

NON-ISOTHERMAL CONSTITUTIVE RELATIONS AND HEAT TRANSFER EQUATIONS OF A TWO-PHASE MEDIUM

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Abstract: In the case of a two-phase medium – such as the soil, which consists of an elastic skeleton and is filled with pore fluids – stress and strain within the medium are dependent on both phases. Similarly, in the case of heat transfer, heat is conducted through the two phases at different rates, with an additional heat transfer between the phases. In the classical approach to modelling a porous medium, it is assumed that the fluid filling the pore space is water, which is incompressible. In the case of gas, the volume of which is strongly dependent on temperature and pressure, one should take this behavior into account in the constitutive relations for the medium. This work defines the physical relations of a two-phase medium and provides heat transfer equations, constructed for a porous, elastic skeleton with fluid-filled pores, which may be: liquid, gas, or mixture of liquid and a gas in non-isothermal conditions. The paper will present constitutive relations derived from the laws of irreversible thermodynamics, assuming that pores are filled with either a liquid or a gas. These relations, in the opinion of the authors, may be used as the basis for the construction of a model of the medium filled partly with a liquid and partly with a gas. It includes the possibility of independent heat transfer through any given two-phase medium phase, with the transfer of heat between the phases.

Key words: heat transfer, porous media, thermal consolidation, constitutive equations

1. INTRODUCTION

The discussion about the behavior of a two-phase medium, composed of a porous skeleton and fluid filling the pores, is an important issue in the mechanics of soils and rocks. Readers can find a description of the rheology of soils and rocks in [15] and [16]. It is shown in these works that, in addition to the attempts to present the behavior of the soil or rock in the form of a rheological model, there have been continuous attempts to present a model describing the behavior of such a medium with regard to the properties of the solid and liquid or gas phases in rock pores. In further discussion, these relations will be used to more fully describe the impact of the seepage on the strain of the two-phase medium composed of an elastic skeleton and slightly compressible fluid. The equations of the model were presented for the first time by Biot [5], with further refinement by Biot [6], [7], and

Biot & Willis [8]; they were then derived from the basic laws of irreversible thermodynamics, taking into account the statements made by Onsager on the work of de Groot & Mazur [13].

The discussion about the constitutive relations also dealt with the search for macroscopic dependencies for the two-phase medium by averaging processes at the pore scale, adopted as a non-uniform scale of the periodic medium using the method of asymptotic homogenization. The non-uniform scale presents constitutive relations in accordance with Bensoussan et al. [4], Auriault et al. [2], Strzelecki et al. [20], Łydźba [17], [18], separately written for the solid and liquid phases of the REV element, additionally assuming the lack of flow on the border of the solid and liquid phases and introducing the condition of periodicity. Thanks to the use of asymptotic homogenization algorithms, one gains a macroscopic system of equations corresponding to those proposed by M.A Biot, with the ability to define the effective parameters of the

medium when the properties of the skeleton and its pore fluid are known; one can also define the geometry at a microscopic scale. The solution from the above-cited work is a significant achievement of theoretical physics.

Classic constitutive relations for the poroelastic medium for isothermal issues were given by Biot [6] and Biot & Willis [8]. Considering also the work of Coussy [10], [11] on the media consisting of a porous elastic skeleton and liquid, the equations can be written as follows:

- The first form of the constitutive relations:

$$\begin{aligned}\sigma_{ij} &= 2N\varepsilon_{ij} + A\varepsilon\delta_{ij} + Q\theta\delta_{ij}, \\ \sigma - \sigma_a &= Q\varepsilon + R\theta,\end{aligned}\quad (1)$$

where N and A are the Lamé parameters μ and λ for the skeleton of the porous medium, R is the bulk modulus of fluid, Q is the coupling coefficient resulting from the interaction between the solid and liquid phases, σ_{ij} is the partial stress tensor in the skeleton, σ is the partial stress in the liquid, ε_{ij} signifies the tensor of strain of the skeleton, $\varepsilon = \varepsilon_{ij}$ is the dilatancy of the skeleton and θ is the dilatancy of the liquid. Moreover, determining the pressure in the liquid as p and the porosity as f_0 , partial stress in the liquid is given as $\sigma = -f_0p$.

The second form of the constitutive relations:

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2N}\sigma_{ij} - \frac{A}{2N(3A+2N)}\sigma_{kk}\delta_{ij} + \frac{\sigma - \sigma_a}{B}, \\ \theta &= \frac{3\sigma_m}{B} + \frac{\sigma - \sigma_a}{R},\end{aligned}\quad (2)$$

where $\sigma_m = \sigma_{kk}/3$.

If we confine ourselves to volume, we use the formula:

$$\begin{aligned}\varepsilon &= \frac{\sigma_m}{K} + \frac{\sigma - \sigma_a}{B}, \\ \theta &= \frac{\sigma_m}{B} + \frac{\sigma - \sigma_a}{R},\end{aligned}\quad (3)$$

where K determines the bulk modulus equal to $(A + 2N/3)$. Using the assumptions for the construction of the relations for gas (dependence resulting from the Boyle–Mariotte law), provided in the work of Strzelecki et al. [20]:

$$\frac{p_a}{1+\theta} = p. \quad (4)$$

With relations taking into account the mutual influence of the phases, the reverse relations (3) take the

form:

$$\begin{aligned}\varepsilon &= \frac{\sigma_m}{K} + \frac{\sigma - \sigma_a}{B}, \\ \theta &= \frac{\sigma_m}{B} + \frac{\sigma_a}{\sigma} - 1.\end{aligned}\quad (5)$$

The issue of constructing physical, non-isothermal relations for the single phase is discussed in detail in a work by Nowacki [19]. Based on the laws of irreversible thermodynamics, the problem of constitutive relations for the two-phase medium consisting of an elastic skeleton and liquids is discussed in detail in Coussy [10], Strzelecki et al. [22], Uciechowska and Strzelecki [24]. In these works the assumption was made that the heat flows evenly through the porous medium, meaning that the temperature of the liquid and the skeleton is the same in any given area.

2. NON-ISOTHERMAL CONSTITUTIVE RELATIONS OF THE EQUATIONS OF HEAT TRANSFER FOR AN OPEN TWO-PHASE SYSTEM

When the first law of thermodynamics is used, it can be shown as:

$$\dot{L} + \dot{Q} = \frac{\partial}{\partial t}(W + K), \quad (6)$$

where: \dot{L} is the work of internal forces, gravitational force and the viscous resistance of fluid; \dot{Q} signifies heat generated by seepage and stress on the porous medium; W is internal energy; K denotes kinetic energy.

