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TESTING FOR EPIDEMIC CHANGES IN THE MEAN OF A MULTIPARAMETER STOCHASTIC PROCESS

Beatrice Bucchia (University in Köln)

1. Introduction

We discuss the problem of detecting epidemic changes of multi-indexed variables over a rectangle in $\mathbb{N}^d$. More precisely, assuming we have observed n^d values $\{x_{\mathbf{j}}: \mathbf{j} \in \{1, \dots, n\}^d\}$ of a random field $\{X_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$ (where $d \in \mathbb{N}$ is fixed and small relative to $n \in \mathbb{N}$), we may ask whether these observations all have the same mean μ_n, or whether there is a rectangle $(\mathbf{k}_0, \mathbf{m}_0] = (k_{0,1}, m_{0,1}] \times \dots \times (k_{0,d}, m_{0,d}]$ over which the mean has changed to a value $\mu_n + \delta_n$. Such a change point problem is the straightforward generalization to the multiparameter case of a one-dimensional change point problem with two change points $0 < k_0 < m_0 < n$. Levin and Kline (1985) coined the term epidemic change for the latter in their paper about the connection between chromosomal abnormalities and the number of spontaneous abortions. In this medical context, the term epidemic change corresponds to a period of normal behavior, followed by a sudden change in patient numbers and finally by a return to normalcy. The change point problem considered here, namely a change in the mean over a rectangle in the index-space of a random field, was also studied by Jaruškovă and Piterbarg [2011] and Zemlys [2008]. In both of these publications, the asymptotic distributions of the considered test statistics are determined by the fact that the random variables are independent and therefore the associated partial sum processes converge weakly to a Wiener process. This observation motivates us to replace the independence assumption by the (weaker) assumption of a functional

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central limit theorem (FCLT). Examples of the problem of detecting imhomogeneity arise in image analysis and in textile fabric quality control (e.g. [Zhang, Bresee 1995]). In particular, the search for an inhomogeneity in the shape of a rectangle might be of interest in the context of rectangular shape object detection problems. For instance, fiding rectangular objects in an image is a step in the detection of buildings or vehicles from aerial imagery [Vinson et al. 2001; Vinson, Cohen 2002; Moon et al. 2002], license plate detection [Kim et al. 2002; Huang et al. 2008] and in the detection of filaments in cryoelectron microscopy images [Zhu et al. 2001].

1.1. The model

First, we introduce some notations. We consider the vector space $\mathbb{Z}^d$ ($d \in \mathbb{N}$) equipped with the usual partial order. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we write $\lfloor \boldsymbol{x} \rfloor = (\lfloor x_1 \rfloor, \dots, \lfloor x_d \rfloor)'$ for the integer part of $\boldsymbol{x}$, $|\mathbf{x}| = (|x_1|, \dots, |x_d|)'$ and $[\mathbf{x}] = x_1 \cdots x_d$. Furthermore, for any integer $k \in \mathbb{N}_0$, we denote $(k, \dots, k)' \in \mathbb{N}_0^d$ by $\underline{\mathbf{k}}$. A rectangle in $\mathbb{R}^d$ is a set of the form

$$(\mathbf{x}, \mathbf{y}] = \{\mathbf{z}\colon x_i < z_i \leq y_i, i = 1, \dots, d\}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ ($(\mathbf{x}, \mathbf{y}] = \emptyset$, if $x_i \geq y_i$ for some $i \in \{1, \dots, d\}$). A rectangle in $\mathbb{Z}^d$ is the intersection of a rectangle in $\mathbb{R}^d$ and the set $\mathbb{Z}^d$. For a function $f\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}^d$, we define the increment of f over a rectangle $(\mathbf{s}, \mathbf{t}] \subset D$ as

$$f(\mathbf{s}, \mathbf{t}] = \begin{cases} \sum_{\boldsymbol{\varepsilon} \in \{0,1\}^d} (-1)^{d - \sum_{i=1}^d \varepsilon_i} f(\mathbf{s} + \boldsymbol{\varepsilon}(\mathbf{t} - \mathbf{s})), & \mathbf{s} < \mathbf{t} \\ 0, & \mathbf{s} \nless \mathbf{t}. \end{cases}$$

For instance, in the case $d = 2$ and $\mathbf{s} < \mathbf{t}$, the increment is

$$f(\mathbf{s}, \mathbf{t}] = f(t_1, t_2) - f(t_1, s_2) - f(s_1, t_2) + f(s_1, s_2).$$

We write

$$\sum_{\mathbf{k} \leq \mathbf{j} \leq \mathbf{m}} x_{\mathbf{j}} = \begin{cases} \sum_{\mathbf{j} \in (\mathbf{k}, \mathbf{m}] \cap \mathbb{Z}^d} x_{\boldsymbol{j}}, & \mathbf{k} < \mathbf{m} \\ \sum_{\mathbf{j} \in \emptyset} x_{\mathbf{j}} = 0, & \mathbf{k} \nless \mathbf{m}. \end{cases}$$

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Assume we have n^d realisations $x_{\mathbf{k}}$, $\mathbf{k} \in \{1, \dots, n\}^d$, $n, d \in \mathbb{N}$, of a real-valued random field $\{X_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$. We want to test

$$H_0: X_{\mathbf{k}} = Y_{\mathbf{k}} + \mu_n \quad \forall\, \mathbf{k} \in \{1, \dots, n\}^d$$

against

$$H_A: \exists\, \underline{\mathbf{0}} \leq \mathbf{k}_0 < \mathbf{m}_0 \leq \underline{\mathbf{n}},\ \lfloor \alpha n^d \rfloor \leq [\mathbf{m}_0 - \mathbf{k}_0] \leq \lfloor (1-\beta) n^d \rfloor:$$
$$X_{\mathbf{k}} = Y_{\mathbf{k}} + \mu_n + \delta_n\, I_{\{\mathbf{k}_0 < \mathbf{k} \leq \mathbf{m}_0\}} \quad \forall\, \mathbf{k} \in \{1, \dots, n\}^d,$$

where $\mu_n, \delta_n \in \mathbb{R}$ are unknown parameters, $\delta_n \neq 0$, and $0 < \alpha < 1 - \beta < 1$. The parameters α and β are used to restrict the possible changes to rectangles that have a certain size. They were used for technical reasons (cf. Section 2.1) but the restriction is nevertheless reasonable since a shifted mean on a set which is too small or too large would be difficult to distinguish. Since the points $\mathbf{k}_0, \mathbf{m}_0 \in \mathbb{Z}^d$ parametrize the set over which the change takes place, we call them the change points. Our main assumption is that the random field $Y = \{Y_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ is centered, weakly stationary and fulfills the FCLT

$$\left\{ \frac{1}{\sigma n^{d/2}} \sum_{\underline{\mathbf{1}} \leq \mathbf{k} \leq \lfloor n\mathbf{t} \rfloor} Y_{\mathbf{k}} \right\}_{\mathbf{t}\in[0,1]^d} \overset{D[0,1]^d}{\longrightarrow} \{W(\mathbf{t})\}_{\mathbf{t}\in[0,1]^d}, n \to \infty, \tag{1}$$

where $0 < \sigma^2 := \sum_{\mathbf{k}\in\mathbb{Z}^d} \mathrm{Cov}(Y_{\underline{\mathbf{0}}}, Y_{\mathbf{k}}) < \infty$, and $\{W(\mathbf{t})\}_{\mathbf{t}\in\mathbb{R}_+^d}$ is a standard Wiener field.

This covers a large class of processes, e.g. i.i.d. (cf. [Wichura 1969] Corollary 1), (positively and negatively) associated and (BL,θ)-dependent (cf. [Bulinski, Shashkin 2007] Theorem 5.1.5), as well as martingale-difference and Roelly (cf. [Poghosyan, Rœlly 1998] Theorem 3) random fields fulfill this assumption under certain conditions.

Example 1.1. *Let* $\{\xi_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$ *be a centered, stationary random field such that* $E[|\xi_{\mathbf{j}}|^q] < \infty$ *for some* $q > 2d$ *and*

$$0 < \rho^2 = \sum_{\mathbf{k}\in\mathbb{Z}^d} Cov(\xi_{\underline{\mathbf{0}}}, \xi_{\mathbf{k}}) < \infty.$$

We assume further that the $\{\xi_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ *fulfill the FCLT (1) with* $\sigma=\rho$. *For* $\mathbf{k}\in\mathbb{Z}^d$ *and real numbers* $\{a(\mathbf{j})\}_{\mathbf{j}\in\mathbb{Z}^d}$ *that fulfill the assumption*

$$\sum_{i_1=0}^{\infty}\cdots\sum_{i_d=0}^{\infty}\sum_{k_1=i_1+1}^{\infty}\cdots\sum_{k_d=i_d+1}^{\infty}|a(k_1,\ldots,k_d)|<\infty,$$

we define

$$Y_{\mathbf{k}}=\sum_{j_1=0}^{\infty}\cdots\sum_{j_d=0}^{\infty}a\,(j_1,\ldots,j_d)\xi(k_1-j_1,\ldots,k_d-j_d).$$

Then Ko et al. [2008] showed that $\{Y_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ *satisfies (1) with*

$$\sigma=\rho\cdot\sum_{i_1=0}^{\infty}\cdots\sum_{i_d=0}^{\infty}a\,(i_1,\ldots,i_d).$$

In the case when the $\{\xi_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$ *are i.i.d., this result was proven by Marinucci and Poghosyan [2001] without the assumption that the* $\{\xi_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$ *fulfill the FCLT themselves.*

2. Testing for epidemic changes in the mean

2.1. The test statistic

The idea to test for a change in the mean is to test for each rectangle $(\mathbf{k},\mathbf{m}]$ whether or not the mean is significantly different from the overall mean on $(\underline{\mathbf{0}},\underline{n}]$ and to reject the null hypothesis if this is the case for any of the rectangles. For each rectangle, the test for difference in the mean is based on a pseudo log-likelihood approach. This approach makes it necessary to restrict the choice of considered rectangles: The weighting function $\sqrt{\frac{[\mathbf{m}-\mathbf{k}]}{n^d}\left(1-\frac{[\mathbf{m}-\mathbf{k}]}{n^d}\right)}$ for $[\mathbf{m}-\mathbf{k}]$ tending to zero or one causes the test statistic to be unbounded even under the null hypothesis. We therefore consider a trimmed test statistic of the following form (cf. [Jaruškovă. Piterbag 2011]):

$$T_n(\alpha,\beta)=\hat{\sigma}_n^{-1}n^{-d/2}\max_{\substack{\underline{\mathbf{0}}\le\mathbf{k}<\mathbf{m}\le\underline{\mathbf{n}}\\ \lfloor\alpha n^d\rfloor\le[\mathbf{m}-\mathbf{k}]\le\lfloor(1-\beta)n^d\rfloor}}\frac{\left|\sum_{\mathbf{k}<\mathbf{j}\le\mathbf{m}}\left(X_{\mathbf{j}}-\bar{X}_n\right)\right|}{\sqrt{\frac{[\mathbf{m}-\mathbf{k}]}{n^d}\left(1-\frac{[\mathbf{m}-\mathbf{k}]}{n^d}\right)}},$$

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where $\bar{X}_n = n^{-d}\sum_{\underline{\mathbf{1}}\le\mathbf{k}\le\underline{\mathbf{n}}} X_{\mathbf{k}}$, $\hat{\sigma}_n$ is an estimator for σ and $0 < \alpha < \beta < 1$ are trimming parameters. It can easily be seen that T_n is independent of μ_n, so that we can assume $\mu_n = 0$ w.l.o.g.

