5, \qquad x_1, x_2, x_3 \ge 0$$

Here, if we give some value to x_1, e.g. $x_1 = 1$, the second level problem has multiple optima and then if we consider these multiple optima, the third level problem has multiple optima corresponding to these optima.

This fact means that the inducible region does not consist of the union of the faces of the polyhedron *S*. Moreover, the optimization problem of the first level decision maker is not well defined. In order to completely evaluate $f_1(1, x_2, x_3)$, it is necessary to give a rule for selecting $x_2 \in M(1)$. Likewise, a rule for selecting x_3 is necessary. If we use the optimistic approach and start with $x_1 = 1$, we notice that we do not get an extreme point of the polyhedron *S* and if we apply the pessimistic approach, then no optimal solution to the problem exists.

Remark 2. It is well known that the Charnes and Cooper (C&C) transformation allows us to reformulate a linear fractional programming (LFP) problem as a linear programming (LP) one [5]. Hence, we consider the applicability of the C & C transformation to reformulate, in a similar way, the LFMP problem as a linear multilevel programming problem. Having this motivation in mind, assume that

$$S = \{(x_1, x_2, ..., x_n) : A_1x_1 + A_2x_2 + ... + A_nx_n \le b, \qquad x_1 \ge 0, \qquad x_2 \ge 0\}$$

where *b* is a vector and $A_1, A_2, ..., A_n$ are matrices of the appropriate dimensions.

For any fixed $(x_1, x_2, ..., x_{n-1}) \in S_{n-1}$, let

$$z = \frac{1}{\beta_{n-1} + d_{n-11}x_1 + d_{n-12}x_2 + ... + d_{n-1n}x_n} \qquad \text{and} \qquad y_n = zx_n$$

Then the *n*th level decision maker has to solve the following LP problem:

$$\begin{aligned}
&\min\left(\alpha_n + c_{n1}x_1 + c_{n2}x_2 + ... + c_{(n-1)n}x_{n-1}\right)z + c_{nn}y_n \\
&\text{s.t. } A_ny_n - (b - A_1x_1 - A_2x_2 - ... - A_{n-1}x_{n-1})z \le 0 \\
&d_{nn}y_n + \left(\beta_n + d_{n1}x_1 + d_{n2}x_2 + ... + d_{n(n-1)}x_{n-1}\right)z = 1 \\
&y_n \ge 0, \qquad z \ge 0
\end{aligned}$$

By embedding this problem in the LFMP problem (3), we get:

$$\begin{aligned}
&\min \frac{\left(\alpha_{n-1}z + c_{(n-1)1}x_1 z + c_{(n-1)2}x_2 z + \ldots + c_{(n-1)(n-1)}x_{n-1}z + c_{(n-1)n}y_n\right)}{\left(\beta_{n-1}z + d_{(n-1)1}x_1 z + d_{(n-1)2}x_2 z + \ldots + d_{(n-1)(n-1)}x_{n-1}z + d_{(n-1)n}y_n\right)}\\
&\text{where } y_n, z \text{ solve}\\
&\min\left(\alpha_n + c_{n1}x_1 + c_{n2}x_2 + \ldots + c_{(n-1)n}x_{n-1}\right)z + c_{nn}y_n \qquad (9)\\
&\text{s.t. } A_n y_n - \left(b - A_1x_1 - A_2x_2 - \ldots - A_{n-1}x_{n-1}\right)z \le 0\\
&d_{nn}y_n + \left(\beta_n + d_{n1}x_1 + d_{n2}x_2 + \ldots + d_{n(n-1)}x_{n-1}\right)z = 1\\
&x_1 \ge 0, x_2 \ge 0, \ldots, x_{n-1} \ge 0, y_n \ge 0, z \ge 0
\end{aligned}$$

Notice that the (n – 1)th level objective function contains the nonlinear terms $x_1z, x_2z, \ldots, x_{n-1}z$. In this case, it definitely makes no sense to consider $y_{n-1} = x_{n-1}z$ as a single variable because $x_1, x_2, \ldots, x_{n-1}$ are the variables controlled by the first, second, …, (n – 1)th level decision makers, respectively, while z is controlled by the nth level one. Since the reformulated problem is apparently more complicated to solve than the original one, it does not seem very tempting to directly use the C&C transformation in the process of solving the LFMP problem. In the next section, we will see that it can be used to solve LFP problems arising in successive iterations of the Kth-best algorithm.

