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ON THE FORMALIZATION AND INTERPRETATION OF THE NOTION OF "EFFICIENCY" – FURTHER EXAMPLES AND COMMENTS

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Abstract. The article is devoted to revealing further kinds of the meaning of the term "efficiency" (and related notions), consequently setting them in mathematical frames. First of all the so called envelope-type efficiency is introduced. This notion is illustrated by several examples derived from the elementary topology, Bayesian statistics, mathematical economics and primer of financial engineering. It seems that the above examples do reflect the essence of this idea in the best possible picture. The next proposition concerns the type of efficiency which was called "collective-type efficiency". It turns out that the reasonable compromise is "better than the best solution" (even in the Nash sense). The quite good class of examples are provided by problems derived from the famous "prisoner's dilemma" and exploitation of common resources. At the end of the paper some complementing thoughts are indicated – in a loose form. They concern the principal conflict between efficiency and equity as well as the problem of economic behaviour in the sphere of scientific research (the balance between two factors: "erudite components" and "creative potentials" of the researchers).

Keywords: efficiency, envelope-type efficiency, collective-type efficiency, efficient frontier, equity, sustainability.

JEL Classification: C02, C60, C70.

1. Introduction

The paper is intended to be the second part of the set of reflections concerning the meaning and formalisation of "efficiency". The above mentioned considerations are collected in two articles – the other one is entitled *On the ways of formalization and interpretation of the notion "efficiency" – introductory remarks and some examples*. These articles which are simultaneously submitted to the same volume of *Mathematical Economics* are complementary. However, it should be noted that the present article makes in fact the third segment of a more extensive discussion, initiated several

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years earlier by the author in the paper On many-sidedness, relativity and complexity of the “efficiency” (as a category) (in Polish: *O wielostronności relatywizmie i złożoności kategorii efektywności*, (Rybicki, 2005a)). In the previous papers the various ways of the attack directed towards “discovering”, explanation and qualifying of the ideas of efficiency, effectiveness and economy (concerning objects and activities) were proposed (and conducted, from different perspectives). The philosophical point of view prevailed in the earliest work, whereas the more formalized (mathematical) style dominates in the two current papers. “Many a name efficiency has” with this, somewhat metaphorical, phrase began the first article of a two-part series of papers concerning the meaning and mathematical formalisations of ideas belonging to a range of such terms as efficiency, effectiveness and economy. The series was, in fact, initiated several years ago in the author’s article from 2005 (Rybicki, 2005a).

In the present paper (as well as in the accompanying, “twin” article (Rybicki, 2010a)) mathematical aspects of the question in the mind prevail (in contrast to the above mentioned “forewarding” work of a “philosophically-praxeological” character). As a consequence of taking a more formal point of view (and the treatment of problems) different forms of classification of “types of efficiency” appeared: basis-type efficiency, sup(inf)-type efficiency, information capacity and linear-similarity efficiency (of (pre)orders) and logical efficiency. Some general comments and representative examples are also given in the previous segment of the series.

In the current part of the diptich we continue to illustrate – by examples – the main idea contained in the first five words of introductions to both articles. In order to achieve the assumed aims, we introduce another way of meaning and interpreting of the key notion of considerations: the envelope-type and collective-type efficiency.

This version of the explanation and formalisation of the concept of efficiency seems to contribute the greatest portion of essential, intrinsic information about this notion and “illuminate the core” of the whole family of related conception including effectiveness and economy (of objects, sets or activities). Anticipating the detailed discussion of (selected) examples we briefly characterize some typical cases of the envelope-type efficiency. There is a set of objects, which themselves are sets (i.e. curves, surfaces, functions) and the same is true about their “envelope” or “optimal boundary”. So the envelope efficiency may be perceived as “set-setting of the sup(inf)-type efficiency” (discussed in the mentioned twin paper). The limiting element of the family of objects (sets) is itself the object of analogous structure ("properly closing" this family).

The notion of envelope-efficiency will be elaborated via several "representative" examples: the Bayes envelope functional (the concept from theory of statistical decision functions, introduced by Herbert Robbins (1955, 1964), the least concave utility function (representing given preferences, the concept of Gerard Debreu (1976), the efficient consumption path (in time). The subsequent theme comes from the game-theoretical modelling of conflictual situations appearing in the occasion of exploitation of common resources. We propose to identify the next "type of efficiency" which we call "collective-type efficiency". It turns out that a reasonable compromise (eventually solution forced by superior authorities or management) can be better than the optimal solution resulting from the classical antagonistic-games. We illustrate this conception by the famous "tragedy of commons" example (given by e.g. Hardin (1968)).

The last concept concerns (in a dynamical setting) the principal conflict between efficiency (effectiveness) and equity (justice). These dilemmas have been present in economics for years. For over half a century it has been elaborated in the context of the theory of economic growth and development as well as in the theory of intergenerational equity (as an autonomic, branch of economics and ethics concerning just, rational and nonmyopic intergenerational distribution of resources). We also mention the classical notion coming from the primer of financial engineering: the efficient frontier (of the financial market) originated from Markowitz efficient frontier (Markowitz, 1959).

At the end of the article we will shortly discuss – in a fictional convention – questions of interrelations between "eruditional and creative factors" during the processes of individuals' scientific activity.