2.2. Behavior under the null and alternative hypotheses

To define a test that has a given asymptotic level, we need to determine the asymptotic behavior of our test statistic under the null hypothesis. We do this in two steps, by first determining its limit variable and then finding an approximation for the tail behavior of the limit distribution.

Theorem 2.1. *Let $\hat{\sigma}_n$ be a (weakly) consistent estimator for σ under H_0. Then under H_0 it holds that for $n \to \infty$*

$$T_n(\alpha,\beta) \xrightarrow{\mathcal{D}} \sup_{\substack{\underline{0}\le s<t\le\underline{1} \\ \alpha\le[t-s]\le 1-\beta}} \frac{|W(\mathbf{s},\mathbf{t}] - [\mathbf{t}-\mathbf{s}]W(\underline{\mathbf{1}})|}{\sqrt{[\mathbf{t}-\mathbf{s}](1-[\mathbf{t}-\mathbf{s}])}}. \tag{2}$$

Approximating the tail behavior of the limit distribution is made easier by the fact that the limit variable is the supremum of a Gaussian field over a compact set. We define

$$C_d(\alpha,\beta) = \int_{\alpha}^{1-\beta} \frac{1}{4^d \xi_d^2 (1-\xi_d)^{2d}} \int_{\xi_d}^{1} \cdots$$

$$\int_{\xi_2}^{1} \frac{(1-\xi_1)(\xi_1-\xi_2)\cdots(\xi_{d-1}-\xi_d)}{\xi_1^2\cdots\xi_{d-1}^2} d\xi_1 \cdots d\xi_{d-1} d\xi_d$$

and consider a random field $\{X(\mathbf{s},\mathbf{t})\}_{(s,t)\in D}$ of the form

$$X(\mathbf{s},\mathbf{t}) = \frac{W(\mathbf{s},\mathbf{t}] - [\mathbf{t}-\mathbf{s}]W(\underline{\mathbf{1}})}{\sqrt{[\mathbf{t}-\mathbf{s}](1-[\mathbf{t}-\mathbf{s}])}},$$

where

$$D = \{(\mathbf{x},\mathbf{y}) \in [0,1]^{2d} : \mathbf{x} < \mathbf{y}, \alpha \le [\mathbf{y}-\mathbf{x}] \le 1-\beta\}.$$

The following theorem is a direct consequence of Theorem 7.1 of [Piterbarg 1996] (cf. also [Jaruškovă 2011] Theorem A.1).

Theorem 2.2. *Let $\phi(u)$ be the density of the standard normal distribution. For $u \to \infty$ it holds that:*

$$P\left(\sup_{(\mathbf{s},\mathbf{t})\in D} X(\mathbf{s},\mathbf{t}) > u\right) \sim C_d(\alpha,\beta)u^{4d-1}\phi(u).$$

This result can be used to obtain an approximation for the tail behavior of the right hand side of (2):

Corollary 2.1. *With the same notations as in Theorem 2, it holds for $u \to \infty$ that*

$$P\left(\sup_{(\mathbf{s},\mathbf{t})\in D} |X(\mathbf{s},\mathbf{t})| > u\right) \sim 2\ C_d(\alpha,\beta)u^{4d-1}\phi(u).$$

The constructed test is consistent under the alternative hypothesis:

Theorem 2.3. *If $|\delta_n|n^d \to \infty$ for $n \to \infty$ and $\hat{\sigma}_n = \mathcal{O}_P(1)$, $\hat{\sigma}_n > 0$, it holds under the alternative $H_{\alpha,\beta}$ that*

$$T_n(\alpha,\beta) \xrightarrow{P} \infty \ for\ n \to \infty.$$

2.3. Long-run variance estimators

In the test statistics presented above, we have used an unspecified estimator for σ^2 in order to show that the main requirements for such an estimator are consistency under the null and stochastic boundedness under the alternative hypothesis. In order to give some idea of possible estimators, we now give an example for an estimator that fulfills our requirements. We apply generalizations of well-known kernel-based variance estimators from the time series literature to our model. In the spirit of our general approach, we consider a nonparametric estimator. In order to shorten notation, we write $r(\boldsymbol{j}) = \mathrm{Cov}\left(Y_{\underline{0}}, Y_{\mathbf{j}}\right)$ and define

$$\hat{r}_X(\boldsymbol{j}) = \frac{1}{n^d}\sum_{k\in N_{\mathbf{j}}}(X_{\mathbf{k}} - \bar{X}_n)\left(X_{\mathbf{k+j}} - \bar{X}_n\right),$$

With $N_{\mathbf{j}} = \left\{k \in \mathbb{Z}^d : \underline{1} \le \mathbf{k}, \mathbf{k}+\mathbf{j} \le \underline{\mathbf{n}}\right\}$. We consider estimators of the form

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$$\hat{\sigma}_n^2 = \sum_{\mathbf{j} \in B_{q-1}} \omega_{q,\mathbf{j}} \, \hat{r}_X(\boldsymbol{j}),$$

where $q = q(n) \in [1, n]$ is an integer with $q = q(n) \to \infty$ and $\lim_{n\to\infty} q/n = 0$, $B_q = \{-q, \dots, q\}^d$ and $\omega_{q,\mathbf{j}}$ is a bounded weight function that fulfills $\omega_{q,\mathbf{j}} \to 1$ for $q \to \infty$. If we assume additional moment and homogeneity conditions on $Y_{\mathbf{k}}$ (cf. [Lavancier 2008] hypothesis H0), a careful reading of the proof of Lemma 1 in [Lavancier 2008] shows that his proof of $\hat{\sigma}_n^2$ converging stochastically to σ^2 remains valid if we replace $|\mathbf{j}|$ by and consider different weight functions (e.g. flat-top kernels as suggested by [Politis, Romano 1996]). This more general case is therefore discussed here.

Lemma 2.1. (*cf. [Lavancier 2008]) for* $\delta_n = 0$*). It holds for* $q = q(n) \to \infty$*with* $\lim_{n\to\infty} q/n = 0$*, that*

$$\hat{\sigma}_n^2 \xrightarrow{P} \sigma^2, \quad n \to \infty,$$

Under H_0 *and*

$$\hat{\sigma}_n^2 = \mathcal{O}_P(1),$$

under H_A*, if* δ_n *and* q *satisfy*

$$\hat{\sigma}_n^2 q^d = \mathcal{O}_P(1),$$

And $|\delta_n| n^{d/2} \to \infty$.

3. Estimation of the change points

In this last section, we want to cover the related problem of estimating the change points. We consider the alternative

$$H_A(\boldsymbol{\vartheta}, \boldsymbol{\gamma}): \bigwedge \underline{\mathbf{0}} < \boldsymbol{\vartheta} < \boldsymbol{\gamma} < \underline{\mathbf{1}}: \mathbf{k_0} = \lfloor n\boldsymbol{\vartheta} \rfloor, \mathbf{m}_0 = \lfloor n\boldsymbol{\gamma} \rfloor,$$

and the "change" δ_n is assumed to be a constant multiple of $n^{-d/2}$, i.e.

$$\delta_n = \delta n^{-d/2}, \qquad \delta \neq 0. \tag{3}$$

Our aim is to estimate the points $\boldsymbol{\vartheta}$ and $\boldsymbol{\gamma}$. Using a similar approach to the one employed by Aston and Kirch (2012), the estimators we consider are points where the maximum of a slightly modified version of our test statistic is reached. To do so, we define

$$\arg\max_B Z = \{\mathbf{a} \in B : Z(\mathbf{a}) = \max_{\mathbf{b}\in B} Z(\mathbf{b})\}$$

for functions $Z: A \to \mathbb{R}$ ($A \subseteq [0,1]^d, d \in \mathbb{N}$ in $D[0,1]^d$ and compact subsets $B \subseteq A$. Furthermore, let

$$K_d = \{(\boldsymbol{s}, \boldsymbol{t}) \in [0,1]^{2d} : \underline{\mathbf{0}} < \underline{\mathbf{s}} < \underline{\mathbf{t}} < \underline{\mathbf{1}}\}$$

and

$$G_{n,d}(\mathbf{s},\mathbf{t}) = \frac{1}{n^d} \sum_{\lfloor n\mathbf{s}\rfloor < i \le \lfloor n\mathbf{t}\rfloor} (X_{\mathbf{i}} - \bar{X}_n)\, I_{K_d}(\mathbf{s},\mathbf{t}).$$

Then $|G_{n,d}| \neq \emptyset$, and arbitrary points $(\widehat{\boldsymbol{\vartheta}}_{\boldsymbol{n}}, \widehat{\boldsymbol{\gamma}}_{\boldsymbol{n}})$ in $\max_{K_d} |G_{n,d}|$ give consistent estimators for $(\boldsymbol{\vartheta}, \boldsymbol{\gamma})$:

Theorem 3.1. *Under $H_A(\boldsymbol{\vartheta}, \boldsymbol{\gamma})$ with δ_n as in* (3), *it holds that*

$$(\widehat{\boldsymbol{\vartheta}}_{\boldsymbol{n}} - \boldsymbol{\vartheta}, \widehat{\boldsymbol{\gamma}}_{\boldsymbol{n}} - \boldsymbol{\gamma}) = o_P(1), \qquad n \to \infty.$$

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VARYING DEMOGRAPHIC ENVIRONMENT IN THE EUROPEAN UNION: INFLUENCE ON ACTUARIAL AMOUNTS

Joanna Dębicka (Wrocław University of Economics

The calculation of reserves for each year of insurance period is made at the moment of policy issue and based on current life tables (LT). During the insurance period life tables are changing. It means that premiums and prospective reserves are changing too, but according to a contract, an insurer cannot change insurance premiums and benefits.

