MATRIX APPROACH TO ANALYSIS OF A PORTFOLIO OF MULTISTATE INSURANCE CONTRACTS

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Summary: This paper discusses the calculation of moments of cash value of future payment streams arising from portfolio of multistate insurance contracts, where the evolution of the insured risk and the interest rate are random. A matrix form for formulas for the first two moments of cash value of the stream of future payments for a portfolio of policies is derived. As an application formulas for insurance premiums are provided. The general theory is illustrated with a case where the rate of interest is modeled by a Wiener and an Ornstein-Uhlenbeck process.

Keywords: multiple state model, cash value of the payment stream, stochastic interest rate, portfolio of policies, modified multiple state model.

1. Introduction

Multiple state modelling is a classical stochastic tool for designing and implementing insurance products. The multistate methodology is intensively used in calculation of premiums and reserves of different types of insurances like life, disability, sickness, marriage or unemployment insurances.

In this paper we focus on the discrete-time model, where insurance payments are made at the ends of time intervals. It practically means that immediate annuity and insurance benefits are paid immediately before the end of the unite time (for example: year or month). Premiums and annuity due are paid immediately after the beginning of the unite time.

The aim of this paper is to give a general approach which can be used to calculate the moments of cash value of future payment streams arising from a portfolio of multistate insurance contract with the finite term of policy. These formulas are usually quite extensive, thus we propose to employ a matrix notation which was used in [3] for the

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analysis of single policies. This notation not only makes calculations easier, but also provides a nice form for important actuarial values both for a heterogeneous and homogenous portfolio. Moreover the matrix form enables us to factorize the double stochastic nature of moments of the cash value of future payment stream.

Note that premium is risk-rated for individuals and groups on the basis of characteristics such as age, sex, occupation and previous health conditions. It also depends on the largeness of a portfolio. It is more difficult to diffuse risks in small groups (like an individual insurance contract) than in large groups. When a portfolio gets larger, the average expected cash value of payment streams is less likely to vary. It appears that the matrix notation helps in analyzing the influence of the policies portfolio size on moments of average payment streams per policy.

The paper is organized as follows. After a brief description of modified multiple state model, we distinguish between different types of cash flows (premiums and benefits) arising from modified multi-state insurance contracts (Section 2). Then in Section 3 we present a matrix form expressions for the first two moments of cash value of the stream of future payments arising from a single policy. Section 4 deals with the analysis of a portfolio of insurance contracts. Some applications including formulas for premiums are provided in Section 5.

2. Modified multiple state model

Following Dębicka [2012] with a given insurance contract we assign a *modified multiple state model*. That is, at any time the insured risk is in one of a finite number of states labelled $1,2,\ldots,N^*$. Let $S^* = \{1,2,\ldots,N^*\}$ be the *state space*. Each state corresponds to an event which determines the cash flows (premiums and benefits). In particular a state may represent such an event as death, disablement, recovery, unemployment, etc. Additionally, by T^* we denote the set of *direct transitions* between states of the state space. Thus T^* is a subset of the set of pairs (i,j) , i.e., $T \subseteq \{(i, j) \mid i = j; i, j \in S^*\}$. The pair (S^*, T^*) is called a *modified multiple state model*.

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In this paper we consider an insurance contract issued at time 0 (defined as the time of issue of the insurance contract) and according to plan terminating at a later time n (n is the term of policy). We focus on discrete-time model. Let $X^*(t)$ denote the state of an individual (the policy) at time t ($t \in T = \{0, 1, 2, \ldots, n\}$). Hence the evolution of the insured risk is given by a discrete-time stochastic process $\{X^*(t); t \in T\}$, with values in the finite set S^*.

The individual's presence in a given state or movement (transition) from one state to another may have some financial impact. We distinguish between the following types of cash flows related to modified multiple state insurance:

– $p_j(k)$ – a period premium amount at time k if $X(k) = j$),
– $\pi_j(k)$ – a premium amount at some fixed time k if $X(k) = j$,
– $b_j(k)$ – an annuity benefit at time k if $X(k) = j$,
– $d_j(k)$ – a lump sum at some fixed time k if $X(k) = j$ (for instance pure endowment),
– $c_{ij}(k)$ – a lump sum at time k if a transition occurs from state i to state j at that time (for discrete-time model it means that $X(k-1) = i \neq j, X(k) = j$ and $X(k+1) \neq j$).

Thus for the modified multistate model all types of cash flows are connected with staying of the process $\{X^*(t)\}$ in a considered state, although the c's correspond to cash flows connected with transitions between states. This situation occurs because of procedure of the modification of multistate model and assumption that a lump $c_j(k)$ sum does not depend on state i and $P(X^*(k+1) = j | X^*(k) = j) = 0$.

Therefore pair (S^*, T^*) not only describes possibilities pertaining to an insured risk, but also informs which transitions the lump sum are connected with, as far as its evolution is concerned.