The *K*th-best algorithm. Bearing in mind that there is an extreme point of S which solves the LFMP problem, an examination of all the extreme points of the polyhedron S constitutes an algorithm that will find the solution of the LFMP problem in a finite number of steps. This is unsatisfactory, however, since the number of extreme points of S is, in general, very large. Nevertheless, in light of Theorem 1, we can propose the Kth-best algorithm, a more successful enumeration scheme, for solving the LFMP problem. This algorithm was first proposed by Bialas and Karwan for solving the linear bilevel programming problem [2].

According to this algorithm, an optimal solution to the relaxed problem (4), $\left(x_1^{[1]}, x_2^{[1]}, \ldots, x_n^{[1]}\right)$ is first considered. If this is a point of IR, then it is an optimal solution of the LFMP problem. If this is not so, the set of its adjacent extreme points, $W^{[1]}$, is considered. Then, the extreme point in $W = W^{[1]}$ which provides the best value of f_1 is selected to test whether it is a point of IR. If it is, then the algorithm finishes. If this is not so, the point is eliminated from W and its adjacent extreme points with a worse value of f_1 are added to W. The algorithm continues by selecting the best extreme point in W with respect to f_1 and repeating the process.

Algorithm

Step 1. Let $\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$ be an optimal solution to problem (4).

Let $W = \left\{\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)\right\}$ and $T = \phi.$

Set $j = 1.$

Go to Step 2.

Step 2. Set $x_1 = x_1^{[j]}$ and solve the second relaxed problem.

Let $\left(x_2^{*}, x_3^{*}, ..., x_n^{*}\right)$ be its optimal solution.

If $x_2^{*} = x_2^{[j]}, x_3^{*} = x_3^{[j]}, ..., x_n^{*} = x_n^{[j]}$, go to step 5; $\left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right)$ is a global optimum for the two level problem.

Otherwise go to step 3.

Step 3. Let $W^{[j]}$ denote the set of adjacent extreme points of $\left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right).$

Let $T = T \cup \left\{\left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right)\right\}$ and $W = \left(W \cup W^{[j]}\right) \setminus T.$

Go to step 4.

Step 4. Set $j = j+1$ and choose $\left(x_1^{[j]}, x_2^{[j]}, ... x_n^{[j]}\right)$ so that

$$f_1\left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right) = \min\left\{f_1\left(x_1, x_2, ..., x_n\right) : \left(x_1, x_2, ..., x_n\right) \in W\right\}.$$

Go to Step 2.

Step 5. Set $x_1 = x_1^{[j]}, \;\; x_2 = x_2^{[j]}$ and solve the third relaxed problem.

Let $\left(x_2^{**}, x_3^{**}, ..., x_n^{**}\right)$ be its optimal solution.

If $x_2^{**} = x_2^{[j]}, x_3^{**} = x_3^{[j]}, ..., x_n^{**} = x_n^{[j]}$, go to step 6; $\left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right)$ is a global optimum for the three level problem.

Otherwise go to step 3.

Step 6. On continuing the process, solve the (n – 1)th relaxed problem.

Let $\left(x_2', x_3', ..., x_n'\right)$ be its optimal solution.

If $x_n' = x_n^{[j]}$, go to step 5. $\left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right)$ is a global optimum for LFMP problem.

Now we give a formal proof of the correctness of this algorithm. For this purpose let $\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$, $\left(x_1^{[2]}, x_2^{[2]}, ..., x_n^{[2]}\right), ...,$ $\left(x_1^{[m]}, x_2^{[m]}, ..., x_n^{[m]}\right)$ denote the m ordered extreme point solutions to the relaxed problem (4), i.e.

$$f_1\left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right) \le f_1\left(x_1^{[j+1]}, x_2^{[j+1]}, ..., x_n^{[j+1]}\right), \qquad j = 1, 2, ..., m-1$$

We will prove that the (j – 1)th best extreme point of S, $\left(x_1^{[j+1]}, x_2^{[j+1]}, ..., x_n^{[j+1]}\right)$, is adjacent to $\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$ or $\left(x_1^{[2]}, x_2^{[2]}, ..., x_n^{[2]}\right)$, ..., or $\left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right)$. Hence, the algorithm successively considers an ordered sequence of extreme points and it is obvious that $\left(x_1^{[k]}, x_2^{[k]}, ..., x_n^{[k]}\right)$ is a global optimum to the two level problem if $k = \min_{j\in\{1, 2, ..., m\}} \left\{j : \left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right) \in IR\right\}$. Likewise, on repeating the process and considering the m ordered extreme points to the (n – 1)th relaxed problem, the algorithm gives the globally optimal solution of the LFMP problem.