In order to determine the strength of force from the viscous resistance of the liquid in a two-phase medium, the findings of Strzelecki et al. [22] and Coussy [11] are used. The function of internal energy dissipation through viscous friction takes the form:

$$2W_d = \int_{\Omega} b v_i^r v_i^r d\Omega, \quad (7)$$

where b is the coefficient of viscous resistance in the fluid, and v_i^r describes the components of the relative speed of fluid, equal to the difference in fluid velocity v_i^l and skeleton velocity v_i^s .

Taking formula (7) as a starting point, one can determine the strength of viscous resistance acting on

the skeleton of the two-phase medium in the following form:

$$M_i^s = \frac{\partial W_d}{\partial v_i^s} = - \int_{\Omega} b v_i^r d\Omega, \quad (8)$$

as well as the viscous resistance interacting with the fluid

$$M_i^l = \frac{\partial W_d}{\partial v_i^l} = \int_{\Omega} b v_i^r d\Omega. \quad (9)$$

The kinetic energy of the two-phase medium is described in accordance with the work of Biot [7] in the following form:

$$2K = \int_{\Omega} (\rho_{11} v_i^s v_i^s + 2\rho_{12} v_i^s v_i^l + \rho_{22} v_i^l v_i^l). \quad (10)$$

If ρ_s and ρ_l denote the density of the skeleton and the liquid, the partial density of skeleton ρ_1 and liquid ρ_2 can be determined basing each of them on the total volume of area Ω . Using the volumetric porosity, denoted by f_0 , the density: $\rho_1 = (1 - f_0)\rho_s$ and $\rho_2 = f_0\rho_l$ can be calculated. Through ρ the density of the two-phase medium will be denoted, equal to: $\rho = \rho_1 + \rho_2$. In accordance with the findings presented by Biot [7], $\rho_{11} + \rho_{12} = \rho_1 > 0$ and $\rho_{12} + \rho_{22} = \rho_2 > 0$ and $\rho_{12} < 0$, which means that ρ_{12} is the density resulting from the mutual dynamic interaction of both phases of the medium.

Formula (10) allows to describe the inertia forces acting on the solid phase of the medium:

$$B_i^s = \int_{\Omega} \left(\rho_{11} \frac{\partial v_i^s}{\partial t} + \rho_{12} \frac{\partial v_i^l}{\partial t} \right) d\Omega, \quad (11)$$

as well as the inertia acting on the liquid:

$$B_i^l = \int_{\Omega} \left(\rho_{12} \frac{\partial v_i^s}{\partial t} + \rho_{22} \frac{\partial v_i^l}{\partial t} \right) d\Omega. \quad (12)$$

The first law of irreversible thermodynamics can be used separately for each phase. Indicators 1 and 2 stand for the respective phases: solid and liquid.

Thus the power of internal forces on the skeleton can be expressed as:

$$\dot{L}_1^A = \int_S (\sigma_{ij} + \sigma\delta_{ij}) v_i^s n_j dS. \quad (13)$$

The power of gravitational forces on the skeleton of the medium:

$$\dot{L}_1^P = \int_{\Omega} (\rho - \bar{\rho}) X_i v_i^s d\Omega. \quad (14)$$

The strength of the viscous resistance of the fluid relative to the skeleton:

$$\dot{L}_1^D = \int_{\Omega} b v_i^r v_i^s d\Omega. \quad (15)$$

Because the force is a scalar quantity, the entire force acting on the skeleton of the medium is:

$$\dot{L}_1 = \dot{L}_1^A + \dot{L}_1^P + \dot{L}_1^D. \quad (16)$$

The rate of heat change in the skeleton of the medium takes the form:

$$\dot{Q}_1 = - \int_{\Omega} q_{i,1}^s d\Omega, \quad (17)$$

where q_i^s are the components of heat flux flowing through the solid phase of the medium.

The derivative of kinetic energy for the solid phase is equal to the power of inertia forces acting on the skeleton of the medium and is:

$$\frac{\partial K_1}{\partial t} = \int_{\Omega} \left(\rho_{11} v_i^s \frac{\partial v_i^s}{\partial t} + \rho_{12} v_i^l \frac{\partial v_i^l}{\partial t} \right) d\Omega. \quad (18)$$

Time derivative of internal energy for the skeleton is:

$$\frac{\partial W}{\partial T} = \int_{\Omega} \dot{w}_1 d\Omega, \quad (19)$$

where \dot{w}_1 denotes the rate of local changes to the internal energy of the skeleton.

Bearing in mind the formulas from (13) to (19), the first law of thermodynamics, in relation to the solid phase of the medium, takes the form:

$$\begin{aligned} & \int_{\Omega} [X_i \rho_1 v_i^s + b v_i^r v_i^s + (\sigma_{ij} + \sigma\delta_{ij}) \\ & \quad + (\sigma_{ij,j} + \sigma_{,i}) v_i^s - q_{i,i}^s] d\Omega \\ & = \int_{\Omega} \left(\dot{w}_1 + \rho_{11} v_i^s \frac{\partial v_i^s}{\partial t} + \rho_{12} v_i^l \frac{\partial v_i^l}{\partial t} \right) d\Omega. \end{aligned} \quad (20)$$

Because, in accordance with Strzelecki [22], the balance equation for the solid phase of the medium has the form:

$$\sigma_{ij,j} + X_i \rho_1 + b v_i^r - \left(\rho_{11} \frac{\partial v_i^s}{\partial t} + \rho_{12} \frac{\partial v_i^l}{\partial t} \right) = 0, \quad (21)$$

the Eq. (20) leads to a local relation allowing to define the speed of local changes to the internal energy of the skeleton:

$$\dot{w}_1 = \sigma_{ij} \dot{\epsilon}_{ij} - q_{i,i}^s. \quad (22)$$

The internal forces in the liquid equal:

$$\dot{L}_2^A = \int_S \sigma (v_i^r - v_i^s) n_i dS. \quad (23)$$

The strength of the forces of gravity in the liquid equal:

$$\dot{L}_2^P = \int_{\Omega} \rho_2 X_i v_i^l d\Omega. \quad (24)$$

The power of the forces of viscous resistance on the liquid is:

$$\dot{L}_2^D = - \int_{\Omega} b v_i^r v_i^l d\Omega. \quad (25)$$

The speed of heat flow in the liquid takes the form:

$$\dot{Q} = - \int_{\Omega} q_{i,i}^l d\Omega, \quad (26)$$

where q_i^l are the components of the vector of heat flow.