2. The envelope-type efficiency

Let us start with the "mathematical object", which may play a role of a bridge connecting the "point-wise" notion of sup-efficient with its "set-wise" version, which we will call envelope efficiency. (The latter notion may be regarded as a generalization of the former one. But, at a bit higher level of abstraction, the converse opinion would be seen as better fitting the true, general logical scheme of a discussed case.) Formally we "only" substitute one-point efficient boundary by a set of points, which we regard as efficient boundary (of a given set) because of their maximality (minimality) property. Passing to details, consider space X equipped with partial order $\preceq$ (in many cases the reasoning can also function for preorders). For the set $B \subset X$ the symbol $\operatorname{Max} B$ (resp. $\operatorname{Min} B$) will denote the set of all maximal (resp. minimal) elements of B (with respect to the relation $\preceq$). So, $b \in \operatorname{Max} B$ if $b \in B$ and there are no elements in B "better" than b. In other words: there is no such $c \in B$ that $b \preceq c$ (different from b).

Now let us consider the linear preorder (preferences) on the plane, generated by square utility. For given two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we define a relation $x \preceq y$ by the following inequality

$$x \preceq y \Leftrightarrow x_1^2 + x_2^2 \le y_1^2 + y_2^2. \tag{1}$$

Let $a > 0$. Put

$$A = \left\{ x \in R^2 : \ x_1^2 + x_2^2 > a \right\}.$$

We can denote $\partial A = \left\{ x \in R^2 : \ x_1^2 + x_2^2 = a \right\}$. The set ∂A plays a role of "greatest lower bound" for primary set A. This is in fact a quarter of the circle, which "effectively" supports the set A. One can notice that ∂A is at the same time an indifference curve (more generally, surface) which is "under" the set A and is located as high as possible (it is "adjacent" to A). On the other hand, if one substitutes the strict inequality in definition of A by a weak inequality, then $\partial A = \min(A)$ (remember that $\preceq$ is a preorder!).

So we have entered, in the most natural way, the area of an envelope-type efficiency. Soon we will describe two important examples of such efficient boundaries – in a bit more detailed form. After discussing these examples we will mention other cases fitting – in our opinion – with questions in mind. Preceding the formal presentation we will comment (briefly) on the main ideas of the first of mentioned subjects.

The concept belongs to the field of the theory of statistical decision functions and was initiated by Herbert Robbins – in the early 1950's century (Robbins, 1955, 1964). The so called Bayesian Envelope Functional (abbrev. BEF) appeared in the context of Bayesian approach to statistics. H. Robbins introduced it in his seminal article presented at the Third Berkeley Symposium on Mathematical Statistics and Probability (1955) when proposing the empirical Bayes methodology for modelling (and solving) statistical problems. The BEF's concern are sets of *a priori* measures corresponding to a given statistical decision problem. It is a function mapping the space of all admissible a priori distributions (of unknown parameters) into the real line. Remember that priors reflect an objective available knowledge or subjective conviction of a decision-maker (economic agent, statistician) about the statistical parameter of the problem in mind (phenomenon, process, experiment). For each "reasonable" *a priori* measure, the BEF is defined as a value of its minimal Bayes risk.

Let us pass to a formal description of the construction of BEF, following, almost literally, the first two pages of Robbins paper (Robbins, 1964, pp. 1, 2). There are several "basic" objects given:

(a) a parameter space Θ (elements $\theta \in \Theta$ are called states of nature);

(b) an action space A (elements $a \in A$ represent possible decisions of statisticians);

(c) a loss function $L: A \times \Theta \to R_+$ (values $L(a,\theta)$ determine losses of statisticians undertaking actions $a \in A$ when the states of nature are $\theta \in \Theta$);

(d) an *a priori* probability distribution q_θ of parameter θ (θ plays a role of a somewhat artificial, "silent" random element – identity function $\theta(\theta) = \theta$ on Θ equipped with a proper σ-algebra S, such that θ is "governed" by q_θ);

(e) an observable random variable X taking values in a measurable space $(\boldsymbol{X}, \boldsymbol{B})$, on which a σ-finite measure μ is defined. We assume that distribution of variable X depends on the "actual" value of state of nature θ, and it has a density f_θ with respect (w.r.) to μ.

The problem is to define the "mechanism" assigning actions from A to the observed realizations of X. In other words we seek for the so called decision function $\delta: \boldsymbol{X} \to A$, such that when we observe $x \in \boldsymbol{X}$, we shall take action $a = \delta(x)$ and thereby incur the loss $L(\delta(x), \theta)$. Because of randomness of X, superpositions $\delta(X)$ also are random variables, as well as functions $L(\delta(X), \theta)$. Assume the existence (for each $\theta \in \Theta$) of the expected losses and denote such expectation

$$R(\delta, \theta) = \int_X L(\delta(x), \theta) f_\theta d\mu(x). \tag{2}$$

The above expression defines the risk corresponding to decision function δ, when we observe random variable X and the state of nature ("pure strategy of nature") is $\theta \in \Theta$. If the parameter θ itself is governed by distribution *a priori* q_θ (on (Θ, S)), than it is possible to consider subsequent expectation which is called the Bayes risk of δ relative to q_θ

$$r(\delta, q_\theta) = \int_{\Theta} R(\delta, \theta)\, dq_\theta(\theta)\,. \tag{3}$$