Theorem 2. Let $\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right)$ be an extreme point of S. There exists an edge path in S from $\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right)$ to $\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$ such that the value of $f_1\left(x_1, x_2, ..., x_n\right)$, $f_2\left(x_1, x_2, ..., x_n\right)$, ..., $f_n\left(x_1, x_2, ..., x_n\right)$ is non-increasing along it.

Proof: Assume for the time being that every extreme point $\left(x_1, x_2 ..., x_n\right)$ adjacent to $\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right)$ satisfies

$$\begin{aligned}
&f_1\left(x_1, x_2, ..., x_n\right) \ge f_1\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right)\\
&f_2\left(x_1, x_2, ..., x_n\right) \ge f_2\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right)\\
&\cdots\\
&f_n\left(x_1, x_2, ..., x_n\right) \ge f_n\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right)
\end{aligned}$$

Hence, $\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right)$ is a local extreme-minimum point of $f_1, f_2, ..., f_n$ in S. Since $f_1, f_2, ..., f_n$ are quasiconcave and explicitly quasiconvex on S, then $\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right)$ is a global minimum of the (n – 1)th relaxed problem, i.e.

$$f_{n-1}\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right) = f_{n-1}\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$$

Therefore, $\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$ and $\left(\overline{x}_1, \overline{x}_2, ..., \overline{x}_n\right)$ are extreme points of the optimum set of the (n – 1)th relaxed problem. Since $f_1, f_2, ..., f_n$ are continuous, quasiconvex and explicitly quasiconcave on S, this set is the convex hull of some extreme points of S. Thus there exists an edge path in this polyhedron from $\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)$ to $\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$. Since all the points of this edge path are from S and have the same value of $f_1\ f_2, ..., f_n$, this is the edge path we are looking for.

Suppose now that there exists at least one extreme point $\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right)$ adjacent to $\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)$ such that

$$\begin{aligned}
&f_1\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right) < f_1\left(\bar{x}_1, \bar{x}_2\ , ..., \bar{x}_n\right)\\
&f_2\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right) < f_2\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)\\
&...\\
&f_{n-1}\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right) < f_{n-1}\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)
\end{aligned}$$

Let us now consider $\left(\hat{x}_1, \hat{x}_2, .., \hat{x}_n\right)$ instead of $\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)$ and repeat the former process. Hence, either there exists an edge path P from $\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right)$ to $\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$ for which all points have the same value of $f_1, f_2, ..., f_n$ and $\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)$ $-\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right) - P$ is the required edge path, or there exists an extreme point $\left(\breve{x}_1, \breve{x}_2, ..., \breve{x}_n\right)$ adjacent to $\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right)$ such that

$$\begin{aligned}
&f_1\left(\breve{x}_1, \breve{x}_2, ..., \breve{x}_n\right) < f_1\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right)\\
&f_2\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right) < f_2\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)\\
&\ldots\\
&f_{n-1}\left(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\right) < f_{n-1}\left(\bar{x}_1, \bar{x}_2, ..., \bar{x}_n\right)
\end{aligned}$$

Next, we consider $\left(\breve{x}_1, \breve{x}_2, ..., \breve{x}_n\right)$ and repeat the process. Since the number of extreme points of S is finite, eventually an edge path will be obtained, along which the value of $f_1, f_2, ..., f_n$ is non-increasing.

Theorem 3. The $(k + 1)$th best extreme point of S $\left(x_1^{[k+1]}, x_2^{[k+1]}, ..., x_n^{[k+1]}\right)$ is adjacent to $\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$ or $\left(x_1^{[2]}, x_2^{[2]}, ..., x_n^{[2]}\right), ...,$ or $\left(x_1^{[k]}, x_2^{[k]}, ..., x_n^{[k]}\right),\ k < m.$