The derivative of the kinetic energy of the fluid is equal to the power of inertia forces acting on the fluid and it is:

$$\frac{\partial K_2}{\partial t} = \int_{\Omega} \left(\rho_{12} v_i^l \frac{\partial v_i^s}{\partial t} + \rho_{22} v_i^l \frac{\partial v_i^l}{\partial t} \right) d\Omega. \quad (27)$$

The derivative of the internal energy of the liquid in the area Ω can take the form:

$$\frac{\partial W_2}{\partial t} = \int_{\Omega} \dot{w}_2 d\Omega. \quad (28)$$

Using the formula (6) as well as the formulas from (23) to (28), the first law of thermodynamics for the fluid takes the following form:

$$\begin{aligned} & \int_{\Omega} [X_i \rho_2 v_i^l - b v_i^r v_i^l + \sigma_{,i} V_i^l + \sigma \theta - q_{i,i}^l] d\Omega \\ & = \int_{\Omega} \left(\dot{w}_2 + \rho_{12} v_i^l \frac{Dv_i^l}{Dt} + \rho_{12} v_i^l \frac{Dv_i^s}{Dt} \right) d\Omega. \end{aligned} \quad (29)$$

In the above relation, the rate of the change of expansion of the liquid is determined by $\dot{\theta}$. Because, in accordance with the law of the conservation of momentum, which is discussed in detail in a work of

Strzelecki and Strzelecki [23], the equation of dynamic equilibrium for fluid takes the form:

$$\sigma_{,i} + X_i \rho_2 - b v_i^r - \left(\rho_{21} \frac{\partial v_i^s}{\partial t} + \rho_{22} \frac{\partial v_i^l}{\partial t} \right) = 0. \quad (30)$$

The Eq. (29) leads to the local relation, allowing to define the rate of the local change of the internal energy of the fluid:

$$\dot{w}_2 = \sigma \dot{\theta} - q_{i,i}^l. \quad (31)$$

It can be assumed that the rate of the change of the internal energy of two-phase medium \dot{w} is equal to the sum of the rate of energy change for each of the phases, \dot{w}_1 and \dot{w}_2 .

Therefore:

$$\dot{w} = \dot{w}_1 + \dot{w}_2. \quad (32)$$

Using the relation (32), as well as the relations (22) and (31), it can be stated that the change of the internal energy of the two-phase medium is:

$$\dot{w} = \sigma_{ij} \dot{\epsilon}_{ij} + \sigma \dot{\theta} - q_{i,i}^s - q_{i,i}^l. \quad (33)$$

The first law of thermodynamics gives the relation between mechanical work and heat. It expresses the balance of energy without imposing limitations on the direction of the process of changing the state of the body. When discussing the classical mechanics of perfectly rigid bodies, one can talk about the mutual conversion of potential energy into kinetic energy assuming, certainly, the system does not have the dissipation of the energy generated due to, for example, the mutual friction of particles. When thermal changes begin to occur in the system, an irreversible process takes place. In this case, the second law of thermodynamics must be used, which imposes significant limitations on the process of changes in the energy system.

In order to describe the irreversible processes, thermodynamics introduces the function of state changes called entropy. To determine the limitation in the direction of changes in the system, the second law of thermodynamics introduces an inequality which states that the entropy change inside the system is always positive or zero in the case of the irreversible process named the Clausius–Duhem inequality. In continuum mechanics, according to Derski [12], de Groot, Mazur [13], the introduced function of specific entropy (S) is measured per unit of volume (Ω) and can be expressed by the formula:

$$S = \int_{\Omega} s d\Omega. \quad (34)$$

Specific entropy is described as:

$$ds = \frac{dq}{T}, \quad (35)$$

where T is the absolute temperature, dq is heat gain per unit of volume, and ds is the exact differential of entropy.

Because, in general, the body may exchange heat with its surroundings, the rate of entropy change is the sum of the rate of entropy change S_z as a result of heat exchange with the surroundings and the rate of internal entropy change, expressed as:

$$\frac{dS}{dt} = \frac{dS_w}{dt} + \frac{dS_z}{dt}. \quad (36)$$

Entropy related to heat exchange with the surroundings is expressed as:

$$\frac{dS_z}{dt} = -\int_{\Omega} \left(\frac{q_i}{T} \right)_{,i} d\Omega. \quad (37)$$

Which leads to the following local equation:

$$s_z = -\left(\frac{q_i}{T} \right)_{,i} \quad (38)$$

As can be seen in (38), entropy S_z can have a positive or negative value, or equal zero, depending on the direction of heat flow and its divergence. Internal entropy S_w behaves differently. Its changes in unit volume must be in accordance with the Clausius–Duhem inequality, which can be expressed as follows:

$$s_w = -\int_{\Omega} \frac{q_i T_{,i}}{T^2} \geq 0. \quad (39)$$

Let us now assume that the temperature of the skeleton of the two-phase medium is equal to the temperature of the pore fluid flowing through it. Defining the absolute temperature of the skeleton as T^s , and the absolute temperature of the fluid as T^l , the rate of entropy change can be defined for the stable phase with this example:

$$\dot{s}_1 = -\frac{q_{i,i}^s}{T^s}. \quad (40)$$

The rate of entropy change for the fluid as:

$$\dot{s}_2 = -\frac{q_{i,i}^l}{T^l}. \quad (41)$$

Utilizing the Clausius–Duhem inequality for each of the phases of the medium:

$$-\frac{q_i^s T_{,i}}{(T^s)^2} \geq 0 \text{ as well as } -\frac{q_i^l T_{,i}}{(T^l)^2} \geq 0. \quad (42)$$

The conduction law for both phases of the medium is obtained:

$$q_i^s = -\lambda^s T_{,j}^s, \text{ as well as } q_i^l = -\lambda^l T_{,j}^l. \quad (43)$$

Introducing the Helmholtz free energy for the two-phase medium gives the equation:

$$F = W - S_1 T^s - S_2 T^l. \quad (44)$$

Taking into account that the Helmholtz energy works with the aid of local free energy:

$$F = \int_{\Omega} \chi d\Omega. \quad (45)$$

The local rate of free-energy change relationship can be defined with the example:

$$\dot{\chi} = \dot{w} - s_1 \dot{T}^s - \dot{s}_1 T^s - s_2 \dot{T}^l - \dot{s}_2 T^l. \quad (46)$$

Taking into account formulas (46) and (33), the following formula is obtained:

$$\sigma_{ij} \dot{\varepsilon}_{ij} + \sigma \dot{\theta} - s_1 \dot{T}^s - s_2 \dot{T}^l - \dot{\chi} = 0. \quad (47)$$

If it is then assumed that local free energy is a function of the tensor of strain on the solid phase ε_{ij} , fluid dilatation θ and absolute temperature for the solid and liquid phases, it can be written that:

$$\chi = \chi(\varepsilon_{ij}, \theta, T^s, T^l). \quad (48)$$

The differential fulfilling in function χ takes the form:

$$\chi = \frac{\partial \chi}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} + \frac{\partial \chi}{\partial \theta} \dot{\theta} + \frac{\partial \chi}{\partial T^s} T^s + \frac{\partial \chi}{\partial T^l} T^l. \quad (49)$$