We can rewrite the last integral as follows

$$r(\delta, q_\theta) = \int_{\boldsymbol{X}} \Phi_q(\delta(x), x)\, d\mu(x) \tag{3'}$$

where we have set

$$\Phi_q(a, x) = \int_{\Theta} L(a, \theta)\, f_\theta(x)\, dq_\theta(\theta). \tag{4}$$

Now let us assume that there exists a decision function δ_q such that for almost every (with respect to measure μ), a.e. (μ) $x \in \boldsymbol{X}$

$$\Phi_q(\delta_q(x), x) = \min_{a \in A} \Phi_q(a, x). \tag{5}$$

Then for any decision function δ

$$r(\delta_q, q_\theta) = \int_{\boldsymbol{X}} \min_{a \in A} \Phi_q(a, x)\, d\mu(x) \le r(\delta, q_\theta), \tag{6}$$

to that, defining

$$\text{(BEF)} \qquad r(q) = r(\delta_q, q_\theta) = \int_{\boldsymbol{X}} \Phi_q(\delta_q(x), x)\, d\mu(x) \tag{7}$$

we have

$$r(q) = \min_{\delta} r(\delta, q_\theta). \tag{8}$$

Any decision function δ_q satisfying condition (5) minimizes the Bayes risk relative to q_θ and is called a Bayes decision function relative to q_θ. The functional r defined by a relation (BEF) is called the Bayes Envelope Functional (for the problem described in the current point). When q_θ is known we can use q_θ and thereby incur the minimum possible Bayes risk $r(q)$.

Despite the fact that the above construction has been running over several steps, it seems to be logically clear and intuitively convincing. These steps follow in a quite natural sequence. The starting point is the strategic game (Θ, A, L) – the "ordinary" two-person game, see also others (Greń, 1972). Here we do not have any possibility of "helping ourselves" through gaining any additional information about state of nature (its strategy) from observations of behaviour of random variable depending on this state.

At the next step we admitted such a possibility: decisions of the second player (the statistician's acts) depend on values of random observables, whose distributions, in turn, are specified through actual (pure) strategies of nature or parameters $\theta \in \Theta$. The original game is substituted by a statistical game $(\boldsymbol{D}, \Theta, R)$, where the first new symbol $\boldsymbol{D}$ denotes the set of all decision functions, mapping the set $\boldsymbol{X}$ (of values of an "auxiliary" random variable X) into the set A (conceivable statistician's actions). The third element of the triple, risk R is defined as an expectation $E_f\left[L(\delta(x), \theta\right]$ (w.r. to distribution f_θ of the observed variable X). We may write $R = R(\delta, \theta)$ because R varies when $\delta \in \boldsymbol{D}$ and $\theta \in \Theta$. The final step of the construction comes to introducing mixed strategies, for the "first player" (nature) or *a priori* distributions (q_θ) on the space of parameters Θ. So we end with extended (on the side of the nature) statistical game $(\boldsymbol{D}, Q, r)$. The explanation of the second and third symbols of the triple is quite natural: Q stands for the space of probability measures on (Θ, S) ("priors" on Θ), while r denotes a subsequent expectation

$$r = E_q\left[R(\delta, \Theta)\right] \quad \text{(w.r. to } \textit{a priori} \text{ distribution } q_\theta \text{ on } \Theta\text{).} \tag{9}$$

In such a manner we have in fact obtained the (real) function of two variables

$$r = r(\delta, \theta) = E_q\left[R(\delta, \theta)\right]; \quad \delta \in \boldsymbol{D},\ q \in Q. \tag{10}$$

In the relation (9) r is called a Bayesian risk of a precedure δ in presence of prior q (or Bayesian risk of a decision function δ with respect to the distribution q). Fixing δ_q as a proper ("opitmal") Bayesian decision function (for varying measures $q \in Q$) leads to a dependence r from the sole argument r: $r = r(q)$. So the "final product" of the above construction obviously fulfils postulates required on envelope-type efficiency: at any prior $q \in Q$ the value of BEF functional $r(q)$ "sticks itself" from above at possibly lowest level, to the "vertical segment" of all Bayes risks, corresponding to the q.

Let us pass to the second example illustrating the discussed form of efficiency. We are going to present in more detail the concept of the least concave utility function among the family of utility representations of given preferences (this notion was introduced and elaborated by G. Debreu (1976)). To this aim we will quote some introductory fragments of the cited paper (Debreu, 1976, pp. 243, 244), see also others (Arrow, 1970; Pratt, 1964; Kihlstrom, Mirman, 1974). Some small modifications of the original article, which will appear further on, are intended to shorten and simplify the presentation (as well as to avoid "photographic" quotations from Debreu). They result in some losses of the formal rigour of consideration (but still keep their mathematical character).

Let us begin with introducing the real valued representation of preferences over the convex subset X of a given real topological vector space *E*. So let $\preceq$ be a linear preorder on X. The function $u: X \to R$ is said to represent this preorder, if the following equivalence holds

$$x \preceq y \Leftrightarrow u(x) \leq u(y); \quad x, y \in X. \tag{11}$$

It should be pointed out that the existence of the representation of preferences makes up itself some kind of efficiency. Preferences in general "Pareto-style" setting, constitute undoubtedly the most complete description of relations among objects in mind (especially multidimensional or even more abstract ones). But on the other hand such complete characterization may be seen as an inconvenient tool for comparison tasks, too "misty" and a little "hidden" due to its complexity (somewhat paradoxically). A much more clear and convincing sound "one--dimensional indexes" – they are simply easier for operating and visualising when one wants to compare some objects. So if an equivalence (11) is true (or synthetic representation of the relative levels of objects, by numbers, is appropriate) then the "Cantorian-line" is the best scale for the orderings of these objects.

