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THE USEFULNESS OF THE TEACHER FROM TAKING A CLASS, AND THE PRODUCTIVITY OF THE PROCESS OF EDUCATION

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Abstract. The author of the article is focusing on the explanation how productivities of the process of the education are being exchanged at the maximization of the so-called model of the profit of the education. This model was introduced and described by Bosworth and Caliendo (2007). The fact that it is taking into consideration the satisfaction of the teacher from lecturing a chosen group pupils is an innovative part of this model. The majority of the article is relying on six theorems presented by Bosworth and Caliendo and on proving two new theorems (one of which is an extension of the results received by the mentioned authors). Received results will be illustrated with proper simulations.

Keywords: production functions of the education, preferences of the teacher, optimum size of the class.

JEL Classification: A20, I20, I21.

1. Introduction

The idea of investment in human capital economically is a very important topic. The objective of this article is to demonstrate how the performance of the education process changes if the teacher maximizes the education profit functions. This function is based on the level of teaching and usability of a teacher from a conducted lesson[1]. The usefulness is understood here as teacher's satisfaction with work with a selected group of students. Spending more time with low-ability students has a negative impact on other students, because "they receive relatively less knowledge". The problem of introducing the teacher's satisfaction parameter into the model

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[1] Thorough explanations in section 2.

of education was presented by Bosworth and Caliendo (2007) in the article *Educational production and teacher preferences*.

The article provides a complete proof for the theorems from 1 to 4 presented by Bosworth and Caliendo, whereas in section 2.3 a new theorem concerning the fluctuation of the average level of education is presented, where the number of pupils is a declining linear function of the number of high-ability students. In the article there also appears a generalization of the theorem on the fluctuation of the average level of education evidenced by Bosworth and Caliendo in the case of the so-called constant ratio of gifted students to less able ones. Received theorems will be supported by relevant simulations which will show, among other findings, how the average level of education changes along with the changes of the number of high-ability students.

When discussing the segregation of pupils with respect to their intelligence it is advisable to refer to the publication of Lazear (2001). By segregating students according to intelligence (abilities), Lazear shows that the efficiency of the learning process increases (within the meaning of Lazear's model).

The article uses the notion of production function in education introduced into economics by Samuel Bowles (1969). This subject is now discussed by many researchers (Akerhielm, 1995; Hanushek, 1979, 1996, 2007; Kruger, 1999, 2003; or Betts, 1999).

The function of production is to be understood as the rate which measures the educational added value. In this case, it will be the knowledge of students conveyed by a teacher.

2. Educational profit model

The aim of the teacher is to optimize the function representing the total profit of classes depending on time devoted to selected students. For the sake of simplicity, the problem was brought down to dividing the students into two types according to their ability to acquire knowledge: high- and less-able students.

Signatures:

L – average share of time (during a lesson or lecture) spent with every low-ability student,

G – share of time spent on carrying out the lesson (lecture),

n – the number of low-ability students in the class,

m – the number of high-ability students in the class,

$E^h(G)$ – production function related to high-ability students,
$E^l(L)$ – production function related to low-ability students.

The assumptions are as follows. The functions E^l and E^h are strictly increasing, concave and $E^h > E^l$. With the above determinations the total profit from classes which expresses both the level of education (teaching) and teacher satisfaction from work can be expressed as $m\theta E^h(G) + n(1 - \theta)E^l(L)$. The equation is called the model (function) of profit of education, where $\theta \in (0, 1)$ denotes the teacher's preference (usefulness) parameter. By preference one should understand the usefulness of a teacher from teaching a specific group of students. Accordingly, the parameter θ represents also the teacher's satisfaction from work with a specific group of students. If the parameter θ is close to 0, the teacher receives high utility from teaching low-ability students. If θ is close to 1, the teacher has greater usability devoting time and effort to more gifted students.

Thus, the teacher's optimization of the function of educational profit could be written as:

$$\max_{L,G} : \ m\theta E^h(G) + n(1-\theta)E^l(L), \tag{1}$$

$$Ln + G = 1. \tag{2}$$

The equation (2) is a result of the assumptions that L and G are shares of the time devoted to each group of students. The problem of fulfilment of the conditions written with formulas (1) and (2) can therefore be reduced to local extremes of functions of many variables. Lagrange's function (see (Fichtenholz, 2005; Gewert, 2005)) for the conditions written with formulas (1) and (2) will be as follows:

$$\ell(L,G) = m\theta E^h(G) + n(1-\theta)E^l(L) - \lambda(-1 + nL + G). \tag{3}$$