So the Eq. (47) leads to:

$$\left(\sigma_{ij} - \frac{\partial \chi}{\partial \varepsilon_{ij}} \right) \dot{\varepsilon}_{ij} + \left(\sigma - \frac{\partial \chi}{\partial \theta} \right) \dot{\theta} + \left(-s_1 - \frac{\partial \chi}{\partial T^s} \right) T^s + \left(-s_2 - \frac{\partial \chi}{\partial T^l} \right) T^l = 0. \quad (50)$$

Which gives:

$$\sigma_{ij} = \frac{\partial \chi}{\partial \varepsilon_{ij}}, \quad \sigma = \frac{\partial \chi}{\partial \theta}, \quad -s_1 = \frac{\partial \chi}{\partial T^s}, \quad -s_2 = \frac{\partial \chi}{\partial T^l}. \quad (51)$$

Let us then expand the function of local free energy in the region of the natural state, meaning a state

where $\varepsilon_{ij} = 0$, $\theta = 0$, $T^s = T_0^s$, $T^l = T_0^l$. Temperature T_0^s and T_0^l is adopted for the skeleton and the absolute temperature of liquid in a natural state, which conventionally can be assumed as equal to the example 0 K. The temperature of both phases of the medium can be represented as the sum of:

$$T^s = T_0 + \mathcal{G}^s \text{ as well as } T^l = T_0 + \mathcal{G}^l, \quad (52)$$

T_0 is freely adopted temperature as the value of the reference temperature, for example, 237.15 K. Assuming that in the natural state:

$$\begin{aligned} \chi(0, 0, T_0^s, T_0^l) = 0, \quad \frac{\partial \chi(0, 0, T_0^s, T_0^l)}{\partial \varepsilon_{ij}} = 0, \\ \frac{\partial \chi(0, 0, T_0^s, T_0^l)}{\partial \theta} = \sigma_a, \end{aligned} \quad (53)$$

$$\frac{\partial \chi(0, 0, T_0^s, T_0^l)}{\partial T^s} = 0, \quad \frac{\partial \chi(0, 0, T_0^s, T_0^l)}{\partial T^l} = 0,$$

As well as introducing the formulas:

$$\begin{aligned} \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = c_{ijkl}, \quad \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial \varepsilon_{ij} \partial \theta} = \beta_{ij}, \\ \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial \theta^2} = \gamma, \quad \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial (T^s)^2} = \xi^s, \\ \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial (T^l)^2} = \xi^l, \quad \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial \varepsilon_{ij} \partial T^s} = \eta_{ij}^s, \\ \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial \varepsilon_{ij} \partial T^l} = \eta_{ij}^l, \quad \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial \theta \partial T^s} = \omega^s, \\ \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial \theta \partial T^l} = \omega^l, \quad \frac{\partial^2 \chi(0, 0, T_0^s, T_0^l)}{\partial T^l \partial T^s} = \xi^s. \end{aligned} \quad (54)$$

The function of free energy can be represented with the formula:

$$\begin{aligned} 2\chi(\varepsilon_{ij}, \theta, T^s, T^l) = 2\sigma_a \theta + c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \\ + \gamma \theta^2 + \xi^s (T^s)^2 + \xi^l (T^l)^2 + 2\beta_{ij} \varepsilon_{ij} \theta \\ + 2\eta_{ij}^s \varepsilon_{ij} T^s + 2\eta_{ij}^l \varepsilon_{ij} T^l + 2\omega^s \theta T^s \\ + 2\omega^l \theta T^l + \xi^{ls} T^s T^l. \end{aligned} \quad (55)$$

In accordance with (51), the constitutive relations and equations of entropy are obtained in both phases of the two-phase medium in the formula:

$$\begin{aligned} \sigma_{ij} = c_{ijkl} \varepsilon_{kl} + \beta_{ij} \theta + \eta_{ij}^s \mathcal{G}^s + \eta_{ij}^l \mathcal{G}^l, \\ \sigma - \sigma_a = \beta_{ij} \varepsilon_{ij} + \gamma \theta + \omega^s \mathcal{G}^s + \omega^l \mathcal{G}^l, \\ -s_1 = \eta_{ij}^s \varepsilon_{ij} + \omega^s \theta + \xi^s \mathcal{G}^s + \xi^{ls} \mathcal{G}^l, \\ -s_2 = \eta_{ij}^l \varepsilon_{ij} + \omega^l \theta + \xi^{ls} \mathcal{G}^l + \xi^l \mathcal{G}^l. \end{aligned} \quad (56)$$

Although the Onsager law of symmetry was not used, constitutive relations and the equation of entropy which fulfils the law were obtained. A four-dimensional tensor of elasticity in the case of an isotropic medium was analyzed in detail by Nowacki [19] and based on the results of these analyses, it can be expressed with the help of two material constants: Young's modulus E and Poisson's ratio ν , or with the two Lamé constants A and N :

$$c_{ijkl} = A \delta_{ij} \delta_{kl} + N (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \quad (57)$$

On the basis of a work by Biot and Willis [8], it can be said that tensor β_{ij} and constant γ , in the case of the isotropic medium, can be expressed with the use of Biot's constants Q and R :

$$\begin{aligned} \beta_{ij} = Q \delta_{ij}, \\ \gamma = R. \end{aligned} \quad (58)$$

Working on the assumption of an isotropic two-phase medium and assuming that the gradient of the temperature causes the volumetric strain of both phases and does not cause shear strain, one are led to the conclusion that the two-phase second-order tensors η_{ij}^s and η_{ij}^l can be expressed through the constant η^s and η^l in the example:

$$\begin{aligned} \eta_{ij}^s = \eta^s \delta_{ij}, \\ \eta_{ij}^l = \eta^l \delta_{ij}. \end{aligned} \quad (59)$$

The constitutive relations and entropy equation after the implementation of the proposed simplifications for the isotropic medium (57), (58) and (59) lead to the equations:

$$\begin{aligned} \sigma_{ij} = 2\mu \varepsilon_{ij} + (\lambda \varepsilon + Q \theta + \eta^s \mathcal{G}^s + \eta^l \mathcal{G}^l) \delta_{ij}, \\ \sigma - \sigma_a = Q \varepsilon + R \theta + \omega^s \mathcal{G}^s + \omega^l \mathcal{G}^l, \\ -s_1 = \eta^s \varepsilon_{ij} + \omega^s \theta + \xi^s \mathcal{G}^s + \xi^{ls} \mathcal{G}^l, \\ -s_2 = \eta^l \varepsilon_{ij} + \omega^l \theta + \xi^{ls} \mathcal{G}^l + \xi^l \mathcal{G}^l. \end{aligned} \quad (60)$$

Next the physical meaning of new material constants $\eta^s, \eta^l, \omega^s, \omega^l, \xi^s, \xi^l, \xi^{ls}$ are considered. After the contraction of the first Eq. (60) the following form is obtained:

$$\sigma_{ii} = 3\sigma_m = (2\mu + 3\lambda)\varepsilon + 3Q\theta + 3\eta^s \mathcal{G}^s + 3\eta^l \mathcal{G}^l. \quad (61)$$

