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Łukasz Radosiński, Andrzej Radosz

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The night sky paradox

"To touch the infinite, just look at the starry sky" - as a columnist expressed the common belief that the world we live in has no boundaries. In other words, the Universe is infinite. Is it really so? As it turns out, the problem is not a trivial one.

A question about the size of the Universe (and its design) accompanied humanity almost from the beginning of its history. This problem was discussed widely in a scientific community but a truly scientific hypothesis was given at the end of the seventeenth century. This became possible thanks to the Copernican revolution that began in the sixteenth century. With the Copernican principle emerged:

The universe is isotropic and homogeneous, always has been, is and will remain the same.

The first part of this principle, otherwise known as *the cosmological principle*, has survived unchanged to the present day and reflects the belief that the Universe has neither distinguished direction, nor the highlighted position in space. Over the following two hundred years no one had an idea about the scale of homogeneity, only tacitly assumed a sufficiently large distance scale. The second part of the Copernican principle has been rejected as a consequence of subsequent discoveries, particularly as a result of the discovery by E. Hubble the expansion of the Universe.

Basing on the Copernican principle one may try to answer the question about the size of the Universe. In 1692 Sir Isaac Newton stated that:

The universe is infinite.

Indeed, if the matter in the Universe would occupy a finite volume, the center of its mass could be determined. Each object would have been so attracted towards this point. As a result of the gravitational pull, a gigantic collapse would happened: the Universe would have ended its existence in the ultimate catastrophe of all the bodies. But the Universe remains unchanged, and therefore it cannot be finite.

Edmund Halley, Newton's ideas fan (and his relative), who directly contributed to their release, said in 1736 an opposite statement. His reasoning was as follows: looking at the night sky one would find the shiny surface of the star. Let us look in slightly different direction: what would we see? Isotropic nature of the infinite Universe suggests that we should see the surface of another, more or less distant star. If hence, the isotropic and homogeneous Universe is infinite, then any point of the night sky should be glittering, filled with a glare of stars. The night sky should be unbearable bright due to shines of more or less distant stars. The night sky is dark, therefore:

The universe may not be infinite, the Universe must be finite (otherwise the night sky should has been bright).

That was the paradox of the Universe (see [1]), which can not be finite (gravitational collapse), but it can not be infinite because then the night sky would be bright. This paradox is known as the *night-sky paradox.*

In 1822, W. Olbers presented Halley's considerations in a different, more rigorous form. Olbers has shown that the isotropic and homogeneous distribution of matter (stars) in an infinite universe leads to the paradoxical result: the energy flux reaching any place in this universe is infinite. In such a world there is no night-sky: the temperature would be infinite. Such a world could not exist.

Olbers reasoning was the following. Imagine an infinite, isotropic, homogeneous universe filled with stars. Instead of various stars, let's consider an average star of luminescence L and area A. Assume that the average density of stars ρ is fixed (in a sufficiently large scale, which for now will be not specified). Apparent brightness of a star placed at a distance *r* from the observer (on Earth) is:

$$B(r) = \frac{L}{4\pi r^2} . \qquad (1)$$

The energy flux from the stars arranged in the layer of thickness dr, is

$B(r)dN_r$ (2)

where the number of stars dN_r is proportional to the volume of the layer dV_r ,

$$dN_r = \rho dV_r = \rho 4\pi r^2 dr \text{ . (3)}$$

Thus, the total energy flux:

$$\int_{r_0}^{\infty} B(r)dN_r = \int_{r_0}^{\infty} \frac{L}{4\pi r^2}\rho dV_r = L\rho\int_{r_0}^{\infty}\frac{1}{4\pi r^2}\left(4\pi r^2 dr\right) \to \infty \qquad (4)$$

reaching the observer is infinite.

This result is not exact, because the distant stars can be partially or completely blocked by the stars situated closer. Then the energy flux coming from stars that populate the sky (completely) is:

$$\int_{r_0}^{R} \frac{A}{r^2}\rho dV_r = A\rho\int_{r_0}^{R}\frac{1}{r^2}4\pi r^2 dr = 4\pi \text{ , (5)}$$

and they are arranged in the distances ranging:

$$R - r_0 = \frac{1}{A\rho} \text{ . (6)}$$

The energy flux is then:

$$\int_{r_0}^{R} B(r)dN_r = \int_{r_0}^{R}\frac{L}{4\pi r^2}\rho dV_r = L\rho\int_{r_0}^{R}\frac{1}{4\pi r^2}4\pi r^2 dr = L\rho\left(R - r_0\right) = \frac{L}{A} \text{ . (7)}$$

Though finite, however, it is equal to the energy flux emitted by a single star, measured in its vicinity!

Energy flux reaching the observer in this world is (almost) infinite!

In such a world the energy flux would be unbearably large and above all, there would be no dark sky. *The night-sky paradox* is the argument that simple idea of the infinite Universe uniformly filled with stars is false and in principle such a universe can not exist. As Stephen Hawking ironically once said, at the beginning of the twentieth century, the only cosmological observation was the statement that "the night sky is dark."

The night-sky-paradox remained unsolved in the system that it arose from, collections of ideas and associated mathematical formalism, called today “classical physics”.

To understand and describe the existence of isotropic and homogeneous universe it is necessary to use the concepts and the language of relativity theory. In the next few chapters we will introduce the concepts of relativity, which allow formulating a model of the Universe.

1. Introduction to the Special Theory of Relativity

1.1. Universal space and time

Describing the Nature in terms of Newton's principles, we work in the classical field of physics. Fundamental concepts of classical physics treat the time and three dimensional space as universal. In Cartesian coordinates XYZ, the distance between points A and B, r_{AB} measured along the straight line is:

$$r_{AB} = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} \equiv \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \ . \qquad (1.1)$$

The distance between the points is independent of the manner of measurement and does not depend on the observer's state. Likewise the time between two events neither depends on the manner of measurement nor on the observer's state.