The aim of the talk was to analyze the influence of change of mortality in the European Union (UE) countries on premiums and prospective reserves in temporary life insurance contracts.

For the analysis of the actuarial amounts we selected European Union countries in which the expected future life-time was changed the most and the least between 1999 and 2009 year (with respect to particular age groups: 20, 40, 60 years old, and sex). 10-years period was chosen, not too short to make changes visible and not too long, so that the changes were not obvious. The first step of research involved analysis of statistics on the difference between the expected future life-time for men/women in 2009 and 1999 in the EU countries. It appeared that, regardless of age, variation range for women is lower than for men and there are no outliers observations. It means that to analyze influence of demographic environment on actuarial amounts it is enough to do it for men. The second step of research was connected with the choice of countries. For the further analysis Lithuania, Poland and Ireland were chosen. The smallest change in the expected future life-time was observed in Lithuania and the biggest in Ireland. Poland appeared to be a country where the change in the expected future life-time was typical.

For the analysis of 10-year temporary life insurance was chosen (such type of insurance contract is sold in each country of the EU). We focused on discrete-time model, where insurance payments are made at the ends of time intervals. Practically it means that insurance benefits are paid immediately before the end of the year. Premiums are paid immediately after the beginning of the year.

Multiple state modeling is a classical tool for designing and implementing insurance products. Among others it is also used to calculate premiums and reserves. In particular, matrix representation of formulas on net premiums and net prospective reserves were used for numerical calculations. For numerical examples, we considered an insurance contract where insurance benefit is equal to 1 and premiums are constant for the whole insurance period. It was assumed that annual interest rate is equal to 2%.

Figure 1 illustrates percentage changes in annual premiums calculated for LT for the years 2000–2008, in relation to the annual premiums calculated for 1999 LT.

It is observed that independently of the age of insured person in Poland and Ireland premiums calculated under current LT were lower than for 1999LT. The dynamics of changes was larger in Ireland than in Poland. In case of Lithuania, situation of 20-year-old persons was similar like in other countries. Interestingly for 40- and 60-year-old persons one can observe that premiums calculated under the current LT were up to 21.5 % higher than for 1999 LT.

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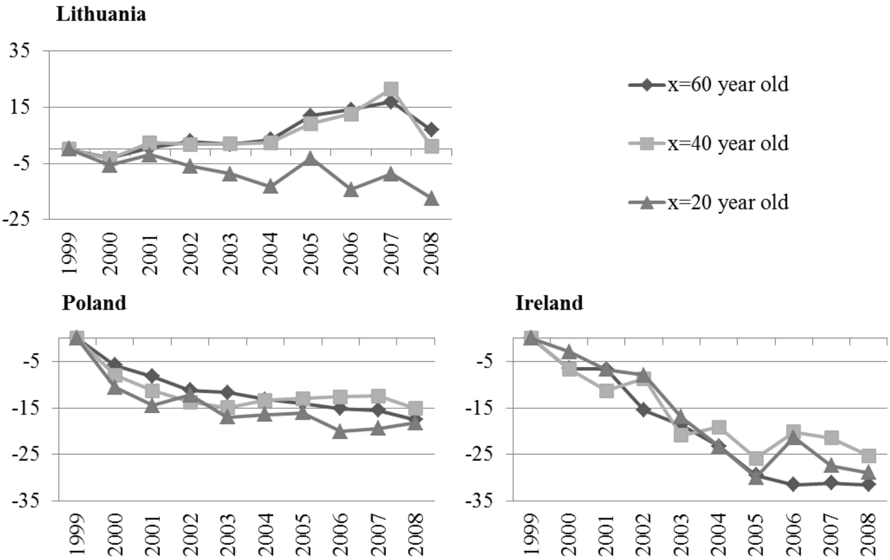

Figure 1. Net premiums

Source: own elaboration.

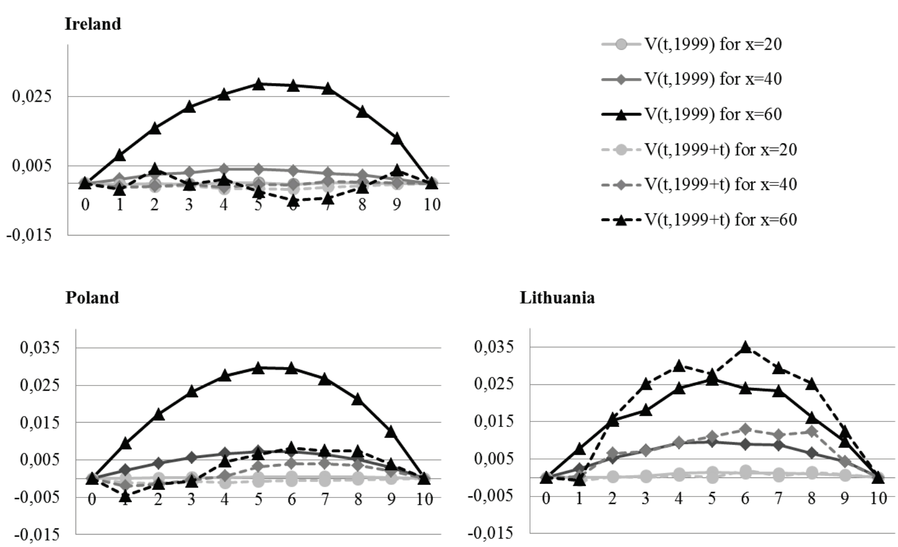

Figure 2. Net prospective reserves

Source: own elaboration.

Figure 2 illustrates net prospective reserves $V_1(t, 1999)$ calculated for 1999 LT and $V_1(t, 1999 + t)$ calculated for current LT ($t = 0,1,2,3,...,10$).

It is observed that in Poland and Ireland net perspective reserves calculated under current LT were higher than for 199LT, because expected future life-time is increasing and this means that actuarial value of benefits is decreasing and simultaneously actuarial value of premiums is increasing. In Lithuania the situation is reverse. The most important effect on the difference between net prospective reserves calculated under current LT and 1999 LT is age at entry. The older a person is, the absolute value of the difference between reserves is greater.

To sum up, because of the demographic environment premiums and reserves in EU countries are quite different. Diversity of dynamics of life tables' parameters may influence actuarial quantities in many ways. In particular, it may lead to underestimation or overestimation of funds needed to cover future benefits. The modelling of cash flows is important both because of the dynamics of changes in interest rates and also because of the dynamics of the life table parameters.

DATA COLLECTION AND ESTIMATION FOR SENSITIVE CHARACTERISTICS AND COMPLEX SAMPLE SURVEYS

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1. Introduction

In surveys with a sensitive characteristic *X* (e.g., income, tax evasion, social benefit fraud, academic cheating behavior), direct questioning causes answer refusal and untruthful responses. If the distribution of *X* is estimated from the responses obtained by direct questioning, a serious bias must be expected. A possible approach for better estimates is the implementation of ingenious survey techniques, which on the one hand protect the interviewees' privacy to increase their cooperation and on the other hand yield data that allow inference on the distribution of the sensitive variable. In this field of research, nonrandomized response procedures are currently emerging.

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2. Diagonal model (DM)

One of these nonrandomized response methods is the diagonal model from [Groenitz 2012], which is suitable to collect data on an arbitrary categorical $X \in \{1,2,...,k\}$. For instance, X may represent income classes. The principle of the DM is that the respondents do not reveal the value of the sensitive X, but give a scrambled response A, which depends on X and a scrambling variable (= auxiliary variable) $W \in \{1,2,...,k\}$. For every A, each X value is still possible. The characteristic W must be nonsensitive, must have a known distribution and must be independent of X. A possible scrambling attribute W can be constructed for the period of the birthday. The following table illustrates the answer schema of the DM for $k = 4$ categories:

X/W	$W=1$	$W=2$	$W=3$	$W=4$
$X=1$	1	2	3	4
$X=2$	4	1	2	3
$X=3$	3	4	1	2
$X=4$	2	3	4	1

We define π_i, c_i and γ_i to be the proportion of units in population having category i for X, W, and A, respectively. Then, the distribution of X, W, and A in the population can be described by vectors $\pi = (\pi_1, \pi_2, ..., \pi_k)^T$, $c = (c_1, c_2, ..., c_k)$, $\gamma = (\gamma_1, \gamma_2, ..., \gamma_k)^T$. With a certain matrix C_0 depending on c, we have $\lambda = C_0 \cdot \pi$. The topic of this talk is to estimate π from the scrambled responses and derive properties of the estimator.

3. Estimation and estimation properties for diagonal model in several sampling designs

Of course, the sampling design must be incorporated in the estimation. The case of simple random sampling with replacement (SRSWR) is treated in [Grönitz 2012]. More complex sampling designs including stratified, cluster, multi-stage and unequal probability sampling are considered in [Grönitz 2013a]. We remark that there are also methods that enable the exploitation of prior information via Bayes technique (see [Grönitz 2013b]), but we do not give details on this issue in this talk.