Then

$$\frac{\partial \ell(L,G)}{\partial G} = m\theta \frac{\partial E^h(G)}{\partial G} - \lambda = 0, \tag{4}$$

$$\frac{\partial \ell(L,G)}{\partial L} = n(1-\theta)\frac{\partial E^l(L)}{\partial L} - \lambda n = 0, \tag{5}$$

$$\frac{\partial \ell(L,G)}{\partial \lambda} = -nL - G + 1. \tag{6}$$

The objective is to examine how both the time spent for the specific group of students and efficiency of the education system change (with maximized education profit model), depending on the number of more or less able pupils, as well as a parameter expressing the usefulness of the teacher. Reducing this idea into a mathematical equation is done by rewriting the following equation (4)-(6) with $G = G(n, m, \theta)$, $L = L(n, m, \theta)$ and $\lambda = \lambda(n, m, \theta)$. For better clarity, the article assumes that the subscript with a variable means the derivative with respect to the variable appearing in the index (similarly the double derivative), e.g.:

$$\frac{\partial l(L,G)}{\partial L} = l_L(L,G), \quad \frac{\partial^2 l(L,G)}{\partial L \partial G} = l_{LG}(L,G)$$

or

$$\frac{\partial^2 l(L,G)}{\partial L^2} = l_{LL}(L,G).$$

Then the relations from the formulas (4)-(6) can be denoted as follows:

$$m\theta E_G^h(G(n,m,\theta)) - \lambda(n,m,\theta) = 0, \tag{7}$$

$$(1-\theta)E_L^l(L(n,m,\theta)) - \lambda(n,m,\theta) = 0, \tag{8}$$

$$1 = nL(n,m,\theta) + G(n,m,\theta). \tag{9}$$

Especially from the formulas (7) and (8) it results that,

$$m\theta E_G^h(G(n,m,\theta)) = (1-\theta)E_L^l(L(n,m,\theta)). \tag{10}$$

The equations (7)-(9) set the qualities of the parameters L and G which maximize the function of education profit. Let k be one of the variables n, m, θ. Then successfully differentiating each of the equations (7)-(9) with respect to k we obtain:

$$(m\theta)_k E_G^h(G(n,m,\theta)) + m\theta E_{GG}^h(G(n,m,\theta))G_k(n,m,\theta) - \lambda_k(n,m,\theta) = 0, \tag{11}$$

$$(1-\theta)_k E_L^l(L(n,m,\theta)) + (1-\theta)E_{LL}^l(L(n,m,\theta))L_k(n,m,\theta) - \lambda_k(n,m,\theta) = 0, \tag{12}$$

$$0 = nL_k(n,m,\theta) + (n)_k L(n,m,\theta) + G_k(n,m,\theta), \tag{13}$$

or in an equivalent matrix notation

$$\underbrace{\begin{bmatrix} m\theta E^h_{GG}(G(n,m,\theta)) & 0 & -1 \\ 0 & (1-\theta)E^l_{LL}(L(n,m,\theta)) & -1 \\ -1 & -n & 0 \end{bmatrix}}_{J} * \underbrace{\begin{bmatrix} G_k(n,m,\theta) \\ L_k(n,m,\theta) \\ \lambda_k(n,m,\theta) \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} -(m\theta)_k E^h_G(G(n,m,\theta)) \\ -(1-\theta)_k E^l_L(L(n,m,\theta)) \\ (n)_k L(n,m,\theta) \end{bmatrix}}_{W}$$

The symbol J^i denotes the matrixes arising from the matrix J by replacing the i column of the matrix J with a column of free variables

$W\ (i \in \{1, 2, 3\})$.

Then, from Cramer's formulas $G_k(n, m, \theta) = |J^1|/|J|$, $L_k(n, m, \theta) = |J^2|/|J|$. The values of the determinants $|J^1|$, $|J^2|$, $|J|$ (easy to calculate) are:

$$|J| = -nm\theta E^h_{GG}(G(n,m,\theta)) - (1-\theta)E^l_{LL}(L(n,m,\theta)), \tag{14}$$

$$\begin{aligned} |J^1| = n[-(1-\theta)_k E^l_L(L(n,m,\theta)) + (m\theta)_k E^h_G(G(n,m,\theta))] \\ + (1-\theta)E^l_{LL}(L(n,m,\theta))(n)_k L(n,m,\theta), \end{aligned} \tag{15}$$

$$\begin{aligned} |J^2| = (1-\theta)_k E^l_L(L(n,m,\theta)) - (m\theta)_k E^h_G(G(n,m,\theta)) \\ + (n)_k L(n,m,\theta) m\theta E^h_{GG}(G(n,m,\theta)), \end{aligned} \tag{16}$$

the level of education of the whole classes (school) is the value of

$mE^h(G) + nE^l(L)$[2].