One can say:

Space and time are universal

in the sense that

Space interval, length of an object, and time interval, time interval between two events, are constant (they are invariant):

$$\begin{array}{l} \Delta l = inv \\ \Delta t = inv \end{array} \ . \ (1.2)$$

1.1.1. Galileo's transformations

Following first Newton's law of motion, frame of reference where a body with no forces acting on it or when applied net force vanishes, is called an inertial frame of reference. Two inertial frames of references are moving with relative constant speed in a straight line. Let us consider two inertial observers K and K' moving with respect each other along axis OX with constant velocity u . Then the relation between these two observers is following:

$$x = x' + ut \qquad y = y' \qquad z = z' \qquad t = t' \ . \ (1.3)$$

One assumes that

$t = t' = 0$,

at the instant O = O'.

More general situation of motion along arbitrary direction leads to the relation

$$\vec{r} = \vec{r}' + \vec{u}t + \vec{r}_0 \qquad t = t' . \qquad (1.4)$$

Relations (1.3) and (1.4), are called the *Galileo's transformations*. It turns out that gravitational interaction is invariant under (1.4) transformation (see problem 1; it should be so since every inertial observer detects the same force, according to the first and second Newton's law of motion). It should be expected that every coupling (second law of motion) should be invariant under Galileo transformation. Galileo's transformations yield law of vector's *addition of velocities*:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\vec{r}' + \vec{u}t + \vec{r}_0\right) = \frac{d\vec{r}'}{dt} + \vec{u} \equiv \vec{v}' + \vec{u} \qquad (1.5)$$

Adding velocities according to (1.5), in spite of verification in many circumstances, in same circumstance loses its universal character, likewise the overall philosophy of universal space and time.

1.2. Speed of electromagnetic wave in the vacuum

In 1865 J. C. Maxwell unified electricity with magnetism. Laws of electromagnetism were presented in the form of four equations, describing electromagnetic field; the distribution of electric charges and currents is the source of electromagnetic field. In the simplest case, the Maxwell's equations may be written in (differential) form:

$$\begin{aligned} &\vec{\nabla} \circ \vec{E} = \frac{\rho}{\varepsilon_0} \qquad &&\vec{\nabla} \circ \vec{B} = 0 \\ &\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad &&\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \qquad (1.6)$$

where $\vec{E}, \vec{B}$ denote electric field and the magnetic induction, $\rho, \vec{j}$ are the densities of electric charge and current density vector, ε_0, μ_0 are constants (dielectric constant/electrical permittivity, magnetic permeability in vacuum). From Maxwell's equations one can read that lines of electric field originate and end up on electric charges („there are electric charges") but lines of magnetic induction form closed loops („there are no magnetic charges").

Although knowledge about electricity and magnetism agreed with physics built upon Newton's laws of motion, equations describing electromagnetic field seem to go beyond this theory. Equations (2.6) *are not invariant* under Galileo transformation. In vacuum magnetic and electric fields satisfy the wave equation:

$$\nabla^2 \vec{E} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \varepsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} \ , \qquad (1.7)$$

from which it results that alternating electromagnetic field (e-m wave) moves in vacuum with speed:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.997 \cdot 10^8 \frac{m}{s} . \qquad (1.8)$$

The velocity of e-m wave does not depend on the state of an observer! Independence of velocity of e-m waves in vacuum on observer, didn't find satisfying explanation in classical physics. The progress was made due to Albert Einstein's observation.

Let us consider the case of a planar monochromatic e-m wave, moving in OX direction. This is a transverse wave: vectors $\vec{E}$ and $\vec{B}$, are mutually perpendicular and they are perpendicular to OX. They oscillate with the same frequency (see Fig.1).

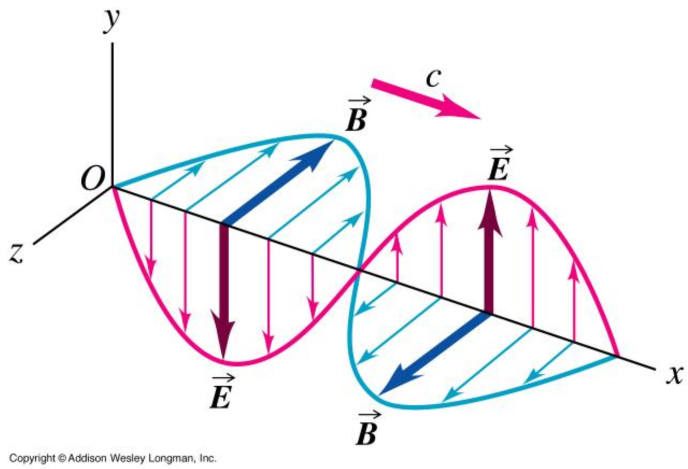

Fig.1. Planar, linearly polarized electromagnetic wave moving along OX axis

Such a picture of oscillating electromagnetic field is observed by static observer for whom the wave travels with speed c. What would be the distribution of electromagnetic field recorded by an observer traveling with velocity c?[1] The answer is given in the picture 1: an observer moving with speed c would see static distribution of electric and magnetic fields perpendicular to each other, having e the same value within y-z planes, perpendicular to the x direction .

This type of the field couldn't exist! The lines B, magnetic field should form closed loops and lines of electric field should originate and end in electric charges – an observer moving with speed of light would see image that couldn't exist.
So, it is not possible to reach speed c.

Moreover Albert Einstein claimed that:

Laws of physics are the same for all inertial observers – there is a democracy between inertial observers.

1.3. The postulates of special theory of relativity

In year 1905 Einstein formulated postulates of the relativity theory:
The laws of the Nature are the same in all inertial systems
The speed of light is the same for all inertial observers

Furthermore in inertial frames of coordinates the light travels in a straight lines, i.e. light rays are straight lines. First postulate has the same meaning in classical physics. Even today the postulate of the invariant speed of light might be (and it is) controversial. But is it really true? It seems that it reflects law of Nature and there is no sense in discussing the reasons behind it.. In this context one may ask for the role of classical physics as a tool describing the Nature .Classical physics is a proper tool to investigate specific area of reality. But it is not a universal tool and trying to describe complex phenomena involving subtle effects beyond our every day life experience, sometimes we must be prepared for developing counterintuitive approaches.