The idea for the estimation is to derive an estimator $\hat{\lambda}$ for λ which involves the sampling design from the observed scrambled answers and to set $\tilde{\pi} = C_0^{-1} \cdot \hat{\lambda}$. We give some examples: For simple random samples (SRSs), $\hat{\lambda}_i$ is the relative frequency of answer $A = i$ in the sample. For stratified SRSs, $\hat{\lambda}_i$ is a weighted relative frequency. For a general with-replacement sample, $\hat{\lambda}_i$ is the Hansen-Hurwitz estimator whereas $\hat{\lambda}_i$ is the Horvitz-Thompson estimator in the case of a general without-replacement sample.

However, the estimator $\tilde{\pi}$ can attain inadmissible values (components outside [0,1] or sum of components unequal to one). For this problem, which is often ignored in the literature, we propose the remedy to compute a modified estimator $\hat{\pi}$ based on $\tilde{\pi}$. In the case of SRSWR or stratified SRSWR, we can apply the expectation maximization (EM) algorithm to obtain $\hat{\pi}$. For other sampling designs, we search the admissible estimate that is closest to the nonmodified estimate, that is, the modified estimate is the solution of a quadratic optimization problem.

To measure precision of the modified estimators, details on the bootstrap variance estimation are also given in this talk. Moreover, we demonstrate a simulation-based method for the investigation of the connection between estimation efficiency in complex sample surveys and the degree of privacy protection. Our simulations illustrate that larger efficiency corresponds to a lower degree of privacy protection and discover optimal model parameters for the diagonal model. Such optimality results are rare in the literature on survey designs for sensitive *X* with an arbitrary number of categories, especially when complex sample surveys are studied.

4. Summary

Privacy-protecting survey designs possess an appealing principle to reduce untruthful answers and answer refusal in surveys with sensitive questions. Such techniques are both methodologically interesting and applicable in practice. In this talk, we have considered the nonrandomized diagonal model, which is suitable for arbitrary categorical sensitive attributes, facilitates the respondents' cooperation and possesses a simple procedure. We have studied different sampling designs which

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often appear in practice and have solved the following problems: First, how can we estimate the distribution of the sensitive attribute? Second, how can we estimate the estimator's variance? Finally, how does the efficiency depend on degree of privacy protection and how can we find optimal parameters of the diagonal model?

References

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PRICING DEPENDENT COMPOUND POISSON PROCESSES

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1. Introduction

The paper is devoted to the compound Poisson process, in which the interclaim time and the neighboring claim amount may be dependent on each other In the classical approach we assume that all random variables and processes are independent. This assumption is unrealistic. In practice some random variables may be dependent. For example, in investigating natural catastrophic events, we meet such variables. The total claim amount on the occurrence of a catastrophe, e.g. the earthquake, and the time elapsed since the previous catastrophe are often dependent. Bigger damages occur when the time between the claims is longer.

The dependent structure is described by some copulas. The values of the insurance premiums based on the moments of the aggregated claim and basic risk measures: VaR and ES, are derived. The exact formulas, approximation and simulations are used to compute these values.

2. Compound Poisson process

We will study the following risk process:

$$S(t) = X_1 + X_2 + \cdots + X_{N(t)},$$

where X_i are the identically distributed claim amounts with the expected values $E(X_i) = 1/\beta$, $N(t)$ is a claim number process generated by a renewal process W_i. We assume that $N(t)$ is a Poisson process, so the interclaim times W_i are identically exponentially distributed, with cumulative distribution function (c.d.f.) $F_W(w) = 1 - e^{-\lambda w}$.

We also assume that (X_i, W_i) are the independent random vectors, but the random variables X_i, W_i may be dependent. We denote by symbol $F(x, t)$ the joint c.d.f. of them. The dependent structure between X_i, W_i may be described by the copula C:

$$F(x, t) = C(F_X(x), F_W(t)).$$

We will use in our paper the Spearman copula defined by the formula

$$C_\alpha(u_1, u_2) = (1 - \alpha)C_I(u_1, u_2) + \alpha C_M(u_1, u_2),$$

where $0 \le \alpha \le 1$. The Spearman copula is a convex combination of independent $C_I(u_1, u_2) = u_1u_2$ and comonotonic (strict dependent) $C_M(u_1, u_2) = \min(u_1, u_2)$ copulas. The parameter α reflects the degree of dependence. It is equal to Spearman coefficient of correlation. If the random variables X, W are comonotonic, then $X = l(W)$, when l is an increasing function, so they have singular joint distribution on $D = \{(x, t)\colon x = l(t)\}$. This copula reflects the positive dependence, between the independence and positive strict, functional dependence.

The Farlie-Gumbel-Morgenstern (FGM) copula is described by the formula

$$C_\theta(u_1. u_2) = u_1u_2 + \theta u_1u_2(1 - u_1)(1 - u_2),$$

where $-1 \le \theta \le 1$. It models the small degrees of dependence only, when the Spearman coefficient of correlation ρ satisfies the relation $-1/3 \le \rho \le 1/3$.

The Clayton copula is done by the formula

$$C_\alpha(u_1. u_2) = (u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1/\alpha},$$

where $\alpha > 0$. It reflects the positive dependence. The Kendal coefficient of correlation τ is equal to $\frac{\alpha}{\alpha+2}$ in this case.

3. Insurance premium

We will study the following form of insurance premium connected with the aggregate claims $S(t)$:

$$\Pi(t) = E(S(t)) + L(t),$$

where $E(S(t))$ is a pure premium and $L(t)$ is a loading for the risk. When $L(t) = cE(S(t)$ we obtain so called the expected value principle, $L(t) = cV(S(t))$ the variance principle and when $L(t) = c\sqrt{V(S(t))}$ we have the standard deviation principle. The constant $c > 0$ is a safety loading.

This insurance premiums are based on the first two moments of $S(t)$. When the dependent structure is described by the Spearman copula, the first moment $\mu_1(t)$ satisfies the following relation [Heilpern in review]

$$\mu_1(t) = E(S(t)) = E(E(X_1 + S(t-w)|\, W_1 = w))$$

$$= \lambda \int_0^t e^{-\lambda w} E(X|W = w)dw + \lambda \int_0^t e^{-\lambda w} \mu_1(t-w)dw,$$

where $E(X|W = w) = \frac{1-\alpha}{\beta} + \alpha l(w)$, so

$$\mu_1(t) = \frac{1-\alpha}{\beta}\left(1 - e^{-\lambda t}\right) + \lambda \int_0^t e^{-\lambda w} l(w)dw + \lambda \int_0^t e^{-\lambda w} \mu_1(t-w)dw$$

and we obtain the Laplace transform of the first moment of $S(t)$

$$\mu_1^*(p) = (1-\alpha)\frac{\lambda}{\beta p^2} + \alpha\lambda\frac{p+\lambda}{p^2} l^*(p+\lambda).$$

The expected value of $S(t)$ is equal to

$$\mu_1(t) = \frac{1-\alpha}{\beta}\lambda t + \alpha\lambda \int_0^t e^{-\lambda w}(1 + (t-w)\lambda) l(w)dw.$$

When the claims are exponentially distributed we have $F_X(x) = 1 - e^{-\beta x}$, $l(w) = \frac{\lambda}{\beta} w$ and

$$\mu_1(t) = \frac{\lambda}{\beta} t - \frac{\alpha}{\beta}\left(1 - e^{-\lambda t}\right).$$

For the Pareto distributed claims we obtain $F_X(x) = 1 - \left(\frac{b}{x+b}\right)^a$ and $l(w) = b(e^{-\lambda w/a} - 1)$

$$\mu_1(t) = \frac{\lambda t}{\beta} + \alpha \frac{a\left(2\lambda t + b\beta\left(e^{\left(\frac{1}{a}-1\right)\lambda t} + \lambda t - 1\right)\right) - a^2\lambda t - (1 + b\beta)\lambda t}{(a-1)^2\beta},$$

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where $\beta = (a - 1)/b$. We obtain the Laplace Transform of the second moment of $S(t)$

$$\mu_2^*(p) = \frac{1}{p^2\beta^2}\lambda(2 - 2\alpha + \beta(\alpha\beta(p+\lambda)(l^2)^*(p+\lambda) + 2p(1-\alpha + \alpha\beta(p+\lambda)l^*(p+\lambda)\mu_1^*(p)))$$

using the formula [Heilpern in review]

$$\mu_2(t) = E(S^2(t)) = E(E((X_1 + S(t-w))^2 | W_1 = w)).$$

For the exponential claims the second moment and variance of $S(t)$ are equal to

$$\mu_2(t) = \frac{2\alpha^2 - 4t\alpha\lambda + t\lambda(2 + t\lambda) - 2e^{-t\lambda}\alpha(\alpha - t(1-\alpha)\lambda)}{\beta^2},$$

$$V\big(S(t)\big) = \frac{2\lambda}{\beta^2}t - \alpha\frac{2\lambda}{\beta^2}t + \alpha^2\frac{1 - 2\lambda t e^{-\lambda t} - e^{-2\lambda t}}{\beta^2}.$$

The variance is a decreasing function of degree of dependence α. When the claims are Pareto distributed we obtain more complicate statements.

Example 1. Let $\lambda = 5$, $\beta = 0.01$, $c = 0.2$ and $t = 2$ when the claims have the exponential distribution and $a = 2$, $b = 100$ for the Pareto distribution. The values of the expected, variance and standard deviation principles are presented in Table 1. We see that if the degree of dependence, described by α, increases, then the values of the insurance premiums decreases.

Table 1. The values of the insurance premiums for the dependent structure described by Spearman copula

	Exponential			Pareto		
α	expected	variance	stand. dev.	expected	variance	stand. dev.
0	1200.000	41000.000	1089.443	1200.000	41000.000	1089.443
0.2	1176.001	33060.001	1060.101	1152.324	37097.070	1045.284
0.4	1152.002	25279.802	1029.744	1104.647	33790.739	1001.619
0.6	1128.004	17659.343	997.829	1056.971	31081.409	958.527
0.8	1104.005	10198.844	963.083	1009.294	28969.078	916.082
1	1080.006	2898.189	919.996	961.618	27453.348	874.358

Source: own elaboration.