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3. Moments of the cash value of future payment streams

Let $cf_j^*(k)$ be the future cash flow payable at time k (k = 0, 1, ..., n) if the process $\{X^*(t)\}$ is in that time in state j ($j = 1,2,\dots,N^*$)

$$cf_j^*(k) = cf_{X^*(k)=j}(k)$$

Since at each time k , all of the above types of cash flows may occur, then

$$cf_j^*(k) = p_j(k) + \pi_j(k) + b_j(k) + d_j(k) + c_j(k). \qquad (1)$$

Let C represents the total payments made up to the end of the term of policy with respect to the individual insurance contract, i.e.,

$$C = \sum_{k=0}^{n} cf_{X^*(k)}.$$

Moreover, let $Y(t)$ denote the rate of interest in time interval [0, t]. Then the discount function $v(t)$ is of the form $v(t) = e^{-Y(t)}$. Thus, the cash value Z of the total payment stream has the following form

$$Z = \sum_{k=0}^{n} cf_{X^*}(k)e^{-Y(k)}. \qquad (2)$$

Let us observe that Z is a random variable, which has a double stochastic nature. It depends on process $\{X^*(t)\}$ and the stochastic interest rate $Y(t)$.

We denote by Z_l the cash value of total payment stream to be paid with respect to the insurance contract for the l -th policy (when its realization is described by process $\{X_l^*(t)\}$), which belongs to portfolio of L insurance policies ($l \in \{1, 2, ..., L\}$).

In order to study the moments of Z we make the following assumptions (see also [4] and [8]):

Assumption 1. The random variables X_l^* for l = 1, 2, ... are independent and identically distributed.

Assumption 2. Conditional on knowing the values of $Y(k)$ for k = 0, 1, 2, ..., n , the random variables Z_l are independent and identically distributed.

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Assumption 3. The random variables X_l^* for l = 1, 2, ... are independent of $Y(t)$.

Assumption 4. All moments of the random discounting function $e^{-Y(t)}$ are finite.

Note that for a portfolio of L policies, the random variables $Z_1, Z_2, \dots, Z_L$ are not independent, since the discount function is the same for each Z_l.

Before presenting formulas for two first moments of Z_l we need to introduce some matrix notation.

Let

$$\begin{aligned} \mathbf{S} &= (1,1,\dots,1)^T \in \mathrm{R}^{N^*} \\ \mathbf{J}_j^* &= (0,0,\dots,\underbrace{1}_{j},\dots,0)^T \in \mathrm{R}^{N^*} \\ \mathbf{I}_{k+1} &= (0,0,\dots,\underbrace{1}_{k+1},\dots,0)^T \in \mathrm{R}^{n+1} \end{aligned}$$

for each $j = 1,2,\dots,N^*$ and k = 0, 1, 2, ..., n .

By $diag(\mathbf{A})$, where $\mathbf{A} = (a_1, a_2, \dots a_{N^*})^T \in \mathrm{R}^{N^*}$ we denote a diagonal matrix

$$diag(\mathbf{A}) = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{N^*} \end{pmatrix}.$$

Furthermore, for any matrix $\mathbf{B} = (b_{ij})_{i,j=1}^{n+1}$ let $Diag(\mathbf{B})$ be a diagonal matrix

$$Diag(\mathbf{B}) = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{n+1\,n+1} \end{pmatrix}.$$

Additionally, in order to describe the probabilistic structure of $\{X^*(t)\}$, for any moment $k \in$ {1, 2, ..., n} let $p_j^*(k) = \mathrm{P}(X^*(k) = j)$. We introduce

$$\mathbf{P}(k) = \left(p_1^*(k), p_2^*(k), \dots p_{N^*}^*(k)\right)^T \in \mathrm{R}^{N^*}$$

and

$$\mathbf{D} = \begin{pmatrix} \mathbf{P}(0) \\ \mathbf{P}(1) \\ \vdots \\ \mathbf{P}(n) \end{pmatrix} \in \mathrm{R}^{(n+1)\times N^*} \tag{3}$$

Moreover for any $k_1, k_2 \in \{0, 1, 2, \dots, n\}$ let

$$\mathbf{P}(k_1, k_2) = \left(p_{ij}^*(k_1, k_2)\right)_{i,j=1}^{N^*},$$

where $p_{ij}^*(k_1, k_2) = \mathrm{P}(X^*(k_1) = i, X^*(k_2) = j)$. In case of $\{X^*(t)\}$ being modelled by a Markov chain, a precise description of the construction of matrices **D** and $\mathbf{P}(k_1, k_2)$ is given in [Dębicka 2002] and [Dębicka 2012].