Let U denote the set of all continuous, concave, real-valued functions u, increasing w.r. to preorder $\preceq$ (according to condition (11)). Such a function $u \in U$ will be called from now on, the "utility function" (u.f.). The important preorder may be introduced in the set U. For $u, v \in U$ we will say that v is more concave than u, if there is a real-valued, concave function f on the set $u(X)$ such that $v = f \circ u$. In words: v can be obtained from u with the help of superposition u with some – concave function f (as an outer factor). The correctness of the above definition is proved in the cited paper. G. Debreu formulated there and proved the following crucial theorem.

Theorem (Debreu, 1976).

If $\boldsymbol{U}$ *is not empty, then* U *has the least element.*

Now we will briefly comment on the result (the author himself placed some remarks, we will outline them). The first corollary from the theorem is a statement that if a preference order is representable by a continuous concave, real-valued utility function, then the least concave utility representing the preorder is yet another instance of a cardinal utility (the above observation provides very significant argumentation in favour of the existence and fairness of the cardinal utilities, as helpful tools in ranking procedures). Remember that cardinal utility is defined as a class of functions related mutually by positive affine transformations

$$u_2 = a \cdot u_1 + b; \quad a, b \in R, \quad a > 0. \qquad (12)$$

Equally important seem to be the next two corollaries arising from the main theorem (also noticed by Debreu). These remarks connect the topic on concavity of utility representations of preferences with the theory of risk aversion founded by K. Arrow and J. Pratt and then generalized by E. Kilhstrom and L. Mirmann (among others, see Arrow, 1970; Pratt, 1964; Kilhstrom, Mirman, 1974). It was conducted in an expected utility setting, coming back to the J. von Neumann and O. Morgenstern (Neumann, Morgenstern, 1944) as well as I. Herstein and J. Milnor (Herstein, Milnor, 1953); see also (Blackwell, Girshick, 1954).

Let us remember that for a set $\boldsymbol{P}$ of probability measures on an open, convex subset X of (commodity vectors) space E the integral representation of preorder (on $\boldsymbol{P}$) is possible (assumming some technical requirements): for each pair $p, q \in \boldsymbol{P}$

$$p \preceq q \iff \int_X u(x) p(dx) \leq \int_X u(x) q(dx), \qquad (13)$$

where u is properly adjusted cardinal utility on X (which means that u is continuous, bounded, increasing real function on X such that both the integrals in (13) exist). The expected utility functional on the right hand of the above inequality makes the ordinal preferences representation of relation on the left hand in (13). Its restriction to the one point-mass probabilities (deterministic distributions) "reconstructs" and, at the same time, defines the utility function u on a set (of points) X.

In this framework the attitude (of a subject) to risk, called "risk aversion" is reflected (and, even defined) as a concavity of "its" (cardinal) utility. It follows from the discussed theorem that there is a least concave kernel of the above integral representation of such preferences, which is carried out by the mentioned superposition of a given utility u with some concave transformation f

$$v = f \circ u . \tag{14}$$

The highly useful (and unexpected) conclusion became possible after careful analysis of the interpretation of the expected utility representation in the light of equality (14): "in this situation one can separate the preferences of the decision-maker for the commodity vectors in X represented by u, from his attitude toward risk described by the strictly increasing continuous concave function f from $u(X)$ to R" (Debreu, 1976). It should be pointed out that the above observation anticipated, in a sense, by several years, the latter attempt in this area (Yaari, 1987; Quiggin, 1990).

3. The collective-type efficiency

The subsequent form of efficiency which plays an important role in mathematical modelling in economics (especially the game theoretical models) is the so called collective-type efficiency (mentioned in Introduction). While analysing and solving *n*-person games, one can encounter some subtle points and difficulties with deciding about "proper" (optimal and, at the same time, "just", satisfying all participants) decisions. The ambiguity of the solution choice provides additional flavour to the problems in question. The area in which such dilemmas appear still increases. So they require urgent solutions in the contemporary world and consequently engage numerous branches of science (a significant role is played, in this context, by mathematical models proposed for the field of sustainable development). There appear conflicts between individual rationality and the common interest, creating the new criteria of rationality and efficiency while exploiting mutual resources. It turns out (somewhat unexpectedly and doubtfully realistic) that the most satisfactory, for all involved subjects as a whole – not egoistically, is to agree to determine (applying the proper mathematical mechanisms) to some compromise distributional politics (strategies). The classical benchmark for this sphere of investigations is the famous "tragedy of commons", for the first time mentioned by A. Hardin which, in turn, makes up the generalization of popular "prisoners dilemma". Let us proceed to referring the formal statement of the problem (and its solutions "in a wine" of Nash and Pareto), according to chapter 11 of the book (Malawski et al., 1997).