From where the value of mean stress σ_m is calculated, it is equal to:

$$\sigma_m = K\varepsilon + Q\theta + \eta^s \mathcal{G}^s + \eta^l \mathcal{G}^l. \quad (62)$$

$K = \lambda + 2\mu/3$ is the modulus of volume strain of the skeleton of the porous medium; thus Q is the coefficient of the influence of liquid volume strain on the skeleton. If it is assumed that the only cause of stress on the skeleton is temperature \mathcal{G}^s , where $\sigma_m = 0, \mathcal{G}^l = 0$ and $\theta = 0$, it can be written:

$$K\varepsilon = -\eta^s \mathcal{G}^s. \quad (63)$$

It can also be stated that:

$$\eta^s = -3K\alpha^s, \text{ so } \eta^s < 0 \quad (64)$$

α^s is the coefficient of linear thermal expansion in the porous skeleton. On the other hand, if the only reason for the deformation of the fluid is the temperature of liquid \mathcal{G}^l , and the temperature of skeleton $\mathcal{G}^s = 0$, where $\sigma_m = 0$ and $\varepsilon = 0$, the following relationship is obtained:

$$Q\theta = -\eta^l \mathcal{G}^l, \quad (65)$$

where α^l is the coefficient of the linear thermal expansion of the liquid. The above formula allows to define the value of coefficient η^l with the equation:

$$\eta^l = -Q\alpha^l, \text{ so } \eta^l < 0. \quad (66)$$

Operating analogously to the second constitutive relationship and assuming that the cause of strain is the case when there is a non-zero relative temperature in the skeleton at a zero relative temperature of the liquid allows to obtain:

$$\omega^s = -3Q\alpha^s, \text{ so } \omega^s < 0, \quad (67)$$

or in the case when the cause of strain is a non-zero relative temperature in the fluid at a zero relative temperature in the skeleton, which allows to define value ω^l :

$$\omega^l = -R\alpha^l, \text{ so } \omega^l < 0. \quad (68)$$

The next considered problem are the entropy equations in the equation system (60), which can be written in the formulas:

$$-s_1 = \eta^s \varepsilon + \omega^s \theta + \xi^s \mathcal{G}^s + \xi^{ls} \mathcal{G}^l, \quad (69)$$

$$-s_2 = \eta^l \varepsilon + \omega^l \theta + \xi^{ls} \mathcal{G}^l + \xi^l \mathcal{G}^l.$$

The Eqs. (69) are differentiated with respect to time:

$$-\dot{s}_1 = \eta^s \dot{\varepsilon} + \omega^s \dot{\theta} + \xi^s \dot{\mathcal{G}}^s + \xi^{ls} \dot{\mathcal{G}}^l, \quad (70)$$

$$-\dot{s}_2 = \eta^l \dot{\varepsilon} + \omega^l \dot{\theta} + \xi^{ls} \dot{\mathcal{G}}^l + \xi^l \dot{\mathcal{G}}^l$$

As is known, $T \frac{ds}{dt}$ is the measure of the amount of heat generated per unit of the volume of the considered medium in the process of temperature change when the deformation of the consolidating medium is fixed (the rate of expansion of the skeleton and liquid is equal to zero). It is defined as the heat of the given component of the medium at constant volume c_v . Because η^s, η^l as well as ω^s and ω^l are negative, meaning $\xi^s < 0, \xi^l < 0$ and $\xi^{ls} < 0, a_3 < 0$ and equals:

$$\xi^s = -\frac{\rho_1 c_v^s}{T_0}, \quad \xi^l = -\frac{\rho_2 c_v^l}{T_0}, \quad \xi^{ls} = -\frac{\rho c_v^{ls}}{T_0}, \quad (71)$$

where c_v^s, c_v^l show specific heat at a constant volume of the skeleton fluid and c_v^{ls} is the specific heat of the thermal coupling of phases.

The entropy Eq. (70) can be written as:

$$-T\dot{s}_1 = T\eta^s \dot{\varepsilon} + T\omega^s \dot{\theta} + T\xi^s \dot{\mathcal{G}}^s + T\xi^{ls} \dot{\mathcal{G}}^l, \quad (72)$$

$$-T\dot{s}_2 = T\eta^l \dot{\varepsilon} + T\omega^l \dot{\theta} + T\xi^{ls} \dot{\mathcal{G}}^l + T\xi^l \dot{\mathcal{G}}^l$$

Taking into account relations (40) and (41), and using the Fourier's law of heat transfer for both phases of the medium (43), the heat transfer equation in the formula is obtained:

$$\lambda^s \nabla^2 \mathcal{G}^s = -T\eta^l \dot{\varepsilon} - T\omega^s \dot{\theta} + \frac{T^s}{T_0} (\rho_1 c_v^s \dot{\mathcal{G}}^s + \rho c_v^{ls} \dot{\mathcal{G}}^l), \quad (73)$$

$$\lambda^l \nabla^2 \mathcal{G}^l = -T\eta^l \dot{\varepsilon} - T\omega^l \dot{\theta} + \frac{T^l}{T_0} (\rho c_v^{ls} \dot{\mathcal{G}}^l + \rho_2 c_v^l \dot{\mathcal{G}}^l)$$

where λ^s is the coefficient of heat transfer of the two-phase medium, and λ^l is the coefficient of the heat transfer of the fluid.

When the pore fluid flows through the two-phase medium, the constitutive relationships can be expressed as below:

$$\sigma_{ii} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon + Q\theta + \eta^s \mathcal{G}^s + \eta^l \mathcal{G}^l) \delta_{ij}, \quad (74)$$

$$\sigma - \sigma_a = Q\varepsilon + R\theta + \omega^s \mathcal{G}^s + \omega^l \mathcal{G}^l.$$

When gas flows through the medium, the constitutive relations take on a non-linear form:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + (\lambda\varepsilon + Q\theta + \eta^s \mathcal{G}^s + \eta^l \mathcal{G}^l) \delta_{ij}, \quad (75)$$

$$\sigma - \sigma_a = Q\varepsilon - \frac{\sigma_a}{1+\theta} \theta + \omega^s \mathcal{G}^s + \omega^l \mathcal{G}^l.$$

The difference is that, in the case of the liquid, the modulus for the strain on the volume is described by the constant R , while, in the case of the gas, there is a non-linear strain of gas equal to $-\sigma_a/(1+\theta)$. The difference is also seen in the definition of the constant ω^l , which in the case of the gas takes the form: $-\sigma_a\alpha^l/(1+\theta)$.