1.3.1. Lorentz's transformations

The postulates of theory of relativity lead us to Lorentz transformations, substituting Galileo's transformations (1.4). Let us again consider the case of inertial observers K and K' moving along axis OX with velocity u. When the centers O and O' of both systems are at the same point $t = t' = 0$ spherical light wave is emitted. In the system K the wave travels with speed *c*:

$$x^2 + y^2 + z^2 = (ct)^2 \qquad (1.9a)$$

and for the observer K' the light wave also travels with speed *c*:

$$x'^2 + y'^2 + z'^2 = (ct')^2 . \qquad (1.9b)$$

Note that:

a) in inertial frames light rays are straight lines so the relation between coordinates K and K' has a linear character:

$$x = a_{11}x' + a_{12}t' \qquad t = a_{21}x' + a_{22}t' \qquad (1.10)$$

b) perpendicular dimensions do not transform,

$$y = y' \qquad z = z'. \qquad (1.11)$$

c) relative velocity of these two frames is u, i.e. O' (x' = 0) travels with speed u

$$dx = a_{12}dt' \qquad dt = a_{22}dt' \Rightarrow a_{12} = ua_{22}. \qquad (1.12)$$

Inserting (1.10 – 12) into equation (1.9a) one obtains (1.9b) iff,

$$a_{11} = a_{22} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} \qquad a_{12} = ua_{22} \qquad a_{21} = \frac{u}{c^2}a_{22} \qquad . \ (1.13)$$

The principal conclusion from previous considerations about the invariant character of the speed of light, is: neither space nor the time are invariant. If it were so, then isotropic and homogenous universe couldn't exist. However, the speed of light is invariant.

Problems

1. Show that gravitation force acting between two masses is invariant under Galileo's transformation.
2. Using Maxwell' equations show that planar electromagnetic wave is transversal.
3. Show that dimension perpendicular to direction of motion doesn't transform
4. Describe three characteristic effects of special theory of relativity:

a) Length contraction

b) Time dilation

c) Adding velocities

5. Apply these derivations to the particular question (below)
6. How much time does a meter stick moving at 0.1c relative to an observer take to pass the observer? The meter stick is parallel to its direction of motion.
7. To A, B's watch seems to run slow. To B does A's watch seem to run fast , run slow, or keep the same time as his own watch?
8. An airplane is flying at 300 m/s. How much time must elapse before a clock in the airplane and one on the ground differ by 1 s.
9. A certain particle has a lifetime of $10^{-7}s$ when measured at rest. How much larger is that time if particle speed is $0.99c$?
10. What is the distance that particle would travel ?

2. Space-time

2.1. Four-vectors

Spatial coordinates together with time coordinate form four dimensional continuum – *spacetime.* The spacetime is not a typical four dimensional space. In order to realize this fact let us consider Lorentz transformation (see [2]). Instead of time coordinate t we will use $x^0 = ct$; then the following notation is applied:

$$\tilde{x} = \left(x^0, \vec{x}\right) \equiv \left(x^0, x^1, x^2, x^3\right) = \left(ct, x, y, z\right) . \qquad (2.1)$$

Lorentz transformation (1.13) in this notation is:

$$x^0 = \gamma x'^0 + \beta\gamma x'^1 \qquad x^1 = \gamma x'^1 + \beta\gamma x'^0 \qquad x^2 = x'^2 \quad x^3 = x'^3 . \qquad (2.1a)$$

It can be expressed in a matrix form as:

$$\tilde{x} = \hat{\Lambda}\tilde{x}' . \qquad (2.2)$$

The transformation matrix is,

$$\hat{\Lambda} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \qquad (2.3)$$

So, it corresponds to the relative motion of two systems K and K' along axis OX with a constant speed, $v \Rightarrow \beta = v/c$. Transformation which corresponds to the relative motions along axes OY and OZ describe symmetric matrices similar to the (2.3), respectively:

$$\hat{\Lambda}_Y = \begin{pmatrix} \gamma & 0 & \beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ \beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (2.3a)$$

and

$$\hat{\Lambda}_Z = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} . \quad (2.3b)$$

In analogy to the vectors in 3-dimensional space in Cartesian coordinates, we introduce a concept of four-vectors in 4-dimensional space-time. Four-vectors $\tilde{A} = \left(A^0, \vec{A}\right)$, will change from one inertial frame of to another one, transforming in the same way as a "radius" four-vector vector (2.1), i.e. according to appropriate transformation matrix $\hat{\Lambda}$ (in the simplest case (2.3) , i.e.

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix} = \begin{pmatrix} \gamma A'^0 + \beta\gamma A'^1 \\ \gamma A'^1 + \beta\gamma A'^0 \\ A'^2 \\ A'^3 \end{pmatrix} . \quad (2.4)$$

2.2. The geometric properties of space-time

In 3-dimentional space, square length of the vector equals a sum of squares of its coordinates:

$$\vec{x}^2 = \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 . \quad (2.5a)$$

The length of vector (2.5a) does not depend upon coordinates choice i.e. is invariant under space rotations, translations and other isometric transformations.

Space-time is a four-dimensional space but a square of length of 4-vector should be chosen as a *difference* between squares of the zero, "time" coordinate and a sum of squares of the three other, "spatial" coordinates:

$$\left(\tilde{x}\right)^2 \equiv \tilde{x} \circ \tilde{x} = \left(x^0\right)^2 - \left(\vec{x}\right)^2 . \quad (2.5b)$$

One can confirm that expression (2.5b) is invariant under Lorentz transformation (see 2.1a)

$$\left(x^0\right)^2 - \left[\left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right] = \left(x'^0\right)^2 - \left[\left(x'^1\right)^2 + \left(x'^2\right)^2 + \left(x'^3\right)^2\right] . \quad (2.6)$$

So, for each 4-vector $\tilde{A} = \left(A^0, \vec{A}\right)$ we define its squared "length" as a difference of squared 'time' coordinate $\left(A^0\right)^2$ and squared of space coordinates $\left(\vec{A}\right)^2$:

$$\tilde{A}^2 = \left(A^0\right)^2 - \left(\vec{A}\right)^2 . \tag{2.7}$$

The length of four-vectors is an invariant quantity: it is the same for all inertial observers. Due to the fact that equation (2.7) may have positive, negative and 0 values, one can distinguish three types of four-vectors:

a) *time-like,* $\tilde{A}^2 = \left(A^0\right)^2 - \left(\vec{A}\right)^2 > 0$,

b) *space-like,* $\tilde{A}^2 = \left(A^0\right)^2 - \left(\vec{A}\right)^2 < 0$,

c) *zero vectors,* $\tilde{A}^2 = \left(A^0\right)^2 - \left(\vec{A}\right)^2 = 0$.