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Barges et al. [2011] studied case when the dependent structure is described by FGM copula. The expected value and variance of $S(t)$ take the following form in this case:

$$\mu_1(t) = \frac{\lambda}{\beta}t - 0.5\theta\left(1 - e^{-2\lambda t}\right)\left(\int_0^{\infty}\left(1 - F_X(x)\right)^2 dx - \frac{1}{\beta}\right),$$

$$V\left(S(t)\right) = 2\frac{\lambda t}{\beta^2} - \frac{1 + 2\lambda t + e^{-2\lambda t}(2\lambda t - 1)}{4\beta^2}\theta - \frac{e^{-4t\lambda} + 4e^{-2t\lambda}t\lambda - 1}{16\beta^2}\theta^2.$$

Example 2. Let $\lambda = 5$, $\beta = 0.01$, $c = 0.2$ and $t = 2$. The values of the insurance premiums for the exponential claims are presented in Table 2.

Table 2. The values of the insurance premiums for the dependent structure described by FGM copula

ρ	Expected	Variance	Stand. dev.
–1/3	1230	51525.0	1125.499
–0.3	1227	50477.5	1121.954
–0.2	1218	47335.0	1111.250
–0.1	1209	44202.5	1100.446
0	1200	41080.0	1089.532
0.1	1191	37967.5	1078.494
0.2	1182	34685.0	1067.098
0.3	1173	31527.5	1055.666
1/3	1170	30475.0	1051.811

Source: own elaboration.

For other copulas we must use the simulation methods. For instance when the dependent structure is done by the Clayton copula, we may use the following procedure:

1) generate two independent exponential ($\lambda = 1$) variates y_1 and y_2;
2) generate gamma ($a = 1/\alpha$, $b = 1$) variate z, independent of y_i;
3) set $u_1 = \left(1 + \frac{y_1}{z}\right)^{-1/\alpha}$ and $u_2 = \left(1 + \frac{y_2}{z}\right)^{-1/\alpha}$;
4) set $x = F_X^{-1}(u_1)$ and $w = F_W^{-1}(u_2)$.

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Example 3. Let $\lambda = 5$, $\beta = 0.01$, $c = 0.2$ and $t = 2$. The values of the insurance premiums for the exponential claims and when the dependent structure is done by the Clayton copula are presented in Table 3.

Table 3. The values of the insurance premiums for the dependent structure described by Clayton copula

τ	Expected	Variance	Stand. dev.
0.01	1216.84	39400.2	1088.63
0.2	1173.70	33723.5	1059.01
0.4	1161.34	26449.7	1039.17
0.6	1133.55	18428.9	1003.76
0.8	1105.86	10680.7	965.731
0.95	1087.63	4825.7	934.358

Source: own elaboration.

4. Calculation of VaR and TVaR

Now we derive the two main risk measure of the aggregate claims $S(t)$: Value at Risk (VaR) and Tail Value at Risk (TVaR). They are done by the following formulas:

$$\mathrm{VaR}_\alpha(S(t)) = \inf\{x\colon F_{S(t)}(x) \geq \alpha\},$$

$$\mathrm{TVaR}_\alpha(S(t)) = E(S(t) \mid S(t) > \mathrm{VaR}_\alpha).$$

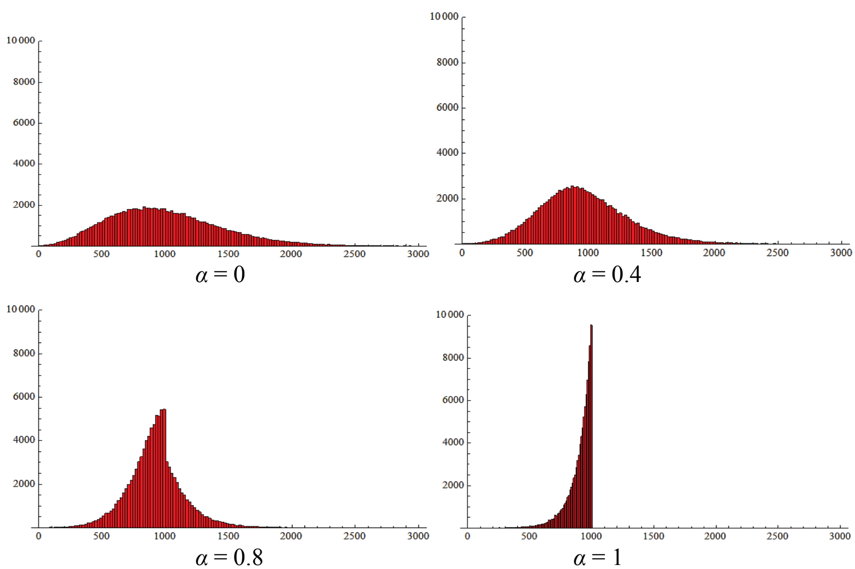

Figure 1. Distributions of $S(t)$ for different degrees of dependence

Source: own elaboration.

When the dependent structure is described by the Spearman copula, we use two methods to this end. First we use the simulation method. The distributions of $S(t)$ for different values of α done by simulation (n = 100 000) are presented in Figure 1.

The second method uses the mixture of Erlang distributions approximation based on the three first moments [Barges et al. 2011]. We approximate for fixed t the distribution of $S(t)$ by the mixture Z of two Erlang distributions with common shape parameter, i.e.:

$$F_{S(t)}(x) \approx F_Z(x) = p_1 F_{Y_1}(x) + p_2 F_{Y_2}(x),$$

where Y_i has gamma distribution $\Gamma(n, l_i)$ and $p_i \geq 0$, $p_1 + p_2 = 1$, $i = 1, 2$. We assume that i-th moment μ_i of Z are equal to such moments of $S(t)$, i.e. $\mu_i = \mu_i(t)$, $i = 1, 2, 3$.

Johnson and Taaffe [1989] obtained the values of parameters of distribution Z. The shape parameter n is the smallest integer satisfying relation

$$n > max\left\{\frac{1}{c^2}, \frac{\frac{1}{c^3} + \frac{1}{c} + 2c - \gamma}{\gamma - c + \frac{1}{c}}\right\},$$

where $c = \frac{\sqrt{m_2}}{\mu_1}$ is a coefficient of variation, $\gamma = \frac{m_3}{(m_2)^{3/2}}$ is a coefficient of skewness, m_i is the i-th central moment of Z, $i = 2, 3$. The scale parameter takes the form

$$l_i = \frac{-B + (-1)^i \sqrt{B - 4AC}}{2A},$$

where $A = n(n + 2)\mu_1 y$, $B = -(nx + \frac{n(n+2)}{n+1} y^2 + (n+2)\mu_1^2 y)$, $C = \mu_1 x$, $y = \mu_2 - \frac{n+1}{n}\mu_1^2$, $x = \mu_1\mu_3 - \frac{n+2}{n+1}\mu_2^2$ and

$$p_1 = 1 - p_2 = \frac{\frac{\mu_1}{n} - \frac{1}{\lambda_2}}{\frac{1}{\lambda_1} - \frac{1}{\lambda_2}}.$$

We investigated in section 3 first two moments of $S(t)$, but now we need the third moment of it. The Laplace Transform of the third moment when the dependent structure is done by the Spearman copula is done by the formula

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$$\mu_3^*(\mathrm{p}) = \frac{1}{p^2\beta^3}\lambda(\alpha\beta^3(p+\lambda)(l^3)^*(p+\lambda) +$$
$$3(2-2\alpha+p\beta((2-2\alpha+\alpha\beta^2(p+\lambda)$$
$$(l^2)^*(p+\lambda))\mu_1^*(\mathrm{p}) + \beta(1-\alpha+\alpha\beta(p+\lambda)l^*(p+\lambda))\mu_2^*(\mathrm{p})))).$$

If the claims have the exponential distribution, then the third moment is equal to

$$\mu_3(t) = \frac{1}{\beta^3}e^{-t\lambda}(3\alpha(2\alpha(2+\alpha)-4-2t(1-\alpha)\alpha\lambda$$
$$+t^2(1-\alpha)^2\lambda^2) + e^{t\lambda}(6\alpha(2-\alpha(2+\alpha)) +$$
$$6t(1-3(1-\alpha)\alpha)\lambda + 3t^2(2-3\alpha)\lambda^2 + t^3\lambda^3)).$$

Example 4. Let the claims have the exponential distribution and $\lambda = 5$, $\beta = 0.01$, $t = 2$. For $\alpha = 0.4$ the probability distribution function of Z is equal to

$$f_Z(z) = 3.45546\times10^{-22}\, e^{-0.0178147\, z}\, z^8 + 9.0297\times10^{-24}\, e^{-0.0090181\, z}\, z^8.$$

Table 4 contains the values of the risk measures obtained by the simulation and approximation methods.

Table 4. The values of VaR and TVaR

	Simulation		Approximation	
α	VaR	TVaR	VaR	TVaR
0	2254.98	2484.62	2259.61	2514.63
0,2	2090.42	2309.79	2095.91	2317.78
0,4	1933.83	2129.65	1914.30	2097.68
0,6	1744.93	1921.56	1727.13	1877.39
0,8	1529.63	1677.39	1484.15	1586.05
1	999.01	999.51	1116.97	1151.50

Source: own elaboration.

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SOME RESULTS OF STOCHASTIC MODELLING OF INTEREST RATE IN LIFE INSURANCE

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1. Introduction

In the traditional actuarial literature, for simplicity, it is assumed that the rate of interest is fixed and the same throughout the years. However, the interest rate that will apply in the future years is, of course, neither known nor constant. Therefore, the stochastic interest rate to the actuarial calculations is applied in the researches. There are presented some results of the doctoral thesis (cf. [Marciniuk 2009]). Some of them have been not published yet.

2. Interest rate models classification

Two ways of stochastic modelling of interest rate: actuarial and financial, are applied. The actuarial and financial ways of modelling of interest rate are distinguished according to applying of interest rate model.

In the actuarial way the expected value and the variance of the discount value of benefit payment are determined in a traditional way, and the technical stochastic interest rate models are applied to the calculations. Therefore, firstly, the technical interest rate as the actuarial model of interest rate is introduced. The technical interest rate is determined by the following formula (cf. [Bowers et al. 1986])

$$i_{t_1,t_2} = \frac{K_{t_2} - K_{t_1}}{K_{t_1}} = \frac{K_{t_2}}{K_{t_1}} - 1 = v_{t_1,t_2}^{-1} - 1,$$

where K_t is a value of the capital at moment t, v_{t_1,t_2}^{-1} – the discounting function from date t_2 to date t_1 $\left(0 \le t_1 \le t_2\right)$.