Then the average level of education is:

$$\mu = \mu(m,n,\theta) = \frac{mE^h(G) + nE^l(L)}{m+n}. \tag{17}$$

By differentiation of the equation (17) with respect to k we obtain

$$\mu_k = \frac{\left[(m_k E^h(G) + mE^h_G(G)G_k + n_k E^l(L) + nE^l_L(L)L_k)(m+n)\right] - \left[(mE^h(G) + nE^l(L))(m+n)_k\right]}{(m+n)^2} \tag{18}$$

From the formula (14) it results that $|J| > 0$. It is the effect of the assumption of the concavity of the function E^l and E^h ($E^l_{LL}(L(n,m,\theta)) < 0$, $E^h_{GG}(G(n,m,\theta)) < 0$).

[2] Production function on the level of the whole class.

2. Theorems on average level of education

Firstly, theorems on the influence of gifted students were put forward. For purposes of sections 2.1 to 2.3, $k = m$ was established[3]. It was assumed that the relation between the number of gifted students and low-ability ones is the following: $n = w - \alpha m$, where:

- w is a certain constant not negative,
- α a certain set constant, can be negative or positive.

After inserting $k = m$ into the formulas (14)-(16) we obtain:

$$L_m(n,m,\theta) = \frac{-\theta E_G^h(G) - \alpha L m \theta E_{GG}^h(G)}{|J|} \tag{19}$$

$$G_m(n,m,\theta) = \frac{n\theta E_G^h(G) - \alpha(1-\theta)E_{LL}^l(L)L}{|J|}. \tag{20}$$

Accordingly, when using the formula (18) and performing some not complicated transformations we obtain:

$$\mu_m = \frac{((1-\alpha)m + w)[E^h(G) + mE_G^h(G)G_m - \alpha E^l(L) + (w-\alpha m)E_L^l(L)L_m]}{[(1-\alpha)m + w]^2} + \frac{-(1-\alpha)[mE^h(G) + (w-\alpha m)E^l(L)]}{[(1-\alpha)m + w]^2}$$

$$\mu_m = \frac{w[E^h(G) - E^l(L)] + ((1-\alpha)m + w)[mE_G^h(G)G_m + (w-\alpha m)E_L^l(L)L_m]}{[(1-\alpha)m + w]^2}.$$

After using the relation (10),

$$\mu_m = \frac{w[E^h(G) - E^l(L)] + ((1-\alpha)m + w)E_G^h(G)[mG_m + L_m(w-\alpha m)m\theta/(1-\theta)]}{[(1-\alpha)m + w]^2}. \tag{21}$$

From the formula (13) $G_m = -nL_m(n, m, \theta) + \alpha L(n, m, \theta)$ or transforming the equation $L_m(n, m, \theta) = -[G_m - \alpha L(n, m, \theta)]/n$. Inserting these equations consecutively one after another into the formula (21) we obtain two equations expressing the value of μ_m

[3] The formulas presented in section 2 will be calculated with $k = m$. For greater clarity I will not mention each time that the corresponding variables are functions of parameters n, m and θ, e.g. $G = G(n, m, \theta)$.

$$\mu_m = \frac{w[E^h(G) - E^l(L)] + ((1-\alpha)m + w)mE_G^h(G)\left[\alpha L + (w - \alpha m)L_m \frac{-1+2\theta}{1-\theta}\right]}{\left[(1-\alpha)m + w\right]^2} \tag{22}$$

$$\mu_m = \frac{w[E^h(G) - E^l(L)] + ((1-\alpha)m + w)E_G^h(G)m\left[G_m \frac{1-2\theta}{1-\theta} + \alpha L\theta / (1-\theta)\right]}{\left[(1-\alpha)m + w\right]^2}. \tag{23}$$

The value of the expression $(-1 + 2\theta)/(1 - \theta) > 0$ for $\theta \in (0.5, 1)$ and $(-1 + 2\theta)/(1 - \theta) < 0$ for $\theta \in (0, 0.5)$, and for $\theta = 0.5$ $(-1 + 2\theta)/(1 - \theta) = 0$. Monotonicity and concavity of the corresponding production functions are equal $E_G^h(G) > 0$, $E_L^l(L) > 0$, $E_{GG}^h(G) < 0$ and $E_{LL}^l(L) < 0$. These comments will be helpful in interpreting the equations in the following three sections.