For each pair of events: A, B (it means for each 2 points in space-time), we may assign a four-vector $\tilde{x}_{AB}$. Its length is called *an interval* (we denote it with a letter τ):

$$\tau_{AB}^2 \equiv \tilde{x}_{AB}^2 = \left(x_{AB}^0\right)^2 - \left(\vec{x}_{AB}\right)^2 . \tag{2.8}$$

Four-vector (time-like vector) $d\tilde{x} = \left(dx^0, d\vec{x}\right)$ determines two infinitesimally close particle’s trajectory points in space-time (events), and its length equals:

$$d\tau = c\,dt\sqrt{1 - \frac{v^2}{c^2}} . \tag{2.9}$$

2.3. Light cone

Let us choose point O in space-time as „here and now”. One can ask which future event we can have influence on and what kind of events in the past would affect us.

For simplification one assumes 2-dimensional space-time: vertical line being axis of time and axis of abscissa one uses for space dimension, x.

Propagation of the light in direction „+” and „–” corresponds to a straight line inclined with angle 45^o to the axis OX. Half-lines (shown in the Picture 2) going outward (upper semi plane) from point O correspond to light emitted, and half-lines going inward to the point O (bottom semi plane) correspond to the light registered ‘here and now’. No signal can travel with a speed greater than *c*, so lines intersecting in the point O are drawing a natural border for signals emitted or registered ‘here’ and ‘now’. One can connect these points in the spacetime (events), belonging to the space between half-lines in top semi plane – *absolute future*; on the other hand one can register only signals emitted from space between half-lines in bottom semi plane – *absolute past*. Following those considerations one can obtain a light cone: the interior part of a cone represents the future and the past seen from point O; the edge of the light cone is an area, in which light rays travel.

Notice an important property of vectors beginning or ending in point O: interior of the cone create a set of time-like vectors, vectors laying on the edge of the cone have light nature, vectors laying out of cone are space-like vectors.

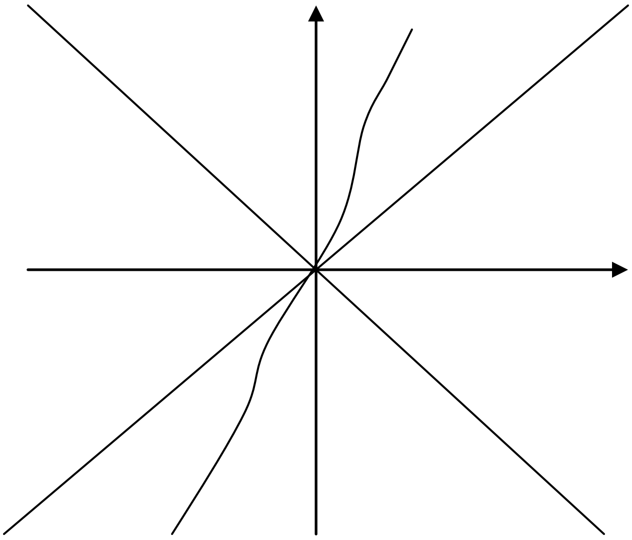

Fig. 2. The curve along the object moves in space-time is called *a word line*. For example: being in the rest we will travel from point O along time axis (upper halfline *ct*). A constant velocity movement corresponds to a halfline inclined with an angle greater than 45; if an object is accelerating (represented in the picture with curved line) then the world line is sufficiently "smooth" (i.e this line does not contain segments too much inclined in the OX axis direction, in other words that its derivative is $\frac{d(ct)}{dx} > 1$).

Above-cited conclusions were related to the two-dimensional space-time model. However, considering 2+1 dimensional space-time that is XOY plane as a two dimensional space and time axis perpendicular to it, the electromagnetic wave emitted from point O has a shape of upturned cone, $t > 0$. Signals arriving to point O came form interior part of cone, $t < 0$. In the moment t, a wave-front of the light wave is constructed as a result of crossing the light cone with a plane t_1- a circle with a radius $c\ t_1$. In four-dimensional spacetime, a light emitted from fixed point has a shape of concentric spheres. To illustrate „light cone" we can follow the analogy with 2+1 dimensional space: if we cut out in this space a „light cone" using „plane" t_1, we obtain a wave-front of the e-m wave. In this case the sphere t_1 *is*

$$x^2 + y^2 + z^2 = \left(ct_1{}^2\right)$$

Using analogies, it is good to remember that in four-dimensional space-time we have four perpendicular to each other 3-dimensional hyper planes, and six 2-dimensional (perpendicular to each other!) planes.

2.4. Interval. Metric tensor

Square of a distance between to infinitesimally close points in space-time $A(\tilde{x})$, $B(\tilde{x}+d\tilde{x})$ i.e. an interval defined as:

$$d\tau^2 = \left(dx^0\right)^2 - \left(d\vec{x}\right)^2 \equiv \left(dx^0\right)^2 - \left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)\right]^2 \qquad (2.10)$$

is invariant i.e. any inertial observer will measure its exactly same value:

$$d\tau_{AB}^2 = \left(dx^0\right)^2 - \left[\left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2\right] = \left(dx'^0\right)^2 - \left[\left(dx'^1\right)^2 + \left(dx'^2\right)^2 + \left(dx'^3\right)^2\right]$$

.