As a result of the lack of data related to effective parameters for the constitutive relationships given above and the heat transfer equations, for the simplification of the model it is necessary to state that heat propagates in the specific medium at the same speed for both phases of the two-phase medium, so:

$$\mathcal{G}^s = \mathcal{G}^l = \mathcal{G}. \quad (76)$$

Constitutive relations for such a case where the pores are filled with liquid can be simplified to:

$$\sigma_{ij} = 2N\varepsilon_{ij} + (A\varepsilon + Q\theta + \eta\mathcal{G})\delta_{ij}, \quad (77)$$

$$\sigma - \sigma_a = Q\varepsilon + R\theta + \omega\mathcal{G}.$$

The constant η is equal to the sum of the constants η^s and η^l and equals:

$$\eta = -(3K\alpha^s + Q\alpha^l). \quad (78)$$

While the constant ω is equal to the sum of the constants ω^s and ω^l and equals:

$$\omega = -(3Q\alpha^s + R\alpha^l). \quad (79)$$

In the case where pores are filled with gas, the following constitutive relation formulas are obtained:

$$\sigma_{ij} = 2N\varepsilon_{ij} + (A\varepsilon + Q\theta + \eta\mathcal{G})\delta_{ij}, \quad (80)$$

$$\sigma - \sigma_a = Q\varepsilon - \frac{\sigma_a}{1+\theta} \theta + \omega\mathcal{G}.$$

The constant α takes the same form as in the case of the fluid, while the constant ω takes the form:

$$\omega = -\left(3Q\alpha^s + \frac{\sigma_a}{1+\theta} \alpha^l\right). \quad (81)$$

The heat transfer equations lead to one equation:

$$\lambda \nabla^2 \mathcal{G} = -T\eta\dot{\varepsilon} - T\omega\dot{\theta} + \frac{T}{T_0} \rho c_v \dot{\mathcal{G}} \quad (82)$$

λ is the averaged coefficient of the two-phase medium, ρ describes the volume density of the medium, and c_v is the averaged specific heat for the analysis of the two-phase medium.

The physical relationships and the heat transfer equation, along with the interpretation of the effective parameters of the simplification model, are in accordance a work by Strzelecki et al. [22].

3. THE NUMERICAL MODEL

The numerical model was created for the two-phase medium using the obtained relationships. The geometry of the model is constituted by a core ground sample with a diameter of 40 mm and a height of 76 mm, made up of an elastic skeleton and a gas (water vapor). Below is presented a comparison of 3 models:

- A) model using one heat transfer Eq. (82),
- B) model using 2 Eqs. (73) of heat transfer without phase coupling – assuming $c_v^{ls} = 0$,
- C) model using 2 heat transfer Eqs. (73) with a full coupling of phases, where c_v^{ls} is calculated according to the thermal parameters of the media, assuming there is no deformation and there is the same temperature in both phases (assuming that the diffusion coefficients for both phases are equal):

$$\rho c_v^{ls} = \frac{\rho_1 c_v^s \lambda^l - \rho_2 c_v^l \lambda^s}{\lambda^s - \lambda^l}. \quad (83)$$

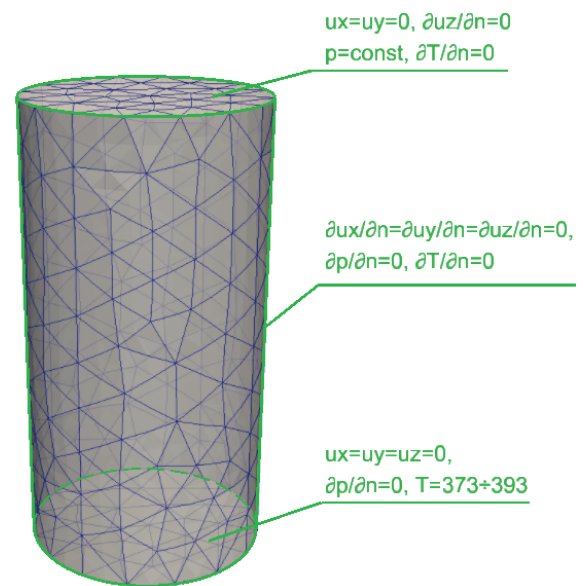


Fig. 1. The outer structure of the element and boundary conditions

The outer surface of the sample was assumed to be an impermeable adiabatic sheath, the initial temperature is 373 K, which then increases to 393 K. The boundary conditions are shown in Fig. 1.

The following parameters for the medium were garnered:

- Porosity: $n = 0.25$.
- Filtration coefficient (initial value) $k_0 = 10^{-8}$ m/s.
- Mechanical parameters of the medium:
 - Shear modulus (Lamé's constant μ) $N = 2.5 \cdot 10^8$ Pa.
 - Biot Modulus A (Lamé's constant λ) for a gas-filled medium $A_g = 5 \cdot 10^8$ Pa,
 - Coefficient determining the influence of stress placed upon gases on the deformation of the skeleton $Q_g = 3 \cdot 10^5$ Pa.

Parameters of the skeleton:

- Linear coefficient of thermal expansion: $\alpha_s = 5 \cdot 10^{-6}$ K.
- Specific heat: $C_v = 700$ J/kgK.
- Heat transfer coefficient: $\lambda_s = 3$ W/mK

Parameters of water vapor:

- Thermal expansion coefficient: $\alpha = 1/T$.
- Specific heat: $C_v = 700$ J/kgK.
- The viscosity and the heat transfer coefficient for gases, as described in Sutherland's formula

$$\mu = \mu_0 \frac{273 + C_s}{T + C_s} \left(\frac{T}{273} \right)^{1.5}, \quad (84)$$

$$\lambda = \lambda_0 \frac{273 + C_s}{T + C_s} \left(\frac{T}{273} \right)^{1.5}, \quad (85)$$

where $\mu_0 = 8.53 \cdot 10^{-6}$ Pa · s, $\lambda_0 = 16.2 \cdot 10^{-3}$ W/mK, $C_s = 673$.

The density of water vapor is described by the following formula, based on the Clapeyron formula:

$$\rho_g = \frac{p}{rT}, \quad (86)$$

where r is the individual gas constant.

- In the case of the model with one equation for temperature (A), the specific heat and heat transfer coefficient are calculated as weighted averages (with density and fraction of a given phase used as a weight, respectively)

The final modeling results, after time $t = 8000$ s for all models, are similar. The displacement field generated by the temperature gradient is shown in Fig. 2.

Differences in modeling result from the heat conduction model – as much as the differences between models A and C were minimal, for model B (without phase coupling), the fluid temperature was significantly different from the skeleton temperature for longer periods. The temperature waveform for the 3 models is shown in Fig. 3.