2.1. Fixed number of low-ability students

In the case $\alpha = 0$, after using the formulas (19) and (20) we obtained $L_m(n, m, \theta) < 0$ and $G_m(n, m, \theta) > 0$, respectively (taking into consideration the fact that under the assumptions made the particular derivatives are positive). This in turn, along with the assumption (22) and taking into account assumption $E^h(G) - E^l(L) > 0$ results in $\mu_m > 0$ for $\theta \in 0, 0.5\rangle$. For $\theta \in (0.5, 1)$ one cannot unequivocally determine the left side of the formula (22), because the equation will divide into two sections, one of which will be positive and the other negative. The described case constitutes exactly the theorems 1 and 2 from the article by Bosworth and Caliendo. The assumptions reflect the situation when the number of less able students is fixed at a constant level. Then the increase of m (in the number of gifted children) causes a decrease of the average time spent for each low-ability student and an increase of lecturing time. This result is in accordance with intuitive guesses.

The average level of education in this case grows when the teacher has greater utility in teaching the less talented students. At the same time the increase in number high-ability students does not have a specified influence on the average educational achievement, unless the teacher has greater utility from working with talented students.

2.2. Fixed ratio of low- to high-ability students

The case of $\alpha < 0$ and $w = 0$ constitutes the assumption of the model from the third and fourth proposition presented in the article of Bosworth and Caliendo (2007). This type of relation tells us that the ratio of low- to high ability students is constant and equals α. From the formula (19) it is visible that $L_m < 0$ as $-\theta E_G^h(G) - \alpha Lm\theta E_{GG}^h(G) < 0$. The figure G_m cannot be unambiguously stated, because from the relation (20) we see that $n\theta E_G^h(G)$ is positive, whereas $-\alpha(1-\theta)E_{LL}^l(L)L$ is negative. From the equality (22) for $\theta \in \langle 0.5, 1)$ results $\mu_m < 0$.

The results obtained show that, as in the previous point, an increase in gifted students causes a decrease of time spent with any low-ability student, while there is no clear effect on the time spent on lecturing. The average learning achievement decreases with the increasing number of talented students if the teacher has greater utility of teaching gifted students.

The above-mentioned relation $\mu_m < 0$ is not necessarily true for models $\alpha < 0$ and $w > 0$. Then in the formula (22) there remains a section

$$w[E^h(G) - E^l(L)],$$

which is positive.

2.3. Linear decrease of low-ability students

The case of $\alpha > 0$ and $w > 0$. This type of relationship will explain what happens if the increase in the number of gifted students in the class causes a linear decline in low-ability students. The formula (20) gives $G_m > 0$ and from (19) we cannot unanimously state the fluctuation of the figure L_m ($-\theta E_G^h(G)$ < 0 and $\alpha Lm\theta E_{GG}^h(G) < 0$). Having this in mind and the equation from the formula (23) it is visible that $\mu_m > 0$ for $\theta \in (0, 0.5\rangle$. This allows us to formulate the following theorem.

Theorem 1. *With the increase in the number of high-ability students (in the linear model of decrease of low-ability students) an increase in lecture time occurs. Also the average level of educational achievement increases if the teacher has greater utility from working with less talented students.*

The results obtained are very similar to the results presented in section 2.1, without condition $L_m < 0$.

2.4. Theorem on influence of teacher preferences

In this section it has been proved that Theorem 6 presented by Bosworth and Caliendo can be generalized. Bosworth and Caliendo present this theorem in the case of the model presented in section 2.2, the ratio of low- to high-ability students is constant and equals α.

Using the formula (18) with $k = \theta$ and corresponding equalities from chapter 1 we obtain

$$\mu_\theta = \frac{(mE_G^h(G)G_\theta + nE_L^l(L)L_\theta)(m+n)}{(m+n)^2} \overset{(10)}{=} E_L^l(L)\left[\frac{G_\theta(1-\theta)/\theta + nL_\theta}{(m+n)}\right]^{(13)}$$
$$= E_L^l(L)G_\theta\left[\frac{(1-\theta)/\theta - 1}{(m+n)}\right] = \frac{E_L^l(L)G_\theta}{n+m}\left[\frac{1-2\theta}{\theta}\right] \quad (24)$$

Also substituting $k = \theta$ into the formulas (15)-(16) we obtain

$$G_\theta(n,m,\theta) = \frac{n[E_L^l(L(n,m,\theta) + m\theta E_G^h(G(n,m,\theta))]}{|J|} > 0$$

$$L_\theta(n,m,\theta) = \frac{-E_L^l(L(n,m,\theta)) - \theta E_G^h(G(n,m,\theta))}{|J|} < 0\,.$$

Hence the dependence of the formula (24) is positive for and negative for $\theta > 0.5$. This allows to formulate the following theorem.