In the financial theory four models of interest rate are distinguished, i.e.: a forward interest rate, a spot interest rate, an instantaneous forward rate, and an instantaneous spot rate. In this abstract two of them are applied, but three definitions must be introduced.

The instantaneous forward rate, in short the forward rate, is given by the formula (cf. [Musiela, Rutkowski 1988])

$$f_{t,T} = -\frac{\partial \ln P_{t,T}}{\partial T}, \qquad 0 \le t \le T,$$

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where $P_{t,T}$ is a price of zero-coupon bond with maturity T at moment t, $0 \le t \le T$.

The instantaneous spot rate, also called the short-term rate, is denoted by r_t and defined as follows (cf. [Jakubowski et al. 2003])

$$r_t = f_{t,t}.$$

The spot interest rate is defined by the following formula

$$R_{t,T} = -\frac{\ln P_{t,T}}{T-t}, \ 0 \le t \le T.$$

In the financial way benefit payments are treated as a stochastic cumulative cash flow $\{dC_t\}_{t \ge 0}$. The moments of the discounted value of these cash flows is valuated under the assumption that arbitrage is not possible. The following formula is used (cf. [Carriere 1999]):

$$E^{\mathbf{Q}}\left(D_t \middle| F_t\right) = E^{\mathbf{Q}}\left(\int_{(t,\infty)} \Lambda_t \Lambda_s^{-1} dC_s \middle| F_t \right),$$

where

$$\Lambda_{t,T}^{-1} = \exp\left(-\int_t^T r_s ds \right)$$

is the discounting process.

The following points are assumed (cf. [Carriere 2004]):

- $\{r_t\}_{t \ge 0}$ is a stochastic process of short-term rate,
- $\{r_t\}_{t \ge 0}$ is defined on a probability space $(\Omega, F, \mathbf{P})$,
- $\mathbf{P}$ is physical measure on a space with a associated history $F_t \subset F$ at time $t \ge 0$,
- $\{r_t\}_{t \ge 0}$ is adapted to the history F_t,
- $\int_t^T |r_s| ds < \infty, t \le T$,
- the another measure $\mathbf{Q}$ exists that is called the martingale measure,
- $\mathbf{Q}$ is equivalent to $\mathbf{P}$.

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3. Applying the deterministic function of spot interest rate

The first the case is considered, when the interest rate is described as a function of time *t*. Four models of this function are used (cf. [Marciniuk 2009]):

- Stoodley model (M.St),
- Nelson-Siegel model (M. N-S),
- Bliss model (M.B),
- Svensson model (M. Sv).

The parameters of these functions are estimated on the basis of real data, which follows from Polish market. The rate of return on Treasury bills and bonds with fixed interest rate from the date of 26.05.08 are used (cf. [http://bossa.pl/notowania/o/ciagle/obligacje/]). These data and the models of spot rate are presented on Figure 1.

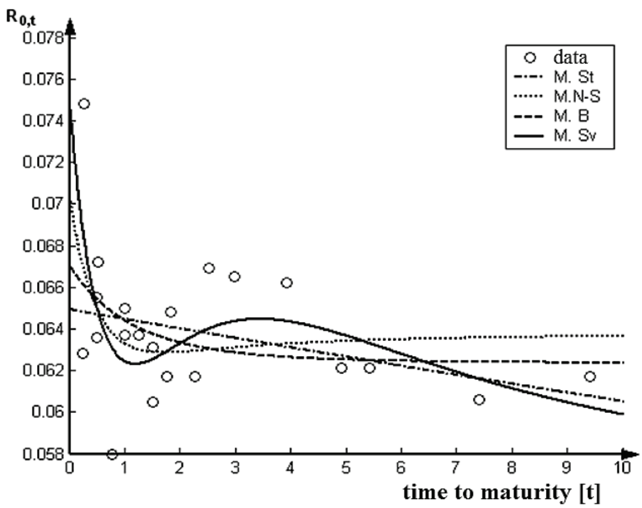

Figure 1. The data and the models of spot interest rate

Source: [Marciniuk 2009].

The results are presented on the basis of pure endowment policy, for 30-year-old women, when benefits is paid in the amount of 10 000 PLN and on the basis of the actuarial calculations in the case of the generalized life annuity payable m $(m > 0)$ times a year at the end of each m-th of the year. The conclusions are as follows:

- the differences in the amount of premium are very small, it is about a few pennies when the benefit is 10 000 PLN,
- the premium is the smallest for $n < 10$ in the case of Svensson function,
- the premium is the smallest for $n \geq 10$ in the case of Nelson-Siegel function,
- the similar results are for the standard deviation of the discounted value of benefit,
- when the benefit is paid at the end of the death day of insured, the premium is higher about 0,23% than the premium in the case when the benefit is paid at the end of month,
- the similar results are for life annuity and installment premium.

4. Applying the models of stochastic interest rate

Four stochastic processes are used as interest rate models in life insurances, i.e. Wiener process, autoregressive process of order one, Vasicek model and Cox-Ingersoll-Ross model.

To calculate the net premium the models parameters should be known. The short-term rate is not directly observed on the financial market. Hence, it must be estimated somehow, e.g. on the basis of WIBOR rate. The forward rate can be also determined on the basis of the treasury bills and the fixed interest bonds. Knowing the forward rate we can calculate the short-term rate. To make it simpler, it is assumed that the data of the short-term rate are known, i.e. the data are simulated. It has been assumed that it was the weekly data observed throughout 20 years (cf. [Marciniuk 2009]).

On the basis of these data the parameters of the short-term rate models have been estimated. In case of the Wiener process, AR(1) process and Vasicek model the maximum likelihood method was used (cf. [Chan et al. 1992]). The general method of moments has been applied in case of CIR model. The packet Solver in Excel program has been used for the estimation. The results of the estimation are as follows:

- AR(1) process – *actuarial model*
  - $d\delta_t = 0.05524 + 0.84598(\delta_{t-1} - 0.05524) + \varepsilon_t$,
    - $\varepsilon_t \sim N(0;0.009375), \delta_0 = 0.04845$,
- Wiener process – *actuarial model*
  - $d\delta_t = 0.0052 dB_t, \delta_0 = 0.04845$,

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- Vasicek model – *financial model*
  - $dr_t = -8.67(r_t - 0.055)dt + 0.04dB_t$,
- CIR model – *financial model*
  - $dr_t = (0.06218 - 1.1254r_t)dt + 0.32\sqrt{r_t}dB_t$.

The actuarial calculations are presented in the case of the generalized life annuity payable m ($m > 0$) times a year at the end of each m-th of the year for women at age 30 and when benefit is paid at height 10 000 PLN. The best model is Vasicek model, so the actuarial values of life annuity are presented in Table 1 only for this model.

Table 1. The actuarial value of life annuity in the case of Vasicek model

n	m = 1 (yearly)	m = 2 (half-yearly)	m = 4 (quarterly)	m = 12 (monthly)	m = 365 (daily)	m = 8760 (hourly)
2	1.947	1.920	1.907	1.898	1.894	1.894
4	3.689	3.639	3.614	3.598	3.590	3.589
6	5.249	5.177	5.141	5.118	5.106	5.106
8	6.644	6.553	6.508	6.478	6.463	6.463
10	7.891	7.783	7.729	7.694	7.676	7.676
12	9.006	8.882	8.821	8.780	8.761	8.760
14	10.002	9.864	9.796	9.751	9.729	9.728
16	10.891	10.740	10.666	10.616	10.593	10.592
18	11.683	11.522	11.442	11.388	11.363	11.362
20	12.389	12.217	12.132	12.076	12.048	12.047

Source: cf. [Marciniuk 2009].

Moreover, we can see in Table 2 the yearly value of net periodic premiums for pure endowment policy for 30-year-old women, when benefit is paid at height 10 000 PLN.

The smallest premium is also for the Vasicek model.

Table 2. The net periodic premiums

model	m = 1 (yearly)	m = 2 (half-yearly)	m = 4 (quarterly)	m = 12 (monthly)	m = 365 (daily)
n =10					
AR(1)	730.19	740.25	745.34	748.76	–
Wiener	757.81	767.05	771.72	774.86	776.39
Vasicek	**726.34**	**736.42**	**741.53**	**744.97**	**746.64**
CIR	897.59	902.62	905.10	906.73	907.53
n =20					
AR(1)	265.03	268.69	270.55	271.80	–
Wiener	291.25	294.78	296.56	297.76	298.34
Vasicek	260.90	264.57	266.42	267.67	268.28
CIR	403.01	405.15	406.22	406.93	407.27

Source: [Marciniuk 2009].

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5. Results and conclusions of applying of stochastic interest rate models

The best of the presented models is the Vasicek model. Similar results were obtained for the AR (1) process. Hovewer, the standard deviation of the discounted value of the benefit is smaller in the first case for all types of insurance. The worst results are obtained for the CIR model. The actuarial values can be calculated for each $t \geq 0$ in the case of the Vasicek model. However, the partition of the year into more than 12 parts does not cause a significant increase in premiums. It makes no sense to pay premiums or benefits more than once a month. Therefore, the use of continuous models, the interest rate is not necessary. In the case of the CIR model and the Wiener process premiums are higher. Theoretically, the insurance company could choose these models, however, the standard deviation of the discounted value of the benefit is also higher in these cases. This increases the risk of incurring higher losses for the insurer. If the premiums are calculated assuming a fixed interest rate, this interest rate should be equal to the long-term interest rate.

In the case of financial modelling of interest rate a zero-coupon bond prices can be taken from the market. This is an advantage. But the premium for the persons at the same age can be different every day. In addition, there are not too many of these bonds on the Polish market, which makes it impossible to calculate the actuarial values of life annuities. Therefore the price of bonds has to be modelled. In order to calculate the standard deviation of the discounted value of benefit the second moment of the discounted process has also to be modelled.