Each member of a group of five farmers has two cows. The whole herd may exploit the common pasture, which efficiency (potential of feedstaff, the volume or quantity of grass) is naturally limited by, say, the number 12. Each farmer can decide how many cows he will graze (for his remaining cows he has to buy the feed, which is a costly solution). It is also quite natural to assume that the efficiency of pasture diminishes, when the number of grazing cows increases – in a sense, proportionally to the number of these cows:

$$P_e = 12 - k \quad (k = 1, 2, ..., 10). \tag{15}$$

From the formal point of view the above conflictual situation may be regarded as a 5-person (antagonistic) game, in which every player has the same set of (pure) strategies $S_i = \{0, 1, 2\}, \ (i = 1, ..., 5)$ reflecting the numbers of cows he decided to graze. The payment function was defined by a formula

$$W_i(q_1, q_2, q_3, q_4, q_5) = q_i \left(12 - \sum_{j=1}^{5} q_j\right), \tag{16}$$

where q_i denotes the number of cows in the pasture belonging to the farmer i. It is easy to see that

(a) all the function W_i have an identical structure – the game is "symmetric" with respect to all players,

(b) the payment for player "i" depends only on two quantities: the number of his own cows and the sum of numbers of the other farmer's cows present on the pasture.

So the discussed problem of the exploitation of common resources can be, without loss of generality formulated as a two-person (antagonistic) game, when the first player is any chosen farmer (i) with his set of strategies $S = S_i; \ i = 1, 2, ..., 5$ and the role of the second player takes over the rest of the group, having at the disposal the set of strategies $O = \{0, 1, 2, ..., 8\}$ and payment function depends on two variables $W : S \times O \to R$

$$(s, o) \in S \times O, \quad (s, o) = \left(q_k, \sum_{j \neq k} q_j\right) \quad \text{(for some } k \text{ vector } (q_1, q_2, q_3, q_4, q_5)\text{)},$$

this dependence is in fact explained in formula (16).

Now we can present the above game in a normal form, using "normal" matrix of dimensions 3×9 (instead of the "terrible", unmanageable "cube" in R^5).

Number of other peoples' cows / Number of farmer's own cows	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	11	10	9	8	⑦	6	5	4	3
2	20	18	16	14	12	10	8	6	④

In this matrix-form game, the columns are labelled by strategies of player 2, rows are labelled by strategies of player 1 and the entry consists of payments corresponding to all possible choices of pairs from $S\times O$.

It could be easily observed that the optimal strategy (for the farmer) in maximin sense is the egoistic strategy – "graze both of your cows". This is the Nash strategy, the best response to each strategy of others, it guarantees the achieving of unique equilibrium – if applied by each of the farmers, independently of the others' behaviour. In addition it is a dominating strategy. But there exists "the better" strategy from a point of view the group as a whole (which, in addition, turns out evidently more attractive for each member of the group, separately)!

Let us observe that according to the unquestioned "individual rationality" the choice of strategy "2" (by each of **five** players) implies the gains for them are described by vector $(4,4,4,4,4)$, whereas if they had agreed previously (unanimously) each one to graze exactly one cow, then the vector of payoffs will be $(7,7,7,7,7)$. The last system of gains is undoubtedly more (mutually) advantageous than the former one. The way of explaining the above apparent paradox is coming back to the original statement of the problem and then to compare 5-tuples of payments. At that time one may immediately see that in the "new" space the vector of gains $(7,7,7,7,7)$ corresponding to the system of strategies $(1,1,1,1,1)$ dominates in a Pareto sense the "old" vector, corresponding to the Nash optimal, equilibrium-type choice of strategies $(2,2,2,2,2)$. The "only" remaining problem is to achieve agreement and not to break the stated rules. So the context of the exploitation of the common goods seems to provide the best illustration of an idea of collective-type efficiency. The alternative (clearly nondemocratic) way of solving such problems comes down to force subjects (by superior authorities) to accept the best solution.

4. Markowitz efficient frontier

This current (very short) point will be dedicated to some "classic" questions from financial engineering (strictly speaking, concerning the basic notion of the portfolio theory initiated by H. Markowitz in works from 1951 and 1959, see (Markowitz, 1959). The subject in mind is the efficient frontier (of a set of securities portfolios). Let us mention the "building blocks" of this commonly known construction. There is given the set of compositions of assets (random variables – risky assets and possible risk-free assets, i.e. bonds). The objects at the decision subject's disposal are expected returns and expected risks of above portfolios calculated from known distributions of variables or estimated. So one may construct the two-dimensional set of admissible effects of chosen strategies (or proportions of components of a built portfolios) in a sense of mathematical expectations. The efficient frontier (EF) for the above problem is defined as a set of such combinations of securities portfolios that maximize an expected return for any level of expected risk (the classic mean-variance approach). Equivalently, one may define EF (in a dual manner) as a collection of such (convex) combinations of admissible securities, which minimizes the expected risk for any level of expected return. When a graphical representation of EF is presented, its traditional name for the obtained curve is the "Markowitz Efficient Frontier" (MEF). As we will soon see, the picture indicates the line built of pairs (of real numbers), the components of which represent respectively x – level of risk and y – the highest level of return corresponding to this risk (according to Markowitz for each level of risk there exists the unique combination of assets giving the highest expected return). The graph MEF shows the above described mechanics in a quite simple way, the optimal portfolios plotted along the curve have the highest expected return possible for the given amount of risk.