Theorem 2. *The average education achievement is an increasing function (of parameter θ) for* $\theta \in (0, 0.5)$ *and decreasing for* $\theta \in (0.5, 1)$. *Thus, the average education achievement reaches maximum, if the teacher finds equally the same satisfaction in teaching both high- and low-ability students.*

The proof uses the relation between *m* and *n*. In particular, the theorem is true when the relations are nonlinear.

3. Simulations

Below there are presented simulations of education profit functions (formula (1)) for the model from sub point 2.3. It has been assumed that $\lambda = 1, \alpha = 1,\ w = 30,\ E^h(G) = 2\sqrt{G}$ and $E^l(L) = \sqrt{L}$. The simulation is to draw the function $m\theta E^h(G) + n(1-\theta)E^l(L)$ for the set n, m, θ and at the change of G from 0 to 1 (formula (2) unambiguously determined value of parameter L). It has been presented what the changes of the average achievement level (calculated for the optimal points) are, and what is the average lecture time for each low-ability student (L), for the same functions. The analysis has been performed for various n, m, θ. In other words, the analysis has been made for the case when the number of students in a class is unchanged (equal to 30) and between the number of high- and low- ability students there occurs the relation $m = 30 - n$.

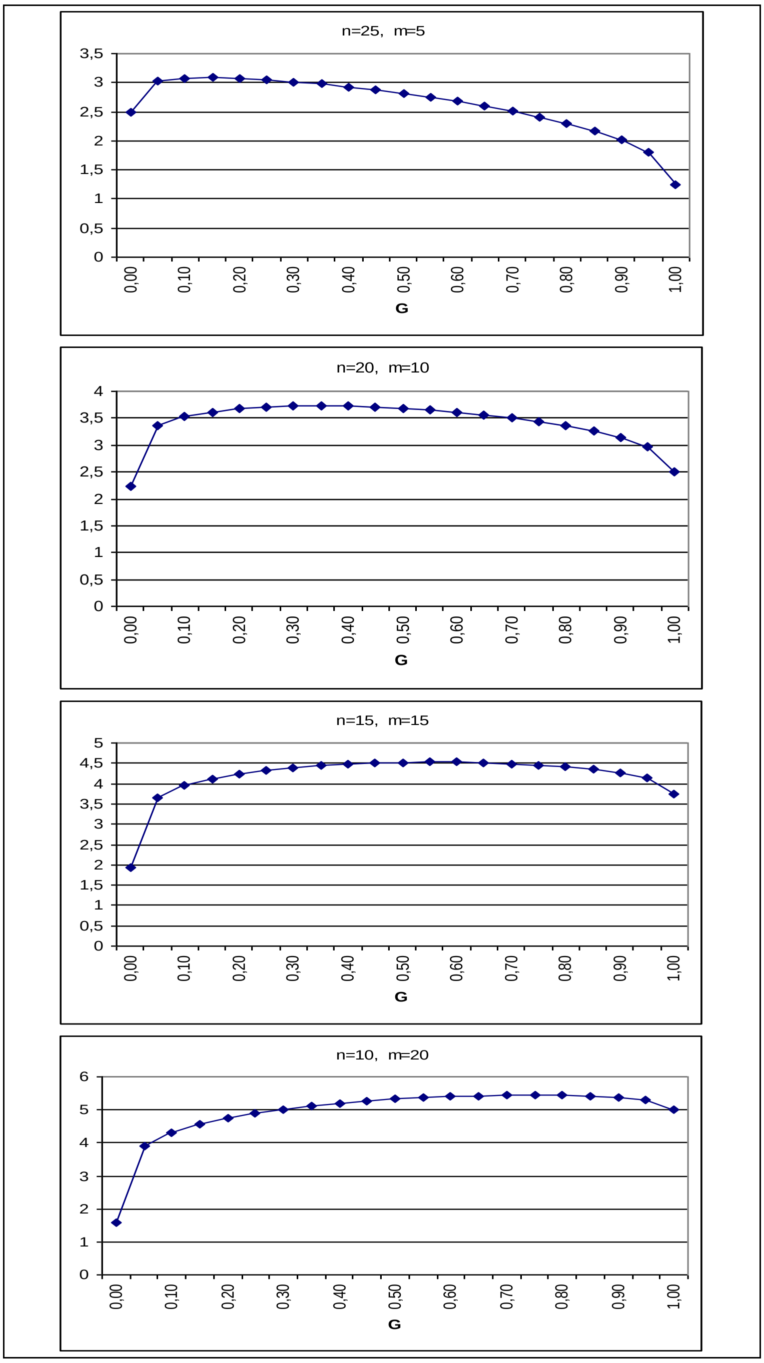

Fig. 1. Profit education function for $\theta = 0.5$ and various n, m

Source: own calculations.