Let **C** denote $(n + 1) \times N^*$ cash flows matrix defined by

$$\mathbf{C} = \begin{pmatrix} cf_1^*(0) & cf_2^*(0) & \cdots & cf_{N^*}^*(0) \\ cf_1^*(1) & cf_2^*(1) & \dots & cf_{N^*}^*(1) \\ \vdots & \vdots & \ddots & \vdots \\ cf_1^*(n) & cf_2^*(n) & \cdots & cf_{N^*}^*(n) \end{pmatrix}.$$

The following notation is useful to describe moments of rate of interest. Let

$$\mathbf{Y} = \left(e^{-Y(0)}, e^{-Y(1)}, \dots, e^{-Y(n)}\right)^T \in \mathrm{R}^{n+1}.$$

and

$$\mathbf{M} = \mathrm{E}(\mathbf{Y}) = (m_0, m_1, \dots, m_n)^T \in \mathrm{R}^{n+1},$$

with $m_k = \mathrm{E}\left(e^{-Y(k)}\right)$. By $\mathbf{R} = \left(r_{ij}\right)_{i,j=1}^{n+1}$ we denote covariance matrix of random vector **Y**, where

$$r_{ij} = \mathrm{Cov}\left(e^{-Y(i)}, e^{-Y(j)}\right).$$

Furthermore, let $\Delta = \left(\delta_{ij}\right)_{i,j=1}^{n+1}$, where

$$\delta_{ij} = \mathrm{E}\left(e^{-Y(i)} e^{-Y(j)}\right) = r_{ij} + m_i m_j.$$

Note that the following identity holds:

$$\Delta = \mathbf{R} + \mathbf{M}\,\mathbf{M}^T.$$

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We refer to [Dębicka 2003] for the exact forms of matrices **M**, **R** and Δ when $Y(t)$ is modelled by Ornstein-Uhlenbeck process or Wiener process.

In the following theorem we present a matrix form for the first two moments of Z (see [Dębicka 2012]).

Theorem 1. For the modified multistate model ($\boldsymbol{S}^*$, $\boldsymbol{T}^*$), if Z, Y (t) satisfy Assumption 1-Assumption 4, then

$$\mathrm{E}(Z) = \mathbf{M}^T Diag(\mathbf{CD}^{\mathrm{T}})\mathbf{S}, \tag{4}$$

$$\mathrm{E}(Z^2) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{n} \mathbf{I}_{k_2+1}^T \Delta^T \mathbf{I}_{k_1+1} \mathbf{I}_{k_1+1}^T \mathbf{CP}(k_1, k_2) \mathbf{C}^T \mathbf{I}_{k_2+1}, \tag{5}$$

$$\mathrm{E}(Z_1, Z_2) = (Diag(\mathbf{CD}^{\mathrm{T}})\mathbf{S})^T \Delta (Diag(\mathbf{CD}^{\mathrm{T}})\mathbf{S}). \tag{6}$$

The matrix form presented in Theorem 1 factorizes the double stochastic nature of Z. Matrices **D** and $\mathbf{P}(k_1, k_2)$ depend only on the distribution of process $\{X*(t)\}$, while **M** , Δ and **R** depend only on the interest rate. Moreover, matrix **C** depends on cash flows and describes the type of the insurance contract.

4. Moments of Z for a portfolio of policies

We consider a heterogeneous portfolio of L policies, where for the l-th policy in the group ($l = 1, 2, ..., L$) we have

- x_l – age at entry,
- n_l – term of policy,
- $\{X_l^*(t)\}$ – the random state occupied by l-th insured at time t,
- $\mathbf{D}_l$ and $\mathbf{P}_l(k_1, k_2)$ – probability vectors described on the distribution process $\{X_l^*(t)\}$,
- $\mathbf{C}_l$ – cash flows matrix.

We assume that the term of each policy in the group is the same and equals n .

Let $Z_{(L)}$ be the random variable representing the total cash value of cumulative payment streams with respect to the entire portfolio of L policies, that is

$$Z = \sum_{l=1}^{L} Z_l.$$

where Z_l denotes the cash value of cumulative payment streams for l-th insured with respect to the multistate insurance contract.

Theorem 2. For the modified multistate model ($\boldsymbol{S}^*$, $\boldsymbol{T}^*$), if Z, Y (t) satisfy Assumption 1 - Assumption 4, then

$$\mathrm{E}\left(Z_{(L)}\right) = \bar{\mathbf{C}}_\Sigma^T \mathbf{M},$$

$$\mathrm{Var}\left(Z_{(L)}\right) = \bar{\mathbf{C}}_\Sigma^T \mathbf{R} \bar{\mathbf{C}}_\Sigma - \sum_{l=1}^{L}\left(\bar{\mathbf{C}}_l^T \Delta \bar{\mathbf{C}}_l - \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} \mathbf{I}_{k_2+1}^T \Delta^T \mathbf{I}_{k_1+1} \mathbf{I}_{k_1+1}^T \mathbf{C}_l \mathbf{P}_l(k_1,k_2) \mathbf{C}_l^T \mathbf{I}_{k_2+1}\right),$$

where $\bar{\mathbf{C}}_l = Diag(\mathbf{C}_l \mathbf{D}_l^T)\mathbf{S}$ and $\bar{\mathbf{C}}_\Sigma = \sum_{l=1}^{L} \bar{\mathbf{C}}_l = diag(\sum_{l=1}^{L} \mathbf{C}_l \mathbf{D}_l^T)\mathbf{S}$.