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THE EMERGENCE OF STATISTICAL SCIENCE

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In 1581 G. Ghislini in his work *Ristretto della civile, politica, statistica e militare scienza* for the first time used the expression "statistical science" (*statistica scienza*) understanding it as a description of the state matter. In 1999 J. Nelder in his paper *From statistics to statistical science* argued that the subject should be renamed statistical science. The aim of this paper is to review a long way of that transformation. As almost all modern sciences, statistics has also its roots in an ancient Greek. It was Aristotle who for the first time described constitutions of 158 Greek cities. Only the constitution of Athens survived till our times. This work was the pattern for the Latine lecture given by H. Conring in Helmsted. Achenwall continued his work, but using German language, he also coined a word "Statistik". Achenwall's book had seven editions and was translated in almost all European languages. In spite of the great popularity the Conring-Achenwall statistics prior to 1800 had an interest "only to antiquarians". English style of statistics, under the name of political arithmetic, became a dominant in Europe. It was initiated by J. Graunt, whose work has been imported to Germany by Süssmilch. The novelty of English school was to emphasize the method rather than description. Graunt's approach was fully scientific: following Baconian philosophy, starting from observations he discovered regularities in social life. The most significant (a crucial step in the taming of chance-in Hacking's words) has been done by A. Quételet. Proclaiming the universality of the rule: *mundum regunt numeri*, he formulated his famous doctrine of statistical law. The basis for this doctrine was the distinction between the constant forces of nature and perturbational one. Statistical determinism and *Qutelesimus* were however criticized in Germany. Statistics as a science of variability and uncertainty needed appropriate theory for the quantification of these concepts. The path breaking contribution in this direction is due to J. Bernoulli's *Ars conjectandi*, alternatively named by him as *stochastics*. Nowadays, in

German-speaking environment this expression is meant as the amalgam of probability and statistics. According to D.V. Lindley (and many others) statisticians in the twenties changed their paradigm (in a Kuhn sense). That change is due to the Fisher revolution, who also distinguished statistics of science and statistics of market-place, i.e. the field of decision making. According to the same Lindley all statisticians will form a united profession in 2020 when all of them will be Bayesians.

IS THE TEST FOR ASYMMETRIC PRICE TRANSMISSION IN A TAR- OR MTAR-FRAMEWORK BIASED BY REQUIRING EVIDENCE FOR COINTEGRATION?

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1. Introduction

Almost all economic processes, like production, refinement or trading, involve some kind of transmission of “input prices” to “output prices”. For example, in the gasoline market the crude oil price is “transmitted” to the end user price at the gas stations. Such a price transmission is said to be asymmetric, if its characteristics differ between periods of increasing and decreasing prices. For example, it is frequently suspected that the oil refining companies, due to their market power, tend to transmit increases in crude oil prices faster to the price at the gas station than they lower the end user price after a decrease. In standard economic theory such an “asymmetric price transmission” (APT) is considered to be the result of a market failure, which should be avoided.

Various statistical methods have been developed to test, based on historical times series of the two prices, if a significant asymmetry in the price transmission can be detected. All approaches are faced with the problem that the price series usually follow integrated, i.e. non-stationary processes. Instead of resorting to models that include short-term changes of the prices only, the cointegration approach focuses on a long-term relationship between the two price series, which, unlike the price series themselves, follows a stationary process. Such a “cointegrating relationship” should exist for economic reasons (there must be something in the price transmission process that keeps the

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two prices tied together), although its precise form may be hard to identify. In the original approach of Engle & Granger [1987], the only type of cointegrating relation allowed was a static linear equation whose stationarity was assessed by a single reversion rate. As this is unable to capture asymmetries in the price transmission, Enders & Granger [1998], Enders [2001] and Enders & Siklos [2001] devised a concept of asymmetric cointegration which allows the cointegrating relation to revert to its long-term equilibrium with two different reversion rates, depending on whether the deviation from the equilibrium is above or below some threshold (TAR-model). A second threshold model was also introduced as the mTAR-model, where the "deviation from the equilibrium" is replaced by its short-term change. The mTAR-model covers, for example, a price transmission where the adjustment rate of the output price depends on whether the input price decreased or increased. Given cointegration with the TAR- or mTAR-model, a formal test for asymmetry is easily devised as an F-test of equality of the two reversion rates.

The Enders/Granger/Siklos approach ends up with a hierarchy of two tests: The primary aim is to reject the null hypothesis "no cointegration" in the test for cointegration. Only if this aim is achieved – i.e. only if there is sufficient evidence for cointegration in the data – will the test for asymmetry be performed. This might not result in a serious problem, if evidence for cointegration were easy to obtain. However, the test for cointegration has a very low power to detect cointegration, so that the test for asymmetry is performed in very special situations only. In this paper I will demonstrate by means of simulations that – at least with the mTAR-model – this causes the test for asymmetry to excessively reject its null hypothesis "symmetry": By insisting on evidence for cointegration one obtains a tendency to false detection of asymmetry. The bias (in the same direction, i.e. towards false detection of asymmetry) becomes dramatic, if the method suggested in Enders & Siklos [2001] to select an optimal switching threshold is applied. This holds for both, the TAR and mTAR-model. Thus, under the threshold optimization, the test for asymmetry in my opinion is (not only biased, but) more or less meaningless.

The performance of the test for asymmetry in an Engle/Granger-like procedure has been investigated in other simulation studies. Von Cramon-Taubadel & Meyer [2001] also diagnose a bias towards excessively rejecting the null hypothesis "symmetry". However, these

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authors deal with the effect of structural breaks, which (from the perspective of the test) is a misspecification, in that the simulated data come from a model the test was not designed for. Hence, their result does not render the test invalid – as in our simulation, where the bias does not originate from a misspecification, but from the implicit conditioning on evidence for cointegration.

There are also papers which point in the (opposite) direction of a tendency to stick with the symmetry hypothesis. The authors usually aim at the conclusion "asymmetry", which they find difficult to obtain with Engle/Granger-like procedures. For example, Galeotti et al. [2003] state (without further reference or simulation) that the tests for asymmetry "are biased toward accepting the null of symmetry in small samples. This fact could explain why the data fail to turn up the asymmetric price adjustments that many commonly suspect." Actually, the authors do not criticize the test for a bias (as we do), but for a low power to detect asymmetry. Honarvar [2010] conducts simulations which support this criticism. His results do not necessarily contradict ours, as he simulates asymmetry in the data (and then finds low power to detect the simulated asymmetry), while we simulate symmetry in the data (and then find excessive rejections of the simulated symmetry). Although both types of errors should be taken into consideration, a low power does not render the test invalid, while excessive rejections of the null hypothesis do. It should be mentioned that Honarvar allows for more types of asymmetric cointegration than the restrictive Enders/Granger/Siklos-model (from whose perspective, the model is again misspecified) and suggests a different method for estimation and testing to account for this. It seems, however, that the suggested method retains the implicit conditioning on evidence for cointegration. Thus, while he aims at improving the ability to detect asymmetry, the problem of excessive rejections of the null of symmetry, invalidating the test, might persist in his approach.

2. Models and tests

The cointegration test suggested by Enders/Granger/Siklos is best understood in the context of the Engle/Granger procedure (EG-procedure), whose aim is to provide evidence for a static linear cointegrating relationship between two integrated time series x_t and y_t (where the cointegration is assessed by a single reversion rate). The EG-procedure first regresses y_t on (a const. +) x_t and then essentially

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performs an (A)DF-test with the residual series z_t. That is, one tests if it is possible to reject

$$H_{null}: \Delta z_t = \varepsilon_t \qquad (+\sum_{j=1}^{k} \gamma_j \Delta z_{t-j}) \quad \text{where} \quad \varepsilon_t : i.i.d., E[\varepsilon_t] = 0,$$

in favor of

$$H_{alt}: \Delta z_t = \rho(z_{t-1} - \overline{z}) + \varepsilon_t \qquad (+\sum_{j=1}^{k} \gamma_j \Delta z_{t-j}) \quad \text{where} \quad \rho < 0 .$$

For the implementation one regresses Δz_t on z_{t-1} (+ constant + lags) and uses the t-statistic of the coefficient ρ of z_{t-1} to decide, if $\rho = 0$ can be rejected in favor of $\rho < 0$. A subtle, but important point is that under H_{null} ($\rho = 0$) the OLS-estimate has non-standard behavior, so that the *t*-statistic does not follow a *t*-distribution (even with normal ε_t or asymptotically). To account for this, the critical values for rejection of $\rho = 0$ must be more extreme (larger in absolute value) than those coming from the appropriate *t*-distribution.

One can interpret the EG-procedure as searching for a sufficiently strong tendency for mean reversion in the residuals z_t, in order to be able to reject the null hypothesis of "no cointegration". As the critical values are more extreme than in a scenario with stationary time series, an extraordinarily strong mean reversion in the cointegration residuals is required for rejection, i.e. to arrive at the conclusion "cointegration". This has the drawback that the power (to detect cointegration) is very low.