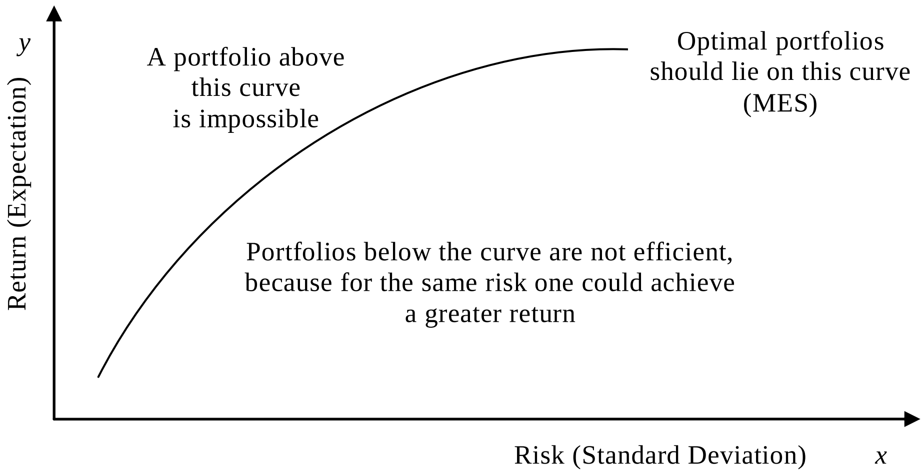

Fig. 1. MEF-graph

Source: http//financial-dictionary.thefreedictionary.com/Efficient+Frontier.

So immediately from the definition (merely!) one can realize that EF makes the object perfectly fitted to conception of envelope-type efficiency (it is no surprise because the Pareto-like "ideology" of construction of this "optimal line").

5. Efficiency, equity and sustainability

The current fragment of the essay will be devoted, according to the announcement made in the Introduction, to a short reflection on some aspects of "efficiency-equity" relations. We will also mention the role the notion of "efficiency" plays in models of economic growth (especially in the neoclassical growth theory framework) as well as in the more general considerations of a sustainable development. Interactions between important (praxeological, social and economic) principles, equity and efficiency, are complicated. The above notions (determining aims of activities and determined by these aims, according to some primary – ethical and praxeological – principles) have met and interlaced from the beginning of economic thought. Generally these requirements remain in the opposition, but the kind ("shade"), strength ("grade") of this relation depends on the mentioned primary assumptions which in turn greatly influence the formal frames and recognized, "time-honoured models". Researchers have coped with these problems for years and they have not achieved agreement in the subject.

As was mentioned, a crucial role is played by definitions or rather formal conventions enabling quantification of "concurring ideas". As long as a modest consensus has been achieved concerning "efficiency" as such (at least in the field of the economic growth, where formulations in the spirit of Pareto optimality are commonly accepted), the question of "proper and convincing" understanding and formalization of the idea of equity (or justice) remains open despite the fact that a variety of proposals were given since more than the last century. Some of these questions were drawn in the paper (Rybicki, 2010a). We are not going to discuss here the above problems extensively, deciding merely to notice selected facts, notions and conceptions according to their connections with and significance for the main subject of the essay, efficiency.

First of all the basic, primary principle should be mentioned: it is an ethical postulate of equal treatment of all generations. Its "static" version demands the so called intragenerational justice. Independently of their "dynamic" or "static" interpretation, the problems of mathematical inconsistency appear when one attempts to be simultaneous achieving both goals. So the important dilemma appeared (and was elaborated, see (Tadenuma, 2002)) "efficiency first or equity first"?, as two principles and rationality of social choice (by the way, the above quoted author proved that the order "equity, efficiency" should be recommended as "less controversial" from the mathematical point of view).

Before passing to a discussion of the efficiency of economic growth and development, an important question should be asked: how intergenerational equity can be operationalized? At this point several notions and mathematical properties might be mentioned: anonymity, impartiability, permutational (finite or infinite) invarianceness, time preferences, impatience, patience, dictatorship of present or future, altruistic (egoistic) treatment of other generations, non envy property, dynastic (paternalistic or non-paternalistic) altruism. The above encountered terms and notions belong to the vocabulary of theory of intergenerational equity and also appear in the sustainability considerations (see among others Tadenuma, 2002; Banerjee, 2006; Koopmans, 1960). They describe ways of evaluations and comparisons of infinite streams of economic objects.

Coming back to the question of interactions "equity-efficiency" in the intergenerational perspective, remember the main idea: sustainability requires (and at the same time implies) that all goods (in the widest possible meaning) have to be shared with future generations. So it leads to the notion of sustainable paths, which should be confronted with standard optimal solutions as described in the traditional theory of economic growth (Asheim et al., 2001). We have to reconcile two “kinds of oppositions”: “the effectiveness versus “hindering” ethical imperative of justice and, at the same time, to combine satisfactory solutions for intergenerational conflicts. Generally so called “impossibility results” prevail (see among others Yoshihara, 2007), however some positive (compromise) propositions (partial solutions) appear too. G. Asheim (Asheim et al., 2001) proposed to restrict the feasible class of technologies (in the neo-classical growth framework to the so called relevant class of technologies and then looked (directly) at the possibility of having intergenerational preferences that are effective, in the sense of having a non-empty set of maximal (Pareto) paths. He introduced that notion of (intergenerational) ethical preferences and developed a justification for sustainability by showing that this axiom of equal treatment (equity) combined with the strong Pareto axiom (efficiency) is sufficient to rule out “wrong”, non-sustainable paths.