Proof. The idea of the proof is analogous to the proof of Theorem 2 in [2], with an observation that for each $l_1, l_2 = 1, 2, \dots, L$

$$\mathrm{E}\left(Z_{l_1} Z_{l_2}\right) = \left(Diag\left(\mathbf{C}_{l_1} \mathbf{D}_{l_1}^T\right)\mathbf{S}\right)^T \Delta\left(Diag\left(\mathbf{C}_{l_2} \mathbf{D}_{l_2}^T\right)\mathbf{S}\right).$$

□

Note that $\bar{\mathbf{C}}_l = \left(\sum_{i=1}^{N^*} cf_i(0)p_i(0), \sum_{i=1}^{N^*} cf_i(1)p_i(1), \dots, \sum_{i=1}^{N^*} cf_i(n)p_i(n)\right)^T$ is a vector where its k-th element denotes the expected value of total cash flows arising from insurance contract, which are realized at moment k. Namely, k-th element of vector $\bar{\mathbf{C}}_\Sigma$ denotes the expected value of total cash flows at moment k for the whole group.

If the portfolio of policies is homogeneous with respect to the age at entry ($x_l = x$), then $\mathbf{C}_l = \mathbf{C}$ and $\mathbf{D}_l = \mathbf{D}$ for $l = 1, \dots, L$ (it also implies that $\bar{\mathbf{C}}_l = \bar{\mathbf{C}}$) . In this case we have the following Corollary to Theorem 2.

Corollary 1. For the modified multistate model ($\boldsymbol{S}^*$, $\boldsymbol{T}^*$), if the portfolio of L policies is homogeneous (with respect to the age at entry and the term of policy) and $\boldsymbol{Z_1}, \boldsymbol{Z_2}, \dots, \boldsymbol{Z_L}$ and Y (t) satisfy Assumption 1--Assumption 4, then

$$\mathrm{E}\left(Z_{(L)}\right) = L\bar{\mathbf{C}}^T \mathbf{M},$$

$$\mathrm{Var}\left(Z_{(L)}\right) = L^2 \bar{\mathbf{C}}^T \mathbf{R} \bar{\mathbf{C}} - \mathbf{L}\left(\bar{\mathbf{C}}^T \Delta \bar{\mathbf{C}} - \sum_{k_1=0}^{n}\sum_{k_2=0}^{n} \mathbf{I}_{k_2+1}^T \Delta^T \mathbf{I}_{k_1+1} \mathbf{I}_{k_1+1}^T \mathbf{C} \mathbf{P}(k_1,k_2) \mathbf{C}^T \mathbf{I}_{k_2+1}\right),$$

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where $\bar{\mathbf{C}} = Diag(\mathbf{C}\mathbf{D}^{\mathrm{T}})\mathbf{S}$.
Proof. The assertion of Corollary 1 is a consequence of the fact that if the portfolio is homogeneous, then $\mathbf{C}_l = \mathbf{C}$ and $\mathbf{D}_l = \mathbf{D}$ for $l = 1, ..., L$. Thus applying Theorem 2 we complete the proof.

□

In particular, we have the following limit theorem for moments of average payment streams per policy for a large homogeneous portfolio.

Theorem 3. For the modified multistate model ($\boldsymbol{S}^*$, $\boldsymbol{T}^*$), if the portfolio of L policies is homogeneous (with respect to the age at entry and the term of policy) and $\boldsymbol{Z_1}, \boldsymbol{Z_2}, \dots, \boldsymbol{Z_L}$ and Y (t) satisfy Assumption 1-- Assumption 4, then

$$\mathrm{E}\left(\frac{1}{L} Z_{(L)}\right) = \bar{\mathbf{C}}^T \mathbf{M}, \tag{7}$$

$$\lim_{L\to\infty} \mathrm{Var}\left(\frac{1}{L} Z_{(L)}\right) = \bar{\mathbf{C}}^T \mathbf{R}\bar{\mathbf{C}},$$

where $\bar{\mathbf{C}} = Diag(\mathbf{C}\mathbf{D}^{\mathrm{T}})\mathbf{S}$.

Proof. Identity (7) straightforwardly follows from Corollary 1 and the additivity property of the expected value operator.

Because

$$\mathrm{Var}\left(\frac{1}{L} Z_{(L)}\right) = \frac{1}{L}\mathrm{Var}(Z_l) + \frac{L-1}{L}\mathrm{Cov}(Z_1, Z_2)\ , \tag{8}$$

then the right side of (9) converges to $\mathrm{Cov}(Z_1, Z_2)$, when L tends to infinity. Using Theorem 1 we have $\mathrm{Cov}(Z_1, Z_2) = \bar{\mathbf{C}}^T \mathbf{R}\bar{\mathbf{C}}$.

□

5. Analysis of a portfolio of health insurances

From the financial point of view, the cash flow $cf_j^*(k)$ is a sum of *inflows* representing an income to a particular fund and *outflows* representing an outgo from a particular fund. Hence

$$\mathbf{C} = \mathbf{C}_{in} + \mathbf{C}_{out}\ ,$$

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where $\mathbf{C}_{in}$ consists only of an income to a particular fund and $\mathbf{C}_{out}$ consists only of an outgo from a particular fund. It is important to calculate the total loss $\underline{L}$ of the insurance contract, defined as the difference between the present value of future benefits and the present value of future premiums. In particular for $\underline{L}$: $\mathbf{C}_{in}$ includes the b's, the d's, the c's and $\mathbf{C}_{out}$ includes the p's and the π's.