Proof: Let $W^{[j]}$ denote the set of adjacent extreme points of $\left(x_1^{[j]}, x_2^{[j]}, ..., x_n^{[j]}\right)$. Let

$$T = \left\{\left(x_1^{[1]}, x_2^{[1]},\ ..., x_n^{[1]}\right), \left(x_1^{[2]},\ x_2^{[2]}, ..., x_n^{[2]}\right), ..., \left(x_1^{[k]}, x_2^{[k]}, ..., x_n^{[k]}\right)\right\}$$

and

$$W = \left(W^{[1]} \cup W^{[2]} \cup \ldots \cup W^{[k]}\right) \setminus T$$

Let $(y_1, y_2, ..., y_n) \in W$ such that

$$\begin{aligned}
&f_1(y_1, y_2, ..., y_n) = \min_{(w_1, w_2, ..., w_n) \in W} \{f_1(w_1, w_2, ..., w_n)\} \\
&f_2(y_1, y_2 ..., y_n) = \min_{(w_1, w_2, ..., w_n) \in W} \{f_2(w_1, w_2, ..., w_n)\} \\
&\ldots \\
&f_{n-1}(y_1, y_2 ..., y_n) = \min_{(w_1, w_2, ..., w_n) \in W} \{f_{n-1}(w_1, w_2, ..., w_n)\}
\end{aligned}$$

Let $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ be an extreme point of S such that

$$(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n) \notin \bigcup_{j=1,2,...,k} W^{[j]}$$

Taking into account the fact that any edge path in S from $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ to $\left(x_1^{[1]}, x_2^{[1]}, ..., x_n^{[1]}\right)$ must contain at least one point of W as an intermediate point, and considering the edge path provided by theorem 2, there exists $(\overline{w}_1, \overline{w}_2, ..., \overline{w}_n) \in W$ such that

$$\begin{aligned}
&f_1(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n) \geq f_1(\overline{w}_1, \overline{w}_2, ..., \overline{w}_n) \geq f_1(y_1, y_2, ..., y_n) \\
&f_2(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n) \geq f_2(\overline{w}_1, \overline{w}_2, ..., \overline{w}_n) \geq f_2(y_1, y_2, ..., y_n) \\
&\ldots \\
&f_n(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n) \geq f_n(\overline{w}_1, \overline{w}_2, ..., \overline{w}_n) \geq f_n(y_1, y_2, ..., y_n)
\end{aligned}$$

Since $(y_1, y_2, ..., y_n)$ minimizes the value of $f_1, f_2, ..., f_n$ over the set of extreme points of S excluding T, then

$$(y_1, y_2, ..., y_n) = \left(x_1^{[k+1]}, x_2^{[k+1]}, ..., x_n^{[k+1]}\right)$$

Theorem 4. The Kth-best algorithm solves the LFMP problem.

Proof. As a consequence of theorem 3, the Kth-best extreme point of the relaxed problems is adjacent to either the 1st, 2nd, …, or $(k - 1)$th extreme point. Thus, upon termination, the algorithm provides the best boundary feasible extreme point, i.e. the optimal solution to the LFMP problem.

As previously was pointed out, it is worth noting that, taking into account the C&C transformation, only linear problems need to be solved when applying the *K*th-best algorithm for solving the LFMP problem [5].

3. Numerical example

Now we illustrate the algorithm with the help of a numerical example.
Let us consider

$$\begin{aligned}
&\min_{(x_1,x_2)} = \frac{1+x_1-x_2+2x_4}{8-x_1-2x_3+x_4+2x_5} \\
&\text{where } \left(x_3, x_4\right) \text{ solve} \\
&\min_{(x_3,x_4)} = \frac{1+x_1+x_2+2x_3-x_4+x_5}{6+2x_1+x_3+x_4-3x_5} \\
&\text{where } \left(x_5, ..., x_8\right) \text{ solve} \\
&\min_{(x_5,x_6,x_7,x_8)} = \frac{2+x_1+x_2-x_3+x_4+x_5}{1-x_1+x_2-x_3+2x_4+x_5} \\
&\text{s.t. } -x_3+x_4+x_5+x_6=1 \\
&2x_1-x_3+2x_4-0.5x_5+x_7=1 \\
&2x_2+2x_3-x_4-0.5x_5+x_8=1 \\
&x_i \geq 0, \quad i=1,2,...,8
\end{aligned}$$