While analyzing the waveforms representing temperature, it can be observed that the line showing temperature change in the skeleton in the coupled model (C) differs only slightly from the corresponding line in the model without the coupling of phases (B) and also from the case in which the medium is treated as homogenous (A); however, significant changes are visible when observing the temperature course in gas. Moreover, the skeleton temperature in the model with the full coupling of phases (C) is also very close to

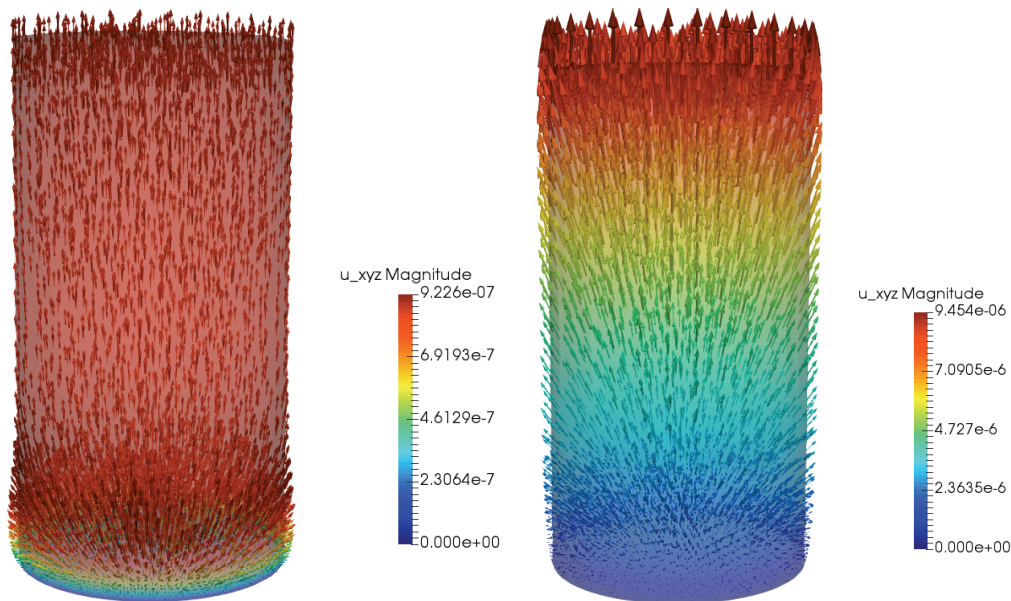


Fig. 2. Displacement: a) after 120 s, b) after 8000 s

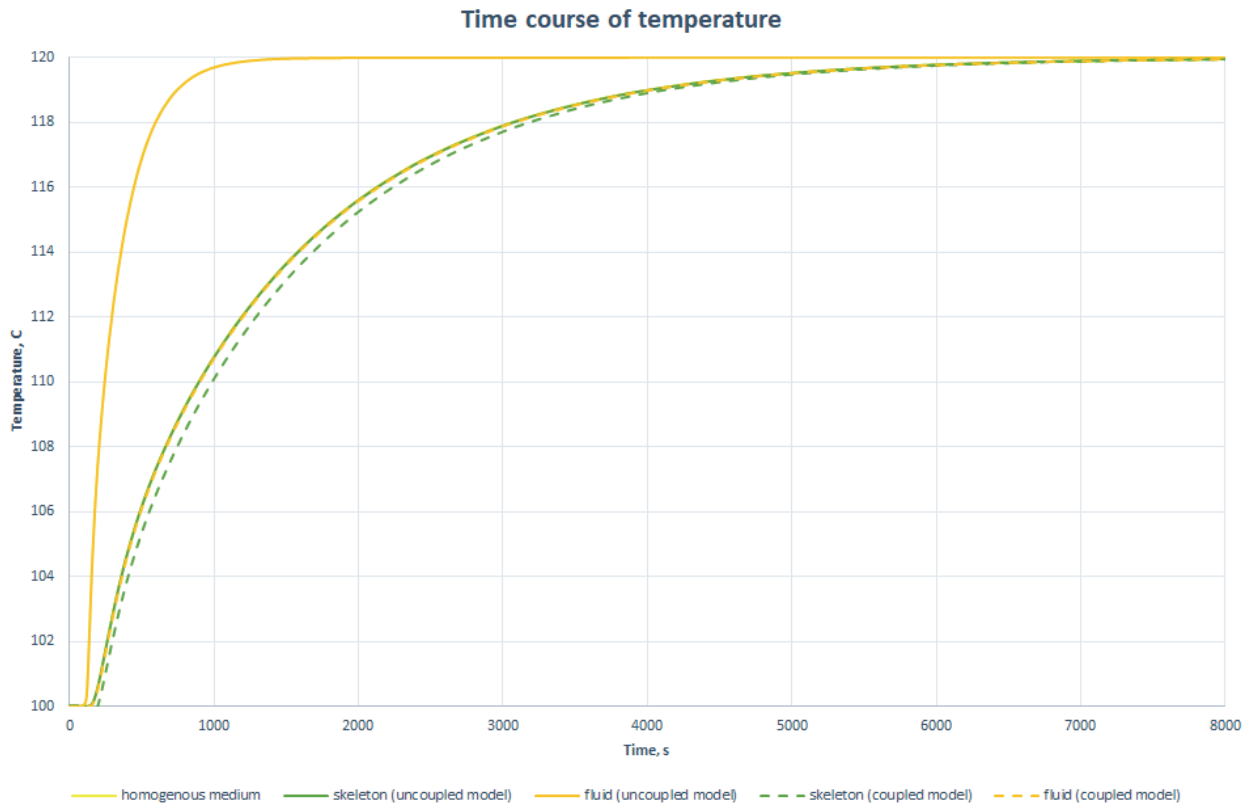


Fig. 3. Time course of temperature at the geometric center of the sample

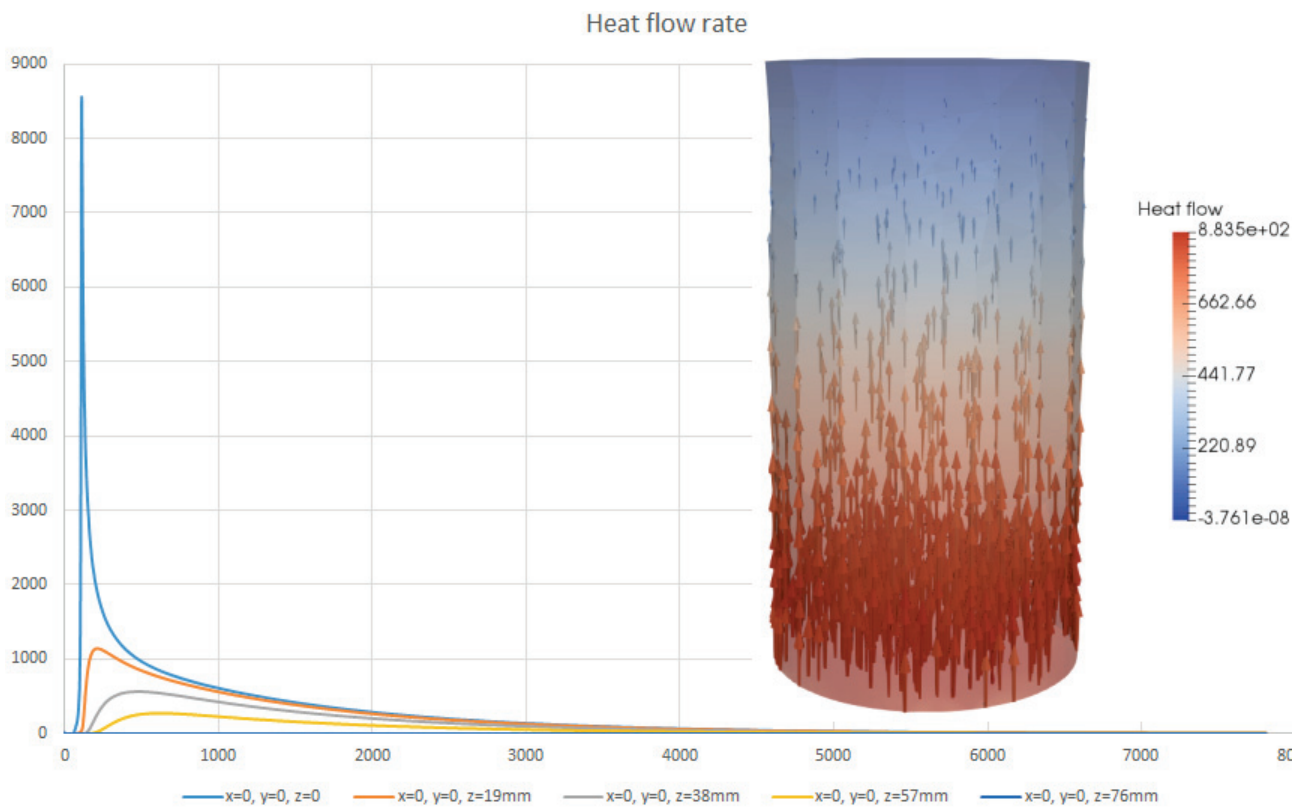


Fig. 4. Amount of heat flowing through the medium at points:
 a) 0,0,0; b) 0, 0, 19 mm; c) 0, 0, 38 mm; d) 0, 0, 57 mm; e) 0, 0, 76 mm

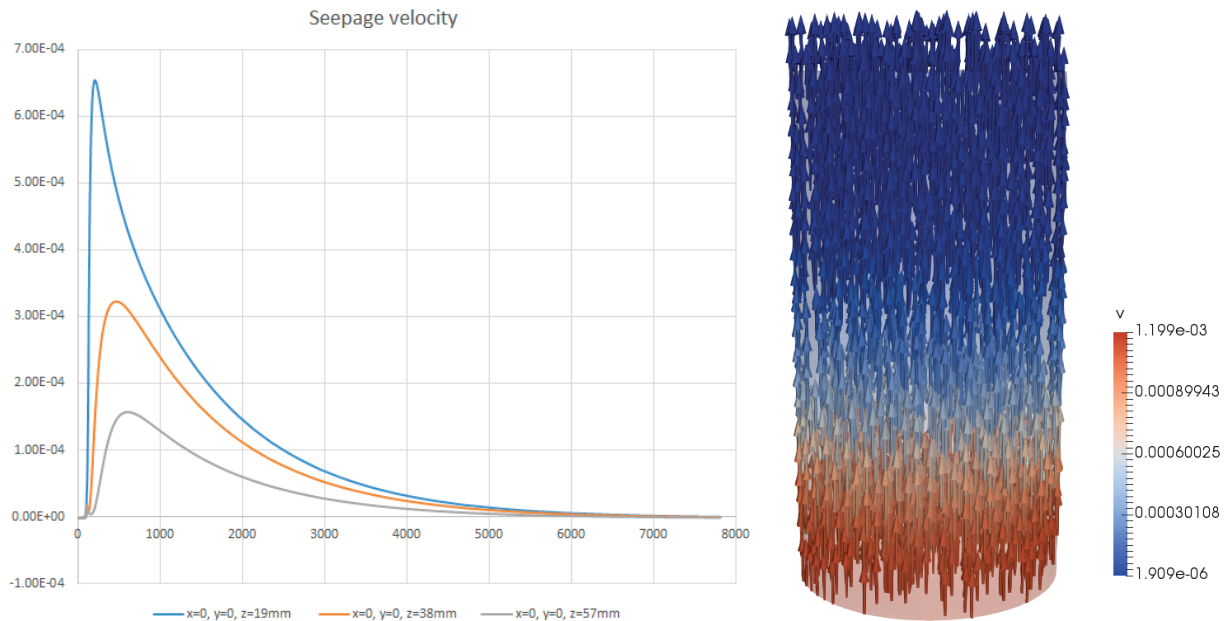


Fig. 5. Seepage velocity; to the left, at points: 0,0,19 mm; 0,0,38 mm; 0,0,57 mm; to the right, distribution of seepage velocity ($t = 180$ s)

that in the model assuming equal temperature in both phases of the medium (A), which confirms the dominant influence of the skeleton on the heat transfer process. Small temperature differences in the skeleton and the fluid in the model result from dilatation. The amount of heat flowing through the medium (model A) is shown in Fig. 4. For calculations using two heat transfer equations (B & C), the results are very close. Moreover, the amount of heat transported by the fluid is negligible compared to the amount of heat transported by the skeleton.