From Theorem 1 and [Dębicka 2012] we may determine the *net single premium* paid in advance at time 0, when $X^*(0) = 1$

$$\pi_1(0) = \mathbf{M}^T Diag(\mathbf{C}_{in}\mathbf{D}^{\mathrm{T}})\mathbf{S}.$$

And the net period premium p payable in advance at the beginning of the unite time (for example at the beginning of the year − an annual premium) during first m units, when $X^*(0) = 1$

$$p = \frac{\mathbf{M}^T Diag(\mathbf{C}_{in}\mathbf{D}^{\mathrm{T}})\mathbf{S}}{\mathbf{M}^T(\mathbf{I}-\sum_{k=m+1}^{n+1}\mathbf{I}_k\mathbf{I}_k^T)\mathbf{D}\mathbf{I}_1}, \tag{9}$$

where the nominator in (9) is equal to the actuarial value of a temporary life annuity due contract.

Frequently the standard deviation of $\underline{L}$ is used as a measure of variability of losses on an individual multistate insurance due to the random nature of the process $\{X^*(t)\}$. Using Theorem 1 (see [Dębicka 2012]) the standard deviation of present value of benefits in case of a single premium is given by

$$\sigma_{\pi_1(0)} = \sqrt{\sum_{k_1=0}^{n}\sum_{k_2=0}^{n}\mathbf{I}_{k_2+1}^T\Delta^T\mathbf{I}_{k_1+1}\mathbf{I}_{k_1+1}^T\mathbf{CP}(k_1,k_2)\mathbf{C}^T\mathbf{I}_{k_2+1} - (\mathbf{M}^T Diag(\mathbf{C}_{in}\mathbf{D}^{\mathrm{T}})\mathbf{S})^2} \tag{10}$$

and for the period premium by

$$\sigma_p = \frac{\sigma_{\pi_1(0)}}{\mathbf{M}^T(\mathbf{I}-\sum_{k=m+1}^{n+1}\mathbf{I}_k\mathbf{I}_k^T)\mathbf{D}\mathbf{I}_1}. \tag{11}$$

Clearly, in the case of a homogeneous group, the net premium per policy does not depend on the size of the portfolio (see (9) and (7)). This observation, however, does not carry over to the volatility. In the following we analyze the influence of the choice of the interest rate model on the net premium and standard deviation of cash value of future payment streams per policy in a homogeneous portfolio of polices.

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Example. We consider a temporary life insurance with 10-year policy term combining sickness insurance which includes annuity benefit. The conditions on sickness insurance contract are analogous to the disability insurance in The Netherlands described in [Gregorius 1993].

Modified multiple state model. We consider sickness in which we allow the influence of illness duration on the recovery and mortality probabilities. In this case the modified multiple state model is

$$(S^*, T^*) = (\{1, 2^{(1)}, 2^{(2)}, 2^{(3)} \dots, 2^{(6)}, 3^+, 3,\} \\ \{(1, 2^{(1)}), (1,3), (2^{(1)}, 1), \dots, (2^{(5)}, 1), (2^{(1)}, 3^+), \dots, (2^{(6)}, 3^+), \\ (2^{(1)}, 2^{(2)}), \dots, (2^{(5)}, 2^{(6)})\})$$

where the meaning of the states is as follows:

1 − the insured is healthy,
$2^{(h)}$ − the insured is sick with illness duration between $h-1$ and h, for $h = 1,2,\dots,5$,
$2^{(6)}$ − the insured is sick and illness duration is at least 6 years,
3^+ − the insured person is dead and the death of the insured occurred not earlier than one year before,
3 − the insured is dead and the death occurred at least one year before.

An illustration of such a model is presented in Figure 1.

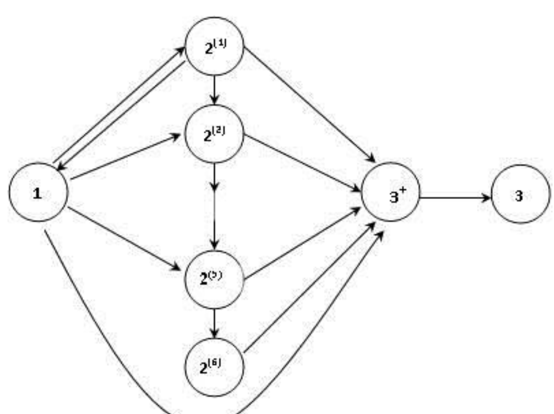

Figure 1. Scheme of the modified disease model with the sickness state split according to duration of illness

Source: own source.

Cash flows. Let the benefit payable at time $k+1$, if the insured person's death occurred in time interval $[k,k+1)$ ($k=0,1,2,\ldots,9$), be equal to 100. Moreover, let the annuity payable for period $[k,k+1)$ ($k=0,1,2,\ldots,9$) if the insured is sick at time $k+1$ be equal to 1.