To allow for an asymmetry in the reversion rate ρ, Enders/ Granger/Siklos (see [Enders, Siklos 2001]) use a TAR-model instead of the ADF-model in the second step of the EG-procedure: The reversion rate ρ is allowed to switch between two different values ρ_+, ρ_-, depending on whether the past residual z_{t-1} is above or below some threshold τ. Enders [2001] and Enders & Siklos [2001] also introduce the so-called mTAR-model, where the switching decision is based on the change Δz_{t-1} rather than z_{t-1}. Thus, they test if it is possible to reject

$$H_{null}: \Delta z_t = \varepsilon_t \qquad (+\sum_{j=1}^{k} \gamma_j \Delta z_{t-j}) \quad \text{where} \quad \varepsilon_t : i.i.d., E[\varepsilon_t] = 0,$$

in favor of

$$H_{alt}: \Delta z_t = \rho_+ \cdot [I_t(z_{t-1} - \overline{z})] + \rho_- \cdot [(1 - I_t)(z_{t-1} - \overline{z})] + \varepsilon_t \quad (+\sum_{j=1}^{k} \gamma_j \Delta z_{t-j})$$

where $\rho_+ < 0$, $\rho_- < 0$ and

$$I_t := \begin{cases} 1 & z_{t-1} \geq \tau \\ 0 & z_{t-1} < \tau \end{cases} \quad (\text{TAR}(\tau) - \text{model})$$

$$\text{or} \quad I_t := \begin{cases} 1 & \Delta z_{t-1} \geq \tau \\ 0 & \Delta z_{t-1} < \tau \end{cases} \quad (\text{mTAR}(\tau) - \text{model})$$

For a given threshold τ, this test can be implemented by regressing Δz_t on $z_t^+ := I_t(z_{t-1} - \overline{z})$, $z_t^- := (1 - I_t)(z_{t-1} - \overline{z})$ ($+lags(\Delta z_{t-j})$) and using the *F*-statistic for H_{null}: $\rho_+ = 0, \rho_- = 0$. Again, the critical values for rejection of $\rho_+ = 0, \rho_- = 0$ are (much) larger than those coming from the appropriate *F*-distribution. For example, the critical value for the 5%-level with $T = 100$ observations is roughly 6, instead of roughly 3. Note that the aim of this test is the same as that of the EG-procedure: To provide evidence for (a more general form of) cointegration. It is somewhat surprising that – even under situations where $\rho_+ \neq \rho_-$ – the power of the test (to detect cointegration) is even lower than that of the original EG-procedure. This means that the test does not achieve its primary aim (to detect cointegration) very well. Given that cointegration ($\rho_+ < 0 \wedge \rho_- < 0$) is confirmed, the test for asymmetry amounts to an *F*-test of $H_0: \rho_+ = \rho_-$, using standard critical values. Note that usage of these critical values assumes cointegration.

In order to improve the ability of the cointegration test to detect cointegration, Enders [2001] and Enders & Siklos [2001] suggested to select the threshold τ by optimizing the AIC of the regression for ρ_+, ρ_- (which is equivalent to maximizing its R^2 and simply means to improve the fit of the cointegration model to the data). In practice, this is achieved by estimating many regressions with fixed τ, where τ runs through to all values of z_t (TAR) or Δz_t (mTAR) (usually a lateral trimming is applied, to avoid τ^* becoming too close to the extreme values). Enders [2001] claims that this is an implementation of a superconsistent TAR-model estimator suggested by Chan [1993)]

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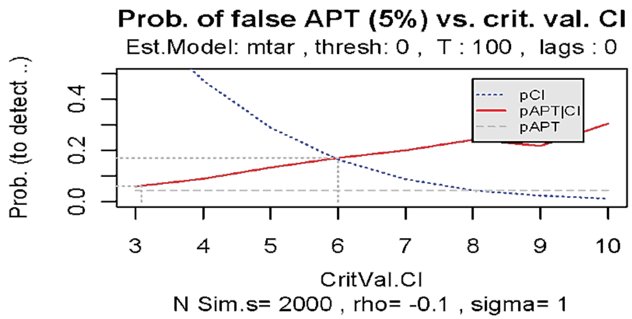
3. Simulations

In order to investigate the dependency of the test for cointegration and the test for asymmetry, we simulate a large number of time series x_t and y_t which cointegrate with symmetric reversion in the cointegration residuals z_t. Thus, the correct conclusions would be to reject the null hypothesis of "no cointegration" (decide "cointegration"), but to not reject the null hypothesis of "no asymmetry" (retain "symmetry"). We report the probability that the symmetry test, performed at the 5%-level, falsely detects asymmetry, given that cointegration was (correctly) detected. That is, we ignore all cases, where the Enders/Granger/Siklos test failed to detect cointegration, as the test for asymmetry is invalid without cointegration (in practice, it is likely that the study based on these data would be aborted, because there was no evidence for cointegration). In the remaining cases, the frequency of (falsely) rejecting the null hypothesis "symmetry" should be below the prescribed significance level of 5%. The curve labelled pAPT | CI in the diagrams shows these probabilities as a function of the critical *F*-value, CritVal.CI, that was used in the test for cointegration. As mentioned above, the valid critical values for the cointegration scenario are much larger than the classical ones from the appropriate *F*-distribution: The valid 5%-critical value is around 6, while the classical one turns out to be slightly above 3. The diagramms also plot the probablitity of (correctly) detecting cointegration as a function of CritVal.CI (dotted curves labelled pCI). For the valid critical value ($\approx$ 6 at 5%-level), this is the power of the test. In all of the simulations shown the length of the time series was $T = 100$ and the (symmetric) model for the CI-residuals z_t was

$$\Delta z_t = \rho z_{t-1} + \varepsilon_t, \qquad \text{where} \quad \varepsilon_t \text{ is i.i.d.: } N(0,\sigma^2),$$
$$(\rho < 0, \text{ but quite close to } 0)$$

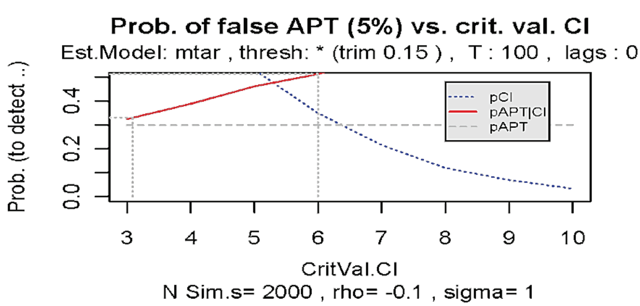
where $\rho = -0.1$ and $\sigma = 1$. Note that we thereby completely conform to the modelling framework of Enders/Granger/Siklos: no misspecification; normally distributed i.i.d. error terms (in particular: no serial correlation in error terms); the number of lags Δz_{t-j} is 0, which is known in advance. In all diagrams 2000 data sets were simulated.

We start by estimating an **mTAR model with a prescribed threshold of $\tau = 0$:**



The results clearly indicate that the combined test much too often falsely finds “asymmetry”. This would already hold when using the classical critical value for detecting cointegration (around CritVal.CI ≈ 3), although here the probability of an α-error in the test of symmetry would be only slightly above the prescribed 5% level. But with the valid critical values for the cointegration test, the actual probability of an α-error for the symmetry test lies above 15%, more than three times as high as the pretended significance level. The striking point here is not that there is a mismatch at all (it is not surprising that in a repeated testing the test which is performed conditional on the outcome of a “primary” test does not match its pretended significance level, except the test statistics were independent of each other), but that the mismatch is always in favor of (falsely) detecting asymmetry and that this discrepancy increases with more evidence for cointegration (larger CritVal.CI). With the large critical values required for the cointegration test to be valid, a considerable amount of excessive rejections of the null hypothesis “symmetry” arises. The diagram also confirms the small power of the cointegration test: At the 5%-level it detects less than 20% of the cointegrations (see curve pCI above CritVal.CI ≈ 6).

We next estimate the **mTAR-model with optimal threshold** $\tau = \hat{\tau}^*$ (with a lateral trimming of 15% to both sides):



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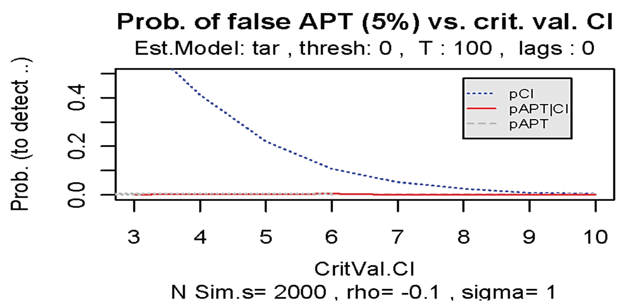
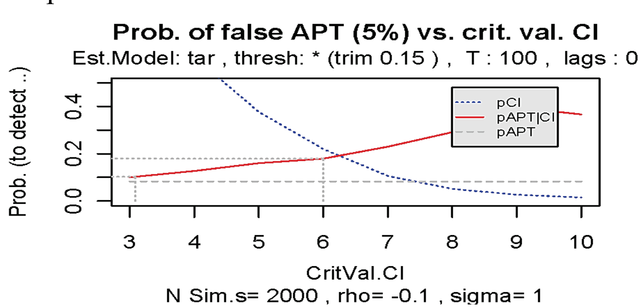
The results are qualitatively similar to those of the previous simulation, but much more pronounced. The APT-test at the 5%-level now (falsely) detects asymmetry in more than half of the cases where evidence for cointegration is found at the 5%-level. At the same time, the ability to (correctly) detect cointegration has improved. The reason for the dramatically increased rate of false rejections of the symmetry hypothesis is that the search for an optimal fitting threshold tends to locate the threshold asymmetrically, i.e. close to the borders of the trimming area.

The result for the **TAR-model with prescribed threshold** $\tau = 0$ is suprisingly different:



We now have almost no false asymmetry classifications (but an even lower power to detect cointegration). I do not try to explain the good performance of the test for asymmetry here, but mention that results like this are only obtained with a perfectly matching specification of the estimated model. If, for example, the TAR-model is estimated with one or more asymmetric lags (which is not a false specification, but an over-specification), the results for the TAR-model become very similar to those of the mTAR-model (too many classifications of "asymmetry").

The results for the **TAR-model with optimal threshold** $\tau = \hat{\tau}^*$ (with lateral trimming of 15%) also resemble those of the mTAR-model with optimal threshold:



4. Conclusions

It was shown that the test of symmetry in the Enders/Granger/Siklos procedure for asymmetric cointegration is confounded towards indicating asymmetry by requiring the data to provide sufficient evidence for cointegration ("Evidence for cointegration provokes artificial asymmetry"). This bias is strongly enlarged by selecting an optimal fitting threshold ("optimizing over the threshold provokes artificial asymmetry"). For purposes of demonstration, we selected scenarios with a substantial bias, which are characterized by a small sample size and relatively small reversion rates. However, except for the TAR-model with perfect compliance of the simulated data to the model specification, a bias in the same direction was observed for all other scenarios simulated (although it tends to become ignorably small with sufficiently large sample sizes and/or reversion rates). The same kind of confoundedness towards asymmetry detection might not be specific to the Ender/Granger/Siklos procedure alone, but latent in other methods for asymmetric cointegration too.

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