If we regard the interval as a bilinear combination of coordinates in a more general form:

$$d\tau_{AB}^2 = \sum_{\alpha,\beta=0}^{3} \eta_{\alpha\beta} dx^\alpha dx^\beta \qquad (2.11)$$

then an object, $\eta_{\alpha\beta}$ is square matrix 4x4. This is a *metric tensor* of inertial observers. That metric tensor has a simple form, which we obtain comparing expressions (2.10) and (2.11):

$$\hat{\eta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \qquad (2.12)$$

There should be pointed out the properties of the metric tensor $\eta_{\alpha\beta}$: the matrix has diagonal form with constant elements with its modulus equal to 1. If the metric tensor would be a unit matrix we would deal with an ordinary four dimensional Euclidean space (in such a space it is impossible to fulfill the light velocity invariance requirement – see problem 2.6). As the actual form is (2.12) the spacetime is called pseudo-Euclidean "1+3" dimensional space. Interval invariance requirement means that *metric tensor is the same in all inertial systems* (see problem 2.7).

2.5. Contravariant and covariant four vectors

Four vectors with indices at the top:

$$A^\mu = \left(A^0, \vec{A}\right) \equiv \left(A^0, A^1, A^2, A^3\right)$$

are called *contravariant vectors*. A metric tensor allows to introduce *covariant vector,* with lower indices:

$$A_\nu \equiv \eta_{\nu\alpha} A^\alpha \qquad (2.13)$$

where the summation is over repeating (upper and lower) indices. This convention will be used hereafter. Coordinates of contravariant and covariant four vectors are simply related:

$$\left(A^\mu\right) \equiv \left(A^0, \vec{A}\right) \Leftrightarrow \left(A_\nu\right) \equiv \left(A^0, -\vec{A}\right) . \qquad (2.14)$$

Using above relation we may present a scalar product of two four vectors as a repeating indices sum:

$$\tilde{A} \circ \tilde{B} = \eta_{\alpha\beta} A^\alpha B^\beta \equiv A_\beta B^\beta \ . \qquad (2.15)$$

Naturally we may consider a metric tensor with indices at the top $\eta^{\alpha\beta}$ defined as inverted $\eta_{\alpha\beta}$ tensor:

$$\eta_{\alpha\beta} \eta^{\beta\mu} = \delta_\alpha^\mu \qquad (2.16)$$

where right hand side of (2.16) is a unit matrix or simply Kronecker's delta.

2.6. Proper time. Four-momentum. Energy.

An interval related with a moving object $d\tau$ is called proper time and may be expressed as follows:

$$d\tau = cdt\sqrt{1-\frac{v^2}{c^2}} \quad . \qquad (2.17)$$

Relativistic 4D kinematics is closely related to the classical kinematics in 3D space. The four-vector of velocity of a particle is defined as follows:

$$\tilde{u} = \frac{d\tilde{x}}{d\tau} = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\vec{v}}{c\sqrt{1-\frac{v^2}{c^2}}} \right). \qquad (2.18)$$

This expression is similar to usual velocity because it has a vector character. On the another hand it is important to remember that a 4-vector is dimensionless and $\tilde{u} = 1$ is unit vector,

$$\tilde{u}^2 = \left(\frac{d\tilde{x}}{d\tau} \right)^2 = 1. \qquad (2.19)$$

Acceleration four vector:

$$\tilde{a} = \frac{d\tilde{u}}{d\tau} \qquad (2.20)$$

is perpendicular to velocity vector:

$$\tilde{u} \circ \tilde{a} = 0 \qquad (2.21)$$

From the physical point of view the most significant quantity is a *momentum four vector*. It turns out that a relativistic momentum is designed by multiplying velocity vector with particle mass and a speed of light (dimension!):

$$\tilde{p} \equiv mc\tilde{u} = \left(\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \right). \qquad (2.22)$$

Space components of the momentum 4-vector (2.22) become classical in the non-relativistic limit:

$$\vec{p}_{rel} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \xrightarrow{(v/c)\to 0} m\vec{v} = \vec{p}_{cl} \quad . \qquad (2.23a)$$

„Zero” (time-component) component of this vector,

$$p^0 = \frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (2.23b)$$

has no analogy in classical mechanics. There are many ways of proving that this is energy of particle moving with velocity *v*. One can consider small velocity limit:

$$\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} \approx mc\left(1+\frac{1}{2}\frac{v^2}{c^2}+\ldots\right) \approx \frac{1}{c}\left(mc^2+\frac{1}{2}mv^2\right) \quad . \qquad (2.24)$$

Hence, time-component of the momentum vector is the energy of the particle:

$$p^0 = \frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{E}{c} \quad . \qquad (2.25)$$

$$E_{tot} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad E_s \equiv E(v=0) = mc^2 \quad E_{kin} = E_{tot} - E_s \qquad (2.26)$$

where in equation (2.26) there are: total energy, invariant mass, and kinetic energy of a body of mass *m*.

Notice that:

$$\tilde{p} \circ \tilde{p} = m^2c^2 = \left(\frac{E}{c}\right)^2 - \left(p^2\right), \qquad (2.27)$$

Relativistic relationship between energy and momentum of body with mass *m* may be yielded from this equation:

$$E = \sqrt{m^2c^4 + c^2p^2} \quad . \qquad (2.28)$$

From this exact formula (2.28), two opposite limits are found:

a) classical,

$$m^2c^4 >> c^2p^2$$

$$E \approx mc^2 + \frac{p^2}{2m}, \qquad (2.29a)$$

b) (ultra)relativistic

$$m^2c^4 << c^2p^2$$

$$E \approx cp. \qquad (2.29b)$$

The relationship (2.29) used for massless objects results in (2.29b) energy and momentum relation for photon.

3. Introduction to General Theory of Relativity

3.1. Equivalence principle

Special Theory of Relativity (STR) allows us to understand phenomena observed from the point of view of inertial observer and henceforth it is intimately related only with inertial frames of reference. In the inertial frame of reference if there are no external forces acting on an object then it travels with constant velocity in a straight line. A straight line is defined by a light ray. The inertial observers are moving with constant velocity in a straight line with respect to other inertial frames of reference. However a common feature in the universe is a circular motion: the Earth is orbiting about the Sun, traveling along its orbit with speed 30 km/s; the Sun (together with all the planets in the solar system) is also orbiting about the center of the Galaxy (with speed 200 km/s). Naturally other planetary systems in other galaxies behave in a similar manner. All circular motions together with gravitational coupling yield non-inertial effects – in such world it is difficult to indicate which frame of reference is inertial. Because of that reason STR explains the nature of very narrow class of observers: since the moment formulation of STR in 1905, Einstein was looking for more general approach – General Theory of Relativity (GTR). A starting point for GTR (1916) was an observation (1908) later called *the equivalence principle* (see [3]).