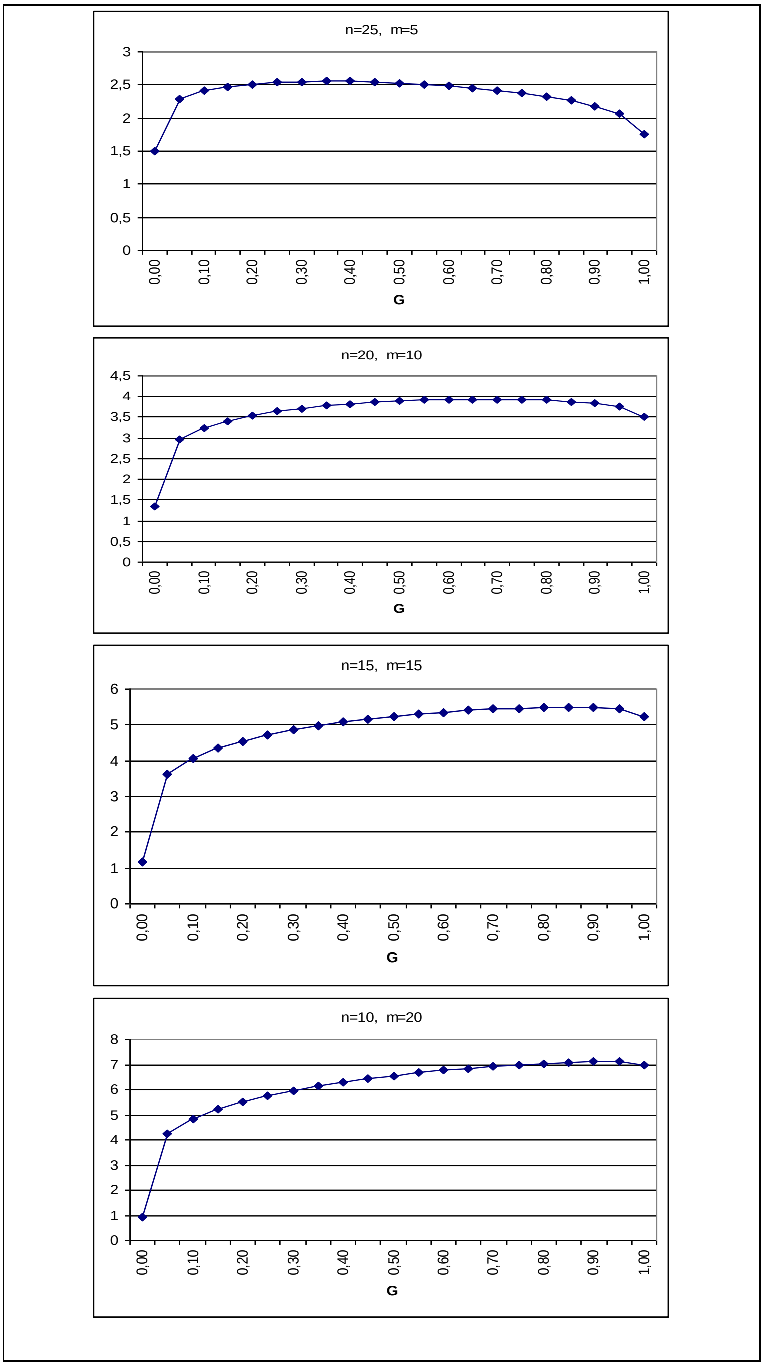

Fig. 2. Profit education function for $\theta = 0.7$ and various n, m

Source: own calculations.