Strictly connected with problems of a just and effective treatment of infinite streams of quantities (consumption, investment, returns, utilities) is the “eternal dispute on discounting”. The arguments of the ethical character intertwine with those reflecting economic principles as well as strictly mathematical requirements. Restricting the historical perspective to the 20th century solely, the discussions trace back from E. von Böhm-Bawerk, F. Ramsey (Ramsey, 1928) and J. Fisher (Fisher, 1930). The basic disagreement concerned the justifying of the use the non-zero discount rate in the classical, utilitarian-type, weighted additive formula

$$U(x)=\sum_{t=0}^{\infty}(1+\rho)^{-t}u(x_t)\,. \tag{17}$$

At is well known, the above formula expresses the way of evaluation of the utility of stream of economic commodities (say, consumption $x=(x_0,x_1,...))$, an infinite time horizon, where function $u(x_t)$ denotes instantaneous utility, obeying standard conditions. At the same time it enables to measure and compare the efficiency of various sequences (of goods, investment strategies, paths of economic growth). Such problems of evaluations (also for continuous time) remain valid in contemporary theories of growth and development, just in order to search for efficient politics and ultimate (efficient) paths of growth.

The breakthrough role of the work of T. Koopmans (1960) should be mentioned. He “redirected” the “debate of Giants” (Ramsey and Fisher) – to the modern fashion (general preorders and their real representation), still remaining the essence of investigations, characterization of "true" efficient economic growth. The relations joining equity, efficiency and discounting are considered by contemporary researchers too, especially these ones engaged in the economics of sustainability (Manne, 1999; Żylicz, 2004; Asheim et al., 2001).

It is also worth to notice the negative dependence linking the social ("too large") inequalities mainly in the spheres of wealth, income, and education with the possibilities to accomplish effective economic growth (Woźniak, 2003; Piotrowska, 2009).

At the end of the paragraph we will give two "samples" of formalizations practiced in modelling problems of the economic growth and the sustainable ("just") development. Following E. Panek (1997) watch, for a moment (superficially), the building blocks of non-stationary, multisector growth model of the Leontief–Gale type. The author proceeds in a standard manner. He begins with defining the technologically feasible production

$$p(t) = \left(k(t), z(t), x(t)\right); \quad t \in \{0, 1, ..., T\}. \tag{18}$$

The vector function (18) maps „time" to positive orthant in $(2n+1)$ dimensional Euclidean spare. (k – process of capital, z – process of employment, x – process of production in n sectors).

At the next step eight conditions (for the process $p(t)$) are formulated and theorem (theorem 6.1, p. 164) proved, which characterize introduced process. At the occasion additional quantities (and notions) are defined such as investment endowment process and consumption process ($i(t)$, $c(t)$ – vector processes in R_+^n). Finally the initial stock of capital is taken (vector $k_0 \in R^n$), and (k_0, T) – feasible growth process is defined. It is a vector process, determined as a four-tuple

$$\left(i(t), k(t), x(t), c(t)\right), \quad t \in \{0, 1, ..., T\} \tag{19}$$

fulfilling earlier formulated conditions. The crucial definition follows, where the Leontief matrix (denoted as $E - A$ for a given input matrix A) appears. Below we will quote this definition, because it's instructive, Pareto-like character, but, first of all for its envelope-type efficiency (in the meaning of the present article).

Definition (Definition 6.2 (Panek, 1997, p. 171)).

(k_0, T) *– feasible growth process* $\left(\bar{i}(t), \bar{k}(t), \bar{x}(t), \bar{c}(t)\right)$, $t \in \{0, 1, ..., T\}$ *is called* (k_0, T)*-efficient growth process, if there does not exist other* (k_0, T) *– feasible process* $\left(\tilde{i}(t), \tilde{k}(t), \tilde{x}(t), \tilde{c}(t)\right)$, $t \in \{0, 1, ..., T\}$ *for which*

(I) $\forall t \in \{0, ..., T\}$ $(E-A)\left(\tilde{x}(t) - \bar{x}(t)\right) \geq 0 \wedge \tilde{i}(t) - \bar{i}(t) \leq 0$,

(II) $\exists t' \in \{0, ..., T\}$ $(E-A)\bar{x}(t') > 0 \Rightarrow (E-A)\left(\tilde{x}(t') - \bar{x}(t')\right) > 0$

(inequalities in (I), (II) can be meant coordinatewise).

In the space of paths (of the (k_0, T) – feasible processes) the (k_0, T)-efficient path fulfil the typical strong – Pareto conditions of maximality (optimality, envelope-efficiency, according to proposed terminology).

Finally a remark that there cannot be forgotten a kind of efficiency (weak, strong) of all turnpike-type trajectories (Samuelson, 1960; McKenzie, 1976).

K. Hellwig and G. Speckbacher (Hellwig, Speckbacher, 1993) considered problems of just intergenerational resource sharing. They proposed an axiomatic approach based on two principles: efficiency and sustainability. Below we remember their reasoning contained in a short introductory fragment, in which the general frames of model are specified and ideas of "simultaneous demands for efficiency and sustainability" presented (Hellwig, Speckbacher, 1993, pp. 223-224).