How can we free ourselves from ubiquitous non-inertial forces (in particular centrifugal force)? In other words, when we may say that an object that no external force is acting on is in rest or moving with constant velocity in a straight line? It turns out that all we have to do is to let ourselves in the gravitational field i.e. to place an observer in free falling frame. A free falling elevator is freed from circulating frame of reference (the Earth) and so there is no centrifugal force acting on it. An object placed in such elevator no force is acting on it at all since a free falling object is moving with gravitational acceleration (given by a gravitational field). In this situation all the forces acting on the object i.e. gravitational pull and inertial force (due to gravitational acceleration of the frame of reference) are mutually canceling each other:

$$\left|\vec{F}_g\right| = \left|\vec{F}_b\right| \qquad (3.1)$$

This equality leads to important implications. Gravitational pull and inertial force may be measured independently. By placing different objects in the same position one finds the same value of acceleration due to gravitational pull. Having properly scaled measuring device we may also measure what is the value of the gravity force acting on an object. By taking quotient of the gravity force, different for the different objects, and the gravitational acceleration, having same value for all the bodies one may define mass as:

$$m_g = \frac{\left|\vec{F}_g\right|}{\left|\vec{g}\right|} \quad . \qquad (3.2)$$

This is *gravitational mass.* On the other hand, by acting on different bodies with fixed force and measuring resulting acceleration we may introduce a new definition of mass:

$$m_i = \frac{\left|\vec{F}_0\right|}{\left|\vec{a}\right|} \quad . \qquad (3.3)$$

This is *inertial mass*. Both definitions (3.2), (3.3) are independent and so both masses, inertial and gravitational one, do not in particular equal each other. Equation (3.1) proves however that

Inertial mass and gravitational mass are equal.

It is the first formulation, among three different ones, of the *equivalence principle*. Another obvious formulation was introduced earlier in this chapter:

In every situation one may always determine locally inertial frame of reference.

Indeed, free falling frame of reference satisfy conditions of the inertial frame. In the next section we will show that, in fact, this frame of reference is *locally inertial* one. One more formulation of equivalence principle may be stated as follows:

Locally inertial observer is not able to distinguish in which gravitational field he is in.

Equivalence principle seems to be one of the basic rules of the Nature.

3.2. Curved space

One of the outcomes of the equivalence principle is that gravitational field *curves* the space. In order to realize this fact all we have to do is to consider the geometry of a light ray in gravitational field. We may assume that a light ray is a straight line i.e. least distance curve. One may ask a question what is the property of such a light ray in a gravitational field? Let us consider two observers where one of them, called LI (local inertial) is placed inside free falling elevator and a second one, called SO (stationary observer) is placed on the surface of the spherical object being the origin of gravitational pull (see [1], [3]). The observers are tracing a light ray emitted from the elevator in position S and traveling towards mirror M. LI observer states that according to the postulate of the special theory of relativity the light ray defines a straight line in the space. This implies that SO observer finds the light coming from S reaches mirror M along curved line. The line curvature is towards gravitational field center. What it is the magnitude of the curvature? In a very crude approximation we may give a following explanation: let us assume that the elevator starts to move at the moment of the light emission and its initial velocity is zero. Then for the SO observer the light ray trajectory is a fragment of the parabola

$$x = ct \qquad y = \frac{1}{2} g t^2 \qquad (3.3)$$

(naturally this considerations should not be treated literally, this is only qualitative assessment, a true geometry of the light ray may be derived from equation of motion of massless object – a equation of motion for a photon). For example in the elevator of 1m width, free falling in a gravitational field, a light ray curvature is of 10^{-16} m order (hence measuring such an effect is impossible).

A relation between curvature, or more generally geometrical properties of spacetime and matter (energy) distribution is described by the General Theory of Relativity and it will presented in further sections. It is worth to point out two facts. Firstly, in earth's conditions the deformation due to gravitational field is very small (it is easy to realize that – see Problem 1). Secondly, matter, i.e. material objects, being a source of gravitational field causes a space deformation and the space geometry affects motion of material objects.

3.3. Equation of motion

Free particle in the locally inertial frame of reference follows straight line; accordingly its equation of motion is (see [2]):

$$\frac{\partial^2 \xi^\alpha}{d\tau^2} = 0 \qquad (3.2)$$

where ξ^α denote Local Cartesian (Local Inertial Frame) coordinates

$$d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta . \quad (3.3)$$

External observer, related to gravitational field is in curved space. His coordinates system x^μ is not rectilinear and appropriate equation of motion using these coordinates is derived, following transformation:

$\xi^\alpha \to x^\mu$,

described by a nonsingular matrix, $\dfrac{\partial \xi^\alpha}{\partial x^\mu}$.

In this global system x^μ gravitational field is described by the metric tensor $g_{\mu\nu}$:

$$d\tau^2 = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta}\left(\frac{\partial \xi^\alpha}{\partial x^\mu} dx^\mu\right)\left(\frac{\partial \xi^\beta}{\partial x^\nu} dx^\nu\right) = g_{\mu\nu} dx^\mu dx^\nu \qquad , (3.4a)$$

$$g_{\mu\nu} \equiv \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \qquad . \qquad (3.4b)$$

Inverse, (covariant) tensor $\hat{g}^{-1}$ is defined as:

$$g^{\alpha\mu} g_{\mu\nu} = \delta^\alpha_\nu \quad . \qquad (3.5)$$

Equation of motion in a global frame of reference is found by applying the transformation,

$$\frac{d\xi^\alpha}{d\tau} = \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau} \qquad .$$

So, its second derivative (the acceleration four-vector)

$$\frac{d^2 \xi^\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu} \frac{dx^\mu}{d\tau}\right) = \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{d^2 x^\mu}{d\tau^2} = 0 \qquad (3.6)$$

Multiplying equation (3.6) by inverse transformation matrix :

$$\frac{\partial x^\nu}{\partial \xi^\beta} \frac{\partial \xi^\beta}{\partial x^\mu} = \delta^\nu_\mu$$

one can obtain the equation of motion, expressed in terms of observer related to the stationary gravitational field.