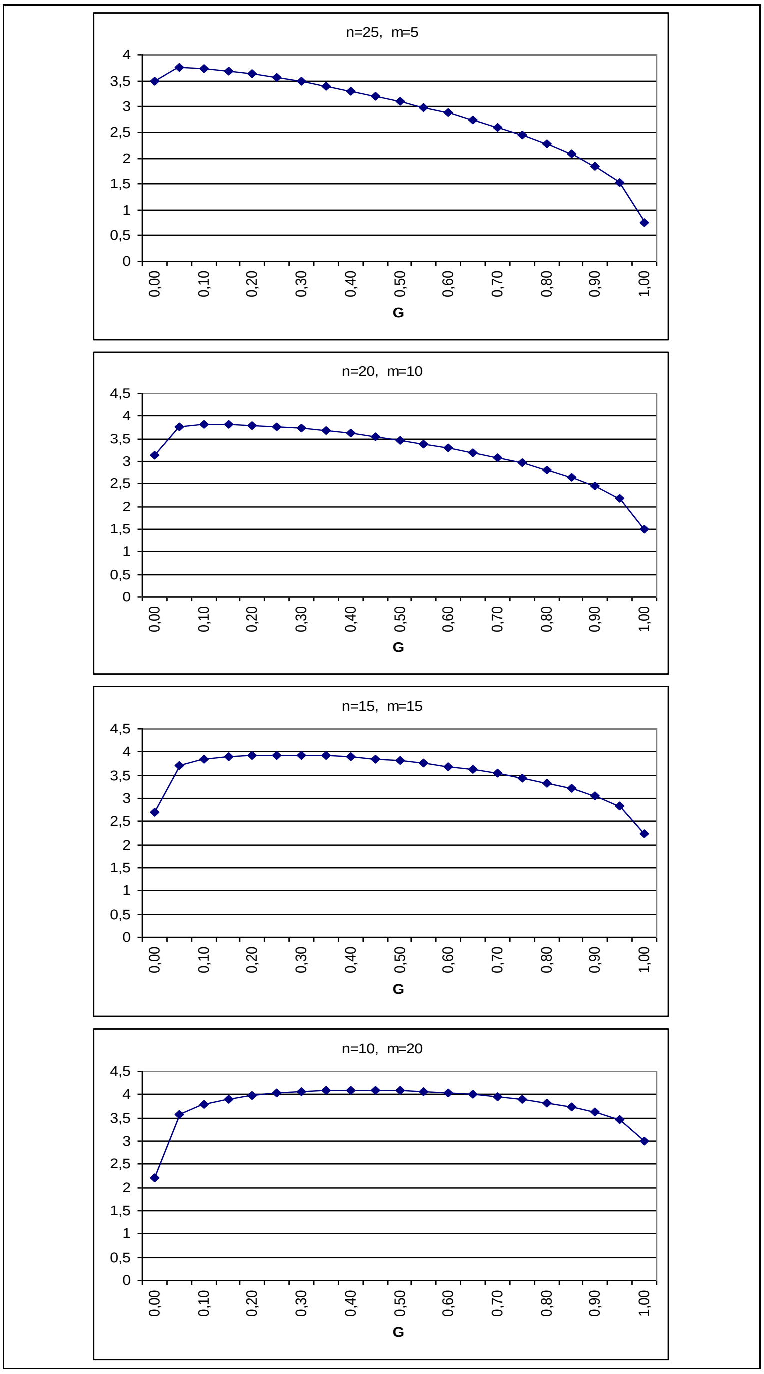

Fig. 3. Profit education function for $\theta = 0.4$ and various n, m

Source: own calculations.

Conducted simulations will illustrate the behaviour of the contemplated values in the case when their directions of interaction are not explicitly described by the theorem 1. Figs. 1-3 show that G (maximizing) at which the education profit function reaches its maximum increases along with the increase of m (confirmed by Theorem 1). Figs. 1-3 show that along with the increase of θ, maximizing G increases as well. The optimal lecture time increases proportionally to the teacher's utility from teaching high-ability students.

Figure 4 shows the average level of educational achievement (for these values G, L which satisfy (1)) the dependency from m. Regardless of θ the average levels of educational achievement are very similar. The proved theorem said that the average education level increases for $\theta \in (0, 0.5)$. In the considered example the growth feature μ is maintained for $\theta > 0.5$. The increase is "almost linear". The curve in "the highest" position is the one obtained for $\theta = 0.5$, thus the highest education productivity is for $\theta = 0.5$ (theorem 2).