In the initial phase of heating the direction of seepage flow is reversed as a result of density differences and a temporary spike in pressure, and as far as the heat transfer process becomes fixed, the sign is changed and stabilized at a level of -2.04×10^{-6} m/s. The graphs of seepage velocity are shown in Fig. 5.

The graph illustrates the cases for simultaneous heat transfer in both phases (A), as well as for the coupling of both phases (C). In the absence of coupling (B), the fluid velocity drops faster, as a result of the lack of heat transfer to the skeleton. This is also associated with slower heating speed in the skeleton.

4. SUMMARY

Starting with the first and second laws of irreversible thermodynamics, the authors obtained constitutive relationships of the two-phase medium consisting of a porous elastic skeleton and a fluid filling

the pores in non-isothermal conditions, as well as heat flow equations. These were gained under the assumption that the process of heat flow in both phases of the medium occurs in parallel, and assuming that there is also a heat transfer between both phases of the medium. The resulting system of equations requires familiarity with the basic mechanical properties of the two-phase medium, known as Biot's bodies, as well as the thermal properties of the solid and liquid phases. The equations for the cases of a gas or liquid filling the pores of the medium are presented separately.

In the case in which the averaged heat flow through the porous medium is considered, the relations are significantly simplified. For various practical issues, such a model may be sufficient from the point of view of the precision required for practical applications.

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