For such an insurance contract, cash flows (benefits) related to (S^*,T^*) (see (12)) are described as follows

$$cf_j^*(k)=\begin{cases} 1 & if \quad j=2^{(1)} \quad and \quad k=1,2,\ldots,10 \\ & \quad\;\; j=2^{(2)} \quad and \quad k=2,\ldots,10 \\ & \quad\;\; \ldots \\ & \quad\;\; j=2^{(6)} \quad and \quad k=6,\ldots,10 \\ 100 & if \quad j=3^{+} \quad and \quad k=1,2,\ldots,10 \\ 0 & \quad\;\; besides \end{cases}. \tag{12}$$

The cash flows matrix consists only of an income to a total loss fund and is given by

$$\mathbf{C}_{in}=\begin{pmatrix} 0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&100&0\\ 0&1&1&0&0&0&0&100&0\\ 0&1&1&1&0&0&0&100&0\\ 0&1&1&1&1&0&0&100&0\\ 0&1&1&1&1&1&0&100&0\\ 0&1&1&1&1&1&1&100&0\\ 0&1&1&1&1&1&1&100&0\\ 0&1&1&1&1&1&1&100&0\\ 0&1&1&1&1&1&1&100&0\\ 0&1&1&1&1&1&1&100&0 \end{pmatrix}.$$

Let p_k denote the premium payable for period $[k,k+1)$ if the insured person is healthy at time k (that is if $X^*(k)=1$). Premiums are paid during the whole insurance period of insurance policy. Thus $p_k\neq 0$ for $k=0,1,2,\ldots,9$.

For the constant period premium p ($p=p_k$ for $k=0,1,2,\ldots,9$) the matrix $\mathbf{C}_{out}$ is given by

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$$\mathbf{C}_{out} = \begin{pmatrix} -p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the net single premium π the matrix $\mathbf{C}_{out}$ is given by

$$\mathbf{C}_{out} = \begin{pmatrix} -\pi & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The probabilistic structure of the model. We assume that $\{X^*(k)\}$ is modelled by a Markov chain [Hoem 1969; 1988; Waters 1984; Wolthuis 1994]. Then matrices $\mathbf{D}$ and $\mathbf{P}(k_1, k_2)$ can be described by a sequence of matrices $\mathbf{Q}^*(0), \mathbf{Q}^*(1), \mathbf{Q}^*(2), \dots, \mathbf{Q}^*(n-1)$, where $\mathbf{Q}^*(k) = \left(q^*_{ij}(\boldsymbol{k})\right)_{i,j=1}^{N^*}$ and $q^*_{ij}(k) = \mathrm{P}(X^*(k+1) = j | X^*(k) = i)$ is a transition probability. For the above described insurance contract, the matrix transition $\mathbf{Q}^*(k)$ has the following form

$$\mathbf{Q}^*(k) = \begin{pmatrix} q^*_{11}(k) & q^*_{12^{(1)}}(k) & 0 & 0 & 0 & 0 & 0 & q^*_{13^+}(k) & 0 \\ q^*_{2^{(1)}1}(k) & 0 & q^*_{2^{(1)}2^{(2)}}(k) & 0 & 0 & 0 & 0 & q^*_{2^{(1)}3^+}(k) & 0 \\ q^*_{2^{(2)}1}(k) & 0 & 0 & q^*_{2^{(2)}2^{(3)}}(k) & 0 & 0 & 0 & q^*_{2^{(2)}3^+}(k) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q^*_{2^{(5)}1}(k) & 0 & 0 & 0 & 0 & 0 & q^*_{2^{(5)}2^{(6)}}(k) & q^*_{2^{(5)}3^+}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^*_{2^{(6)}2^{(6)}}(k) & q^*_{2^{(6)}3^+}(k) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The probability $q^*_{ij}(k)$ not only depends on the future-lifetime of the insured but also on morbidity and probability incurring on misadventure. Assume that the mortality rates are based on Polish 2009 mortality tables for woman, see e.g. [Rocznik Demograficzny 2010] Probability on morbidity and probability incurring on misadventure are based on formulas described for disability insurance in The Netherlands (see [Gregorius 1993]).

For numerical illustrative purposes we consider insurance contracts for insured persons at the age of 20, 40 and 60 years ($x =$

20,40,60) respectively. Then matrices $\mathbf{D}$ and $\mathbf{P}(k_1, k_2)$ are calculated upon [Gregorius 1993] and [Rocznik Demograficzny 2010]. For example matrices $\mathbf{D}_x$ have the following form:

$$\mathbf{D}_{20} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0{,}994184 & 0{,}005566 & 0 & 0 & 0 & 0 & 0 & 0{,}000250 & 0 \\ 0{,}992447 & 0{,}005793 & 0{,}003211 & 0 & 0 & 0 & 0 & 0{,}000260 & 0{,}000250 \\ 0{,}991818 & 0{,}006054 & 0{,}003409 & 0{,}002707 & 0 & 0 & 0 & 0{,}000260 & 0{,}000510 \\ 0{,}991424 & 0{,}006333 & 0{,}003632 & 0{,}002891 & 0{,}002361 & 0 & 0 & 0{,}000261 & 0{,}000770 \\ 0{,}991129 & 0{,}006627 & 0{,}003872 & 0{,}003100 & 0{,}002536 & 0{,}002207 & 0 & 0{,}000282 & 0{,}001031 \\ 0{,}990777 & 0{,}006935 & 0{,}004128 & 0{,}003325 & 0{,}002733 & 0{,}002378 & 0{,}002206 & 0{,}000303 & 0{,}001313 \\ 0{,}990207 & 0{,}007257 & 0{,}004400 & 0{,}003567 & 0{,}002948 & 0{,}002572 & 0{,}004582 & 0{,}000334 & 0{,}001616 \\ 0{,}989391 & 0{,}007592 & 0{,}004687 & 0{,}003825 & 0{,}003178 & 0{,}002783 & 0{,}007151 & 0{,}000376 & 0{,}001950 \\ 0{,}988327 & 0{,}007941 & 0{,}004991 & 0{,}004100 & 0{,}003426 & 0{,}003011 & 0{,}009930 & 0{,}000397 & 0{,}002326 \\ 0{,}986973 & 0{,}008303 & 0{,}005312 & 0{,}004392 & 0{,}003691 & 0{,}003256 & 0{,}012936 & 0{,}000439 & 0{,}002723 \end{pmatrix}$$

$$\mathbf{D}_{40} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0{,}984785 & 0{,}013895 & 0 & 0 & 0 & 0 & 0 & 0{,}001320 & 0 \\ 0{,}973608 & 0{,}014324 & 0{,}011203 & 0 & 0 & 0 & 0 & 0{,}001478 & 0{,}001320 \\ 0{,}963610 & 0{,}014824 & 0{,}011711 & 0{,}010618 & 0 & 0 & 0 & 0{,}001669 & 0{,}002798 \\ 0{,}953165 & 0{,}015358 & 0{,}012288 & 0{,}011160 & 0{,}010244 & 0 & 0 & 0{,}001871 & 0{,}004467 \\ 0{,}942002 & 0{,}015903 & 0{,}012905 & 0{,}011772 & 0{,}010817 & 0{,}010209 & 0 & 0{,}002105 & 0{,}006338 \\ 0{,}929755 & 0{,}016452 & 0{,}013542 & 0{,}012429 & 0{,}011463 & 0{,}010791 & 0{,}010185 & 0{,}002348 & 0{,}008443 \\ 0{,}916363 & 0{,}016998 & 0{,}014195 & 0{,}013111 & 0{,}012157 & 0{,}011433 & 0{,}020922 & 0{,}002622 & 0{,}010791 \\ 0{,}901774 & 0{,}017537 & 0{,}014857 & 0{,}013814 & 0{,}012882 & 0{,}012121 & 0{,}032261 & 0{,}002915 & 0{,}013413 \\ 0{,}885932 & 0{,}018066 & 0{,}015525 & 0{,}014533 & 0{,}013633 & 0{,}012840 & 0{,}044240 & 0{,}003237 & 0{,}016328 \\ 0{,}868793 & 0{,}018579 & 0{,}016195 & 0{,}015264 & 0{,}014406 & 0{,}013585 & 0{,}056877 & 0{,}003577 & 0{,}019565 \end{pmatrix}$$

$$\mathbf{D}_{60} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0{,}954876 & 0{,}034684 & 0 & 0 & 0 & 0 & 0 & 0{,}010440 & 0 \\ 0{,}909159 & 0{,}034669 & 0{,}034283 & 0 & 0 & 0 & 0 & 0{,}011449 & 0{,}010440 \\ 0{,}862932 & 0{,}034554 & 0{,}034224 & 0{,}033842 & 0 & 0 & 0 & 0{,}012559 & 0{,}021889 \\ 0{,}816304 & 0{,}034332 & 0{,}034061 & 0{,}033736 & 0{,}033360 & 0 & 0 & 0{,}013759 & 0{,}034448 \\ 0{,}769393 & 0{,}033997 & 0{,}033789 & 0{,}033523 & 0{,}033202 & 0{,}032832 & 0 & 0{,}015057 & 0{,}048207 \\ 0{,}722332 & 0{,}033542 & 0{,}033399 & 0{,}033195 & 0{,}032934 & 0{,}032619 & 0{,}032256 & 0{,}016458 & 0{,}063265 \\ 0{,}675282 & 0{,}032965 & 0{,}032888 & 0{,}032748 & 0{,}032548 & 0{,}032291 & 0{,}063609 & 0{,}017945 & 0{,}079723 \\ 0{,}628383 & 0{,}032260 & 0{,}032250 & 0{,}032175 & 0{,}032038 & 0{,}031842 & 0{,}093822 & 0{,}019563 & 0{,}097668 \\ 0{,}581802 & 0{,}031424 & 0{,}031482 & 0{,}031472 & 0{,}031399 & 0{,}031265 & 0{,}122633 & 0{,}021292 & 0{,}117231 \\ 0{,}535666 & 0{,}030456 & 0{,}030577 & 0{,}030633 & 0{,}030624 & 0{,}030553 & 0{,}149750 & 0{,}023217 & 0{,}138523 \end{pmatrix}$$