Solution. First, we solve the first relaxed problem and get the optimal solution (0, 0.75, 0, 0, 1, 0, 1.5, 0). Now we fix $x_1 = 0$ and $x_2 = 0.75$ and solve the second level problem. We get its optimal solution $\left(x_3, \ldots, x_8\right)$= (0, 0.5, 0, 0.5, 0, 0). The solution (0, 0.75, 0, 0, 1, 0, 1.5, 0)$\notin(IR)_1$ with $f_1 = 0.0192$ but the solution (0, 0.75, 0, 0, 1, 0, 1.5, 0) $\in (IR)_1$. Now we find all the adjacent extreme points, together with the value of the objective function at these points. We get the following extreme points: $(0, 0, 1, 0, 2, 0, 3, 0)$ with $f_1 = 0.0588$, $(0, 0, 0, 0, 1, 0, 1.5, 1.5)$ with $f_1 = 0.0769$, (0.75, 0.75, 0, 0, 1, 0, 0, 0) with $f_1 = 0.0816$, $(0, 0.9, 0, 0.6, 0.4, 0, 0, 0)$ with $f_1 = 0.1226$, $(0, 0.5, 0, 0, 0, 1, 1, 0)$ with $f_1 = 0.2$.

We do a second iteration, as $(0, 0, 1, 0, 2, 0, 3, 0) \notin (IR)_1$ with $f_1 = 0.0588$ and search for the extreme points. We repeat the process and on the fourth iteration we get the optimal solution $(0.75, 0.75, 0, 0, 1, 0, 0, 0) \in (IR)_1$ with $f_1 = 0.0816$.

Now we solve the second relaxed problem by fixing $x_1 = 0.75,\ \ x_2 = 0.75.$ We get the optimal solution $(0.75, 0.75, 0, 0,1, 0, 0, 0) \notin (IR)_2.$ Now we fix $x_3 = 0,\ x_4 = 0$ and solve the third level problem. We get the solution $(x_5, x_6, x_7, x_8) = (0, 1, 0, 0).$ Now the solution $(0.75, 0.75, 0, 0, 0, 1, 0, 0) \notin (IR)_2.$ Hence, we go on searching the adjacent extreme points by solving

$$
\begin{aligned}
& x_5 + x_6 = 1 \\
& -0.5x_5 + x_7 = -0.5 \\
& -0.5x_5 + x_8 = -0.5 \\
& x_i \geq 0, \qquad i = 5, 6,\ \ldots, 8
\end{aligned}
$$

We get only one extreme point with positive values, which is (0.75, 0.75, 0, 0, 1, 0, 0, 0) with $f_2 = 0.7777.$

4. Conclusion

It is worth pointing out that the proof of Theorem 1 is mainly based on the fact that the objective functions of the first (n – 1) levels are explicitly quasimonotonic, otherwise we could not prove the existence of a boundary feasible extreme point. Hence, we can conclude that Theorem 1 is still valid for more general problems.

Indeed, let us consider the quasiconcave multilevel programming problem, in which $f_1, f_2, \ldots, f_n$ are continuous functions; f_1 is quasiconcave on *S*; f_2 is quasiconcave on $S(x_1)$ for all $x_1 \in S_1, \ldots,$ likewise f_n is quasiconcave on $S(x_1, x_2, \ldots, x_{n-1}).$ *S* is a polyhedron, which is assumed to be nonempty and bounded, and the $M(x_i)$ are single-valued $\forall(x_1, x_2, \ldots, x_{i-1}) \in S_{i-1}.$ This model includes, as important particular cases, a wide class of multilevel problems where the objective functions are linear, fractional (ratios of concave nonnegative functions and convex strictly positive functions) or multiplicative (the product of a set of concave functions, each strictly positive) [8].

As noted previously, for this problem Calvete and Gale proved that *IR* is formed by the union of connected faces of *S* [3]. Hence, there exists an extreme point of the polyhedron *S* that solves it. Under the additional assumption that the objective functions of the first (n – 1) levels are explicitly quasimonotonic, the proof of Theorem 3 can be replicated step by step to show that there exists a boundary feasible extreme point that solves the quasiconcave problem. Notice that we do not require any additional assumption on the nth level objective function, so that this result is still valid for multilevel problems in which the first level objective function is linear or linear fractional and the objective functions of the other levels are linear, fractional or multiplicative. The same can be said with regard to the Kth-best algorithm. Under the mentioned assumptions, an optimal solution to the quasiconcave multilevel problem can be obtained by checking the best of the extreme points adjacent to all previously analyzed extreme points, until a feasible solution is found.

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