$$\frac{d^2x^\lambda}{d\tau^2}+\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0 \quad . \tag{3.7}$$

One may ask a question: in what way information about gravitational field was introduced to relativistic equation of motion? In order to extract this knowledge we have to eliminate coordinates of a local inertial observer. Let us introduce three index parameter $\Gamma^\lambda_{\mu\nu}$ *affine connections*, or so-called *Christoffel's* symbols:

$$\Gamma^\lambda_{\mu\nu}=\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu} \quad . \tag{3.8}$$

Affine connection is not a tensor (see Problem 1). One can write down the equation of motion of a particle freely falling in the gravitational field

$$\frac{d^2x^\lambda}{d\tau^2}+\Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}=0 \; .$$

3.4. Chistoffel's symbols. A metric tensor.

Let us begin from a general description of transformation properties of vectors and tensors. Transformations between contra- and covariant vectors are performed using a metric tensor:

$$\begin{aligned} A_\mu &= g_{\mu\nu}A^\nu \\ B^\alpha &= g^{\alpha\beta}B_\beta \end{aligned} \quad . \tag{3.9}$$

A metric tensor allows lowering or rising of indices in higher rank tensors:

$$D_{\mu\nu}=g_{\mu\alpha}g_{\nu\beta}D^{\alpha\beta}.$$

A vector or tensor is defined as a quantity transforming in appropriate manner with the change of coordinates. If we change our set of coordinates from y to x, then a local transformation is described by a (nonsingular) matrix:

$$y^\alpha \to x^\beta \qquad \left[\frac{\partial y^\alpha}{\partial x^\beta}\right]. \tag{3.10}$$

Hence,

$$\{dy^{\alpha}\}\to\{dx^{\beta}\} \qquad dy^{\alpha}=\left[\frac{\partial y^{\alpha}}{\partial x^{\beta}}\right]dx^{\beta}$$

$$\{A^{\alpha}\}\to\{A'^{\beta}\} \qquad A^{\alpha}=\left[\frac{\partial y^{\alpha}}{\partial x^{\beta}}\right]A'^{\beta} \qquad (3.11)$$

$$T^{\alpha\beta}\to T'^{\mu\nu} \qquad T^{\alpha\beta}=\frac{\partial y^{\alpha}}{\partial x^{\mu}}\frac{\partial y^{\beta}}{\partial x^{\nu}}T'^{\mu\nu} .$$

Christofell's symbols $\Gamma^{\lambda}_{\mu\nu}$ may be expressed by the elements of metric tensor and its derivatives:

$$\Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}g^{\alpha\mu}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\gamma}}+\frac{\partial g_{\gamma\mu}}{\partial x^{\beta}}-\frac{\partial g_{\beta\gamma}}{\partial x^{\mu}}\right) . \qquad (3.12)$$

Relativistic equation of motion of the particle not exposed to the fields (interactions) other than gravitational represented by a metric tensor $g_{\mu\nu}(x)$, takes the form

$$\frac{d^2x^{\lambda}}{d\tau^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}=0, \qquad (3.13)$$

where Christofell's symbols are defined in (3.12) . Path (3.13) of the free particle in the spacetime is called *geodesic*.

Geodesics is a counterpart of the straight line in the special theory of relativity and we may interpreted it as straight line in spacetime. This fact is reflected when we apply variational approach. What is the path that a particle is following between points A and B such a path corresponds to the *minimal action* value:

$$\delta S_{AB}([x(\tau)])=\delta\int_A^B d\tau=\delta\int_A^B\frac{d\tau}{dp}dp=\delta\int_A^B\sqrt{g_{\mu\nu}\frac{dx^{\mu}}{dp}\frac{dx^{\nu}}{dp}}dp=$$

$$\int_A^B\frac{1}{2}\left\{\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\delta x^{\sigma}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}+g_{\mu\nu}\left(\frac{d\delta x^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}+\frac{dx^{\mu}}{d\tau}\frac{d\delta x^{\nu}}{d\tau}\right)\right\}d\tau= \qquad (3.14)$$

$$\int_A^B\frac{1}{2}\left\{\frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\delta x^{\sigma}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}+2\frac{d}{d\tau}\left(g_{\mu\nu}\delta x^{\mu}\frac{dx^{\nu}}{d\tau}\right)-2\delta x^{\mu}\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^{\nu}}{d\tau}\right)\right\}d\tau=0 .$$

The second term in the last equation vanishes at the boundaries as:

$$\delta x^{\mu}(A)=\delta x^{\mu}(B)=0,$$

and the last term is expressed as:

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^{\nu}}{d\tau}\right)=\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}}\frac{dx^{\lambda}}{d\tau}\frac{dx^{\nu}}{d\tau}+g_{\mu\nu}\frac{d^2x^{\nu}}{d\tau^2} . \qquad (3.12)$$

Hence a variation of "an action" (trajectory) equals to:

$$\int_A^B \delta x^\sigma \left\{ \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \left(\frac{\partial g_{\sigma\nu}}{\partial x^\lambda} \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau} + g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} \right) \right\} d\tau$$

and vanishes on trajectory satisfying minimal action (extreme point) condition:

$$\frac{1}{2} g^{\xi\sigma} \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \left(g^{\xi\sigma} \frac{\partial g_{\sigma\nu}}{\partial x^\lambda} \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau} + g^{\xi\sigma} g_{\sigma\nu} \frac{d^2 x^\nu}{d\tau^2} \right) = 0 \quad ,$$

$$\frac{d^2 x^\xi}{d\tau^2} + \frac{1}{2} g^{\xi\sigma} \left(\frac{\partial g_{\sigma\nu}}{\partial x^\lambda} + \frac{\partial g_{\sigma\lambda}}{\partial x^\nu} - \frac{\partial g_{\lambda\nu}}{\partial x^\sigma} \right) \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau} = 0 \quad . \qquad (3.13)$$

The last equation satisfies the principle of minimal „proper time" or in other words, „minimal length in the spacetime" and corresponding trajectory may be referred to as a „straight line". Let us underline two important facts. Firstly affine indices do not form a tensor (see Problem 3). Secondly, equation of motion (3.13) only seemingly reminds classical equation of motion in four dimensional space where gravitational force is substituted by a metric tensor and its derivatives. We have no knowledge about metric tensor of a gravitational field. It has to be found separately and it is a crucial problem in general theory of relativity: having given matter, and in essence, energy and momentum distribution what is the metric tensor? This problem was solved by Albert Einstein in 1916. We will follow Einstein's solution for the metric tensor with some simplifications: we will present a scheme coming from the weak gravitational assumption followed by final form of the equation.