The authors model an economy in which a single (productive) good can be either consumed or saved. They introduce a very simple (anyway, sufficient to bear and demonstrate assumed aims and satisfactory solutions) dynamical model, containing such quantities as stocks of good at the beginnings of successive time periods (K_t), sizes of subsequent populations (N_t – given exogenously), quantities of units of good consumed by them (C_t) and residual, quantities of savings $(S_t = K_t - C_t)$. The assumption of productivity of good can be described in a standard (general) manner $K_{t+1} = f(S_t, N_t)$.

They posed the problem of finding an intertemporal allocation of consumption, regarding the sequence $C_1, C_2, ...$ to satisfy the following principles.

E. Efficiency: "The consumption path should be efficient in the sense that it is not possible to increase consumption during any period without reducing consumption during at least one other period".

S. Sustainability: "During every period, resources should be consumed in a manner consistent with the maintenance of future consumption possibilities".

We will only add that the authors formulate and proved theorem which assures (under mild analytical regularity conditions on the function f) existence and uniqueness of (property formalized) sustainable consumption path. So efficiency and equity can coexist and interplay: they "meet in sustainability".

6. Final remarks and conclusions

In the end of the article we present – in a somewhat lighter tone – a supposition combining elements of psychology of creativity and economics (or rather praxeology). To this aim we introduce a notion "the eruditively – creative efficiency" (effectiveness). What does this curious term mean? We propose it to name the skill of finding the optimal ration ("balance") between the scientific conscience and unhindered, free processes of thinking. On one of the extremes we meet careless dilettantes, who bravely attack untouched territories of science without any necessary professional qualification (and, truly minimal, sense of decency). One can easily see that such efforts are inevitably inefficient and, what is more, even pointless (nevertheless such habits are too often observed in today's scientific practices). On the other hand, "too much knowledge packed in one's mind" may, paradoxically, weaken the "power and freshness" of intellectual creativity – simply, because of the restricting and narrowing of the researcher's horizons. And, consequently, it diminishes scientific efficiency (as well as effectiveness). As is commonly known, "academic official" benchmarking may negatively influence scientific discoveries (as well as genuine attempts in the world of arts).

The above loose observations reveal the reason for which scientists themselves should be keenly interested in the efficiency (as well as effectiveness and economy) of their investigations. They indicated a potential danger of losing the principal (definitional!) property of a scientific activity, when "going to the extremes" in a matter of going in to research work, its style, technicalities, philosophy. The old, universal Greek principle of a “golden mean” (*aurea medicritas*) seems to suit, apply and sound especially convincing in this context. So we should be careful in our own patch!

What else may be added, or rather stated as a résumé of considerations conducted through the two articles? First of all, various situations (cases) in which the notion of “efficiency” (or any related term) appears in a natural way, have been found and demonstrated. Secondly, the identification of basic “kinds of efficiency” was done, together with their differentiation and characterization with the help of a mathematical “arsenal of tools”. “The efficiencies” became selected, placed together and subordinated to adequate formal bencharks. It turned out necessary to depart from traditional “quotient-like” indices of efficiency (effectiveness), dominating in textbooks on the theory of the firm, management and calculations of rates of return.

It came down to reveal and “rediscover” some basic forms of “functioning” (formally and practically) of this category: the set-type efficiency, the point-type efficiency (including an important sort of the latter, somewhat more general: envelope-type efficiency) and so called collective-type efficiency. The last one appears as a result of the reflection concerning the rationale of remodelling the original antagonistic-game setting of some socio-economic problems to the cooperative formulations of these tasks. Such a modification bears fruit in increasing payments for all the participants of “the game”, which substantially improves the effectiveness of undertakings. The above aspects of efficiency become especially important in the light of constraints and limitations rapidly depleting more and more resources, necessary for the sustainability of mankind as a whole.

Strictly connected with the mentioned questions are problems of efficiency met at the domain of economic growth and development theories. In classic (neo-classic) growth models efficient strategies (of investment – saving – consumption spheres) “produce” efficient paths of growth (optimal, in a sense, and closely related to the envelope-efficiency idea). When some additional aspects, reflecting intergenerational equity and sustainability, are taken into consideration, then some “mixed, compromise” criteria become valid. Somewhat unexpectedly the intrinsic conflict between justice and efficiency diminishes, because the reasonable, non-myopic and complex approach to these questions leads to realizing that justice takes just what plays a decisive role as a factor guaranteeing sustainable development.

A relatively extensive bibliography has been included in the presented essays. It might be thought as unjustified and evidently too elaborated – but only from the purely mathematical point of view and “mathematical habits” (mathematicians quote papers and monographs when they effectively and "explicitly" make use of the cited technical topics). We aimed though at thoroughly demonstrating a variety of problems involving the idea of efficiency. On this occasion we will add some words about the convention of presentation. From a purely mathematical point of view the best decision would be to qualify all the discussed cases as optimality problems or, more subtly, problems of seeking maximal elements with respect to "proper, reasonable" preorders. The awareness of the above is however equally important as a conscience of the danger caused by the temptation to decide on such "mechanical abstraction", inevitably leading to the loss of possibility of "portraying" amply specific properties of discussed ("real" as well as "mathematical") problems.

The fact of existence of many situations involving efficiency considerations is not especially surprising, but the extent of the possible contexts (in which "efficiency" is present, though somewhat hidden) turn out unexpectedly large. So the Polish version of a title of the book telling "Mendelsohns' love story" might (after a bit modification) play a role of the "Leitmotif" for the written series of essays.

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