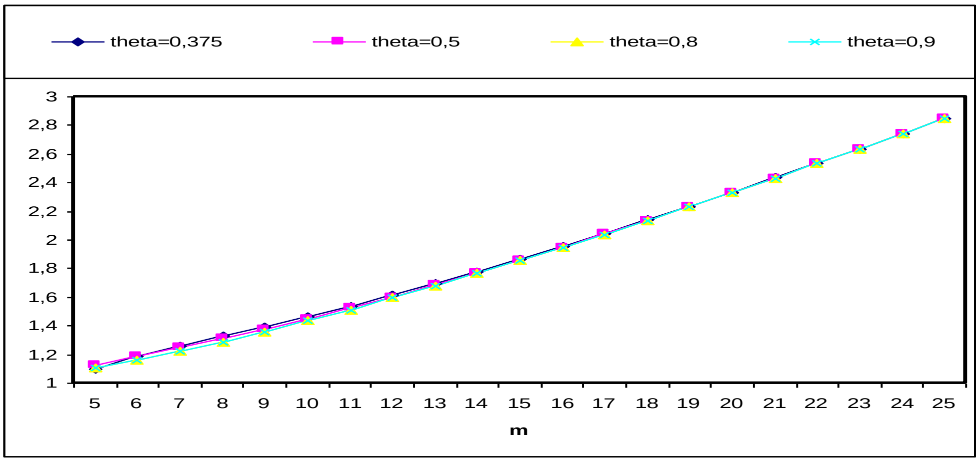

Fig. 4. Average level of educational achievement for various θ and m from 5 to 25

Source: own calculations.

Much more visible differences are for the time spent with each low-ability student. Figure 5 shows that the bigger θ is, the lower the curves presenting the time spent with each low-ability student are. It must be noticed that L is a decreasing function (of parameter m) regardless of the considered θ.

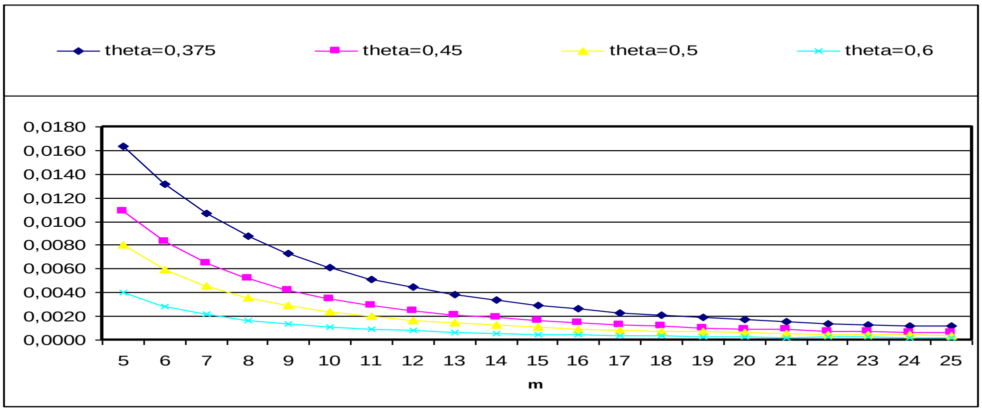

Fig. 5. Time spent with each low-ability student for various θ and m from 5 to 25

Source: own calculations.

4. Conclusions

The research presented herein can be utilized primarily by teachers in planning their lessons. Every teacher should know what gives him/her greater satisfaction: teaching gifted students or helping the less talented ones. Because, when considering the maximizing of the model of education profit the teacher knows in what "direction" will the average level of education go, and how will the time spent with the groups of students change.

The obtained results (section 2.3), which are an extension of the theorems presented in Bosworth's and Caliendo's article, when the relationship between the number of students in the considered groups is linear and declining. Theorem 1 applies, for example, in the case of a fixed number of students in a class (e.g. $m = w - n$). In the Polish education system we very often deal with schools where the size of the class is constant and amounts to 30 students.

In applications it is also important to determine the production function of education E^h as well as E^l. World literature widely describes the possibility of estimating the usages of this production function. The concept of "Educational Production Function" (1969) has gained wide publicity since the publication of the work of Samuel Bowles, *Educational Production* Function (1969). At a later stage, this subject was also discussed by Kruger, Hansuhk and Lazear.

The conducted simulations are based on the average level of education and the time spent with each of the low-ability students (it depends significantly upon the teacher's preference). The parameter of teacher preference is determined for a given teacher in advance (it is difficult to change the teacher's satisfaction with teaching a selected group of people). Theorem 2 can be useful especially for schools that recruit teachers and should be followed by the principle that we get the best training when the teacher has the same usefulness of teaching high- and low-ability students. In determining the value of this parameter one may find psychological knowledge most useful (e.g. asking a teacher relevant questions), also observation of the behaviour of a teacher during the lesson, as well as the opinion of the students can be useful.

The parameter θ can also be interpreted as a skill of conveying knowledge to a chosen group of students. Dividing students into gifted ones and the less able is not necessary. For example, you can divide students who are "visual students" and those who learn better by just listening to the lesson. Then the teacher has to divide the available time between these two groups. Thus obtained results can be applied to many other cases that may be presented in a form of optimization presented by the formulas (1) and (2).

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