The interest rate. We assume that the mean interest rate $\mu = 0{,}02$ and the volatility $\sigma = 0{,}015$. Thus, for $Y(t) = Y_W(t)$ we have

$$Y_W(t) = 0{,}012W(t) + 0{,}02t,$$

and for $Y(t) = Y_{OU}(t)$ we have

$$Y_{OU}(t) = 0{,}015255 \int_0^t U(s)ds + 0{,}02t,$$

where $\{U(s), s \geq 0\}$ is the Ornstein-Uhlenbeck process with the covariance function $R(t) = \exp(-0{,}1t)$ (cf. [8]). We refer to [Dębicka 2003] for the exact forms of matrices $\mathbf{M}$, Δ and $\mathbf{R}$ when $Y(t)$ is mod-

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elled by Ornstein-Uhlenbeck process or Wiener process. For example $\mathbf{M}_W$ for $Y(t) = Y_W(t)$ has the following form

$\mathbf{M}_W$ = (1 0,980 0,961 0,942 0,924 0,905 0,888 0,870 0,853 0,836 0,820)^T

and for $Y(t) = Y_{OU}(t)$ we have

$\mathbf{M}_{OU}$ = (1 0,980 0,961 0,943 0,925 0,907 0,890 0,873 0,857 0,841 0,826)^T

Results. Tables 1, 2 and 3 present the net single premium and its standard deviation of a single policy, the net period premium and its standard deviation of a single policy and the limiting standard deviation of the average benefit per policy as L tends to infinity respectively.

Table 1. Net single premium for health insurance contract ($n = 10$)

$\pi_1(0), \sigma_{\pi_1(0)}$	$\pi_1(0)$		$\sigma_{\pi_1(0)}$	
Interest rate	$Y_W(t)$	$Y_{OU}(t)$	$Y_W(t)$	$Y_{OU}(t)$
x=20	0.46230	0.46386	5.06034	5.09097
x=40	2.65716	2.66652	13.32713	13.41627
x=60	12.24299	15.912572	28.66655	32.57454

Source: own source.

Table 2. Constant net premiums for health insurance contract ($n = m = 10$)

p, σ_p	p		σ_p	
Interest rate	$Y_W(t)$	$Y_{OU}(t)$	$Y_W(t)$	$Y_{OU}(t)$
x=20	0.05088	0.05095	0.55689	0.55917
x=40	0.30633	0.30683	1.53640	1.54376
x=60	1.65036	2.16904	3.86425	4.44023

Source: own source.

Notice that for the each age at issue, premiums are larger for the rate of interest modelled by $Y_{OU}(t)$ (see Table 1 and Table 2). In Table 1 and Table 2 we can also observe that, independently of the age at entry, the standard deviation of Z_l is larger for $Y(t) = Y_{OU}(t)$ than for $Y(t) = Y_W(t)$.

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Table 3. The standard deviation of the average benefit per policy for a large portfolio (as L tends to infinity)

Interest rate	$Y_W(t)$		$Y_{OU}(t)$	
Type of premium	p	$\pi_1(0)$	p	$\pi_1(0)$
x=20	0.00164	0.01491	0.00418	0.03805
x=40	0.01011	0.08769	0.02585	0.22467
x=60	0.05177	0.38402	0.17527	1.28582

Source: own source.

Table 3 is concerned with the standard deviation of the average benefit per 10-year health insurance contract $\frac{1}{L}Z_{(L)}$ of a large portfolio (when $L \to \infty$). It appears that, for large portfolios, the standard deviation of the average benefit per policy is larger for the rate of interest modelled by the Ornstein-Uhlenbeck process ($Y(t) = Y_{OU}(t)$), than by the Wiener process ($Y(t) = Y_W(t)$). Notice that, due to Theorem 3, the expected value of the average benefit per policy agrees with the one presented in Table 1 and Table 2.

We can observe that, when a portfolio gets larger, the standard deviation of the average benefit per policy is less likely to vary (compare Table 1 and Table 2 with Table 3).

The use of the matrix notation not only simplified formulas of net premium and standard deviation of cash value per policy, but also due to the factorization in the formulas, the numerical implementations are less complex.

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MACIERZOWE PODEJŚCIE DO ANALIZY PORTFELA UBEZPIECZEŃ WIELOSTANOWYCH

Streszczenie: Celem artykułu jest analiza przepływów pieniężnych wynikających z realizacji portfela umów ubezpieczeń wielostanowych przy założeniu, że proces opisujący zmiany stanów podczas trwania umowy ubezpieczenia jest niejednorodnym w czasie łańcuchem Markowa, a losowa stopa procentowa jest modelowana przez proces stochastyczny o stacjonarnych przyrostach (np. proces Wienera lub scałkowany proces Ornsteina-Uhlenbecka). W tekście zaproponowany został zapis macierzowy wielkości aktuarialnych istotnych w analizie wpływu wielkości portfela polis na wysokość składki i narzutu bezpieczeństwa dla grupy zarówno jednorodnej, jak i niejednorodnej. Uzyskane wyniki zilustrowano na przykładzie portfela ubezpieczeń zdrowotnych.

Słowa kluczowe: model wielostanowy, przepływy pieniężne, stochastyczna stopa procentowa, portfel polis, zmodyfikowany model wielostanowy.