3. 5. Weak-field approximation

Following relativistic equation of motion (3.13) correct in every situation i.e. in any gravitational field let us consider the case of the quasiclassical limit, of a slow motion in a static, weak gravitational field. Metric tensor of the weak gravitational field $g_{\mu\nu}$ differs slightly from metric tensor $\eta_{\alpha\beta}$ of an inertial observer:

$$g_{\mu\nu} = \eta_{\alpha\beta} + h_{\alpha\beta} \qquad (3.14)$$

where tensor h is a small perturbation,

$$|h_{\alpha\beta}| << 1. \qquad (3.15)$$

Equation of motion (3.13) in the lowest order of magnitude, takes the form:

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0, \qquad i = 1,2,3 \qquad (3.16a)$$

$$\frac{d^2x^0}{d\tau^2} = 0 \qquad . \qquad (3.16b)$$

These equations follow the form of affine connections (see Problem 3) and an assumption of slow motion. Eq. (3.16b) has the solution

$$\frac{dx^0}{d\tau} = const \qquad (3.17)$$

and

$$\Gamma^i_{00} = \frac{1}{2}\eta^{i\nu}\left(\frac{\partial h_{0\nu}}{\partial x^0} + \frac{\partial h_{0\nu}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^\nu}\right) = \frac{1}{2}\frac{\partial h_{00}}{\partial x^i} \quad . \qquad (3.18)$$

Therefore, (3. 16a), in this approximation*, takes the form*:

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00} = -\frac{1}{2}\frac{\partial h_{00}}{\partial x^i}\,. \qquad (3.19)$$

Comparing with the classical equation of motion:

$$\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}\varphi \quad . \qquad (3.20)$$

One finds one of the components of the metric tensor:

$$g_{00} = 1 + h_{00} = 1 + 2\varphi \qquad (3.21)$$

in the weak field approximation. Gravitational potential φ satisfies Laplace' equation:

$$\nabla^2\varphi = 4\pi G\rho(\vec{r}) \qquad (3.22)$$

and this may be regarded as an indication for the equation of "motion" of the metric tensor should fulfill, i.e. Einstein's equation.

4. Einstein's equation

An equation satisfied by a metric tensor should have a tensor form, in other words it should satisfy general covariance principle. In guessing its form we use equation (3.22); rewriting it helps finding right hand side of the equation. Returning to the previous notation:

$$g_{00} = 1 + 2\frac{\varphi}{c^2}\,, \qquad (4.\ 1)$$

one receives

$$\nabla^2 g_{00} = \frac{8\pi G}{c^4}\rho c^2\,. \qquad (4.\ 2)$$

And the right hand side is proportional to the energy density, i.e. to T_{00}, 00-element of the *energy momentum tensor*, $T_{\mu\nu}$.

One can guess then non-relativistic , weak-field approximation for the equation satisfied by the metric tensor is:

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}\,, \qquad (4.\ 3)$$

where $G_{\alpha\beta}$ jest linear combination of the metric tensor, its first and second derivatives. As it follows from the general covariance principle (see [2]), equation for arbitrary gravitational fields hold of the same form:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad (4.\ 3a)$$

where $G_{\mu\nu}$is a tensor that for a weak-fields is reduced to $G_{\alpha\beta}$. What is the form of a tensor $G_{\mu\nu}$? There are plenty of possibilities in choosing a form of $G_{\mu\nu}$ reducing to in weak fields to $G_{\alpha\beta}$; similarly there are plenty of ways leading to the right (the only one?) form of a tensor [2]. We will skip this procedure and give here a final form of $G_{\mu\nu}$ tensor , „geometro-dynamics" tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad (4.\ 4)$$

where $R_{\mu\nu}$ denotes *Ricci tensor* and R is its trace,

$$R = R^{\mu}_{\mu} \qquad R^{\alpha}_{\beta} = g^{\alpha\gamma} R_{\gamma\beta} \quad . \qquad (4.\ 5)$$

Ricci tensor is a contraction of the *Riemann's tensor* defined as a nonlinear function of the affine connections,

$$R^{\lambda}_{\mu\nu\kappa} = \frac{\partial\Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial\Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\lambda}_{\mu\gamma}\Gamma^{\gamma}_{\kappa\nu} - \Gamma^{\lambda}_{\mu\gamma}\Gamma^{\gamma}_{\kappa\nu}, \qquad (4.\ 6)$$

$$R_{\mu\kappa} = R^{\nu}_{\mu\nu\kappa}. \qquad (4.\ 7)$$

Finally *Einstein's field equation* takes the form:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}. \qquad (4.\ 8)$$

Einstein field equation (4.8) is rather complex system of ten, second order, partial differential equations. It may be shown that out of this ten equations, six of them are independent. Simplifications are possible due to the symmetry properties. In the next sections we will study examples of Einstein equations related to cosmological problems.

4.1. Energy-momentum density tensor

Let derive an energy-momentum tensor form starting from the special theory of relativity where an energy-momentum tensor of a system of non-interacting particles may be easily found. Density of momentum and density of the momentum current (flux) define an energy-momentum tensor. Energy-momentum density is:

$$T^{\mu 0}(x) = \sum_n p^{\mu}_n \delta^3(\vec{x} - \vec{x}_n(t)) \qquad (4.9a)$$

and corresponding current is

$$T^{\mu i}(x) = \sum_n p^{\mu}_n \frac{d\vec{x}^{\,i}_n}{dt} \delta^3(\vec{x} - \vec{x}_n(t)). \qquad (4.9b)$$

Both this definitions may be viewed as a a definition of energy-momentum density tensor:

$$T^{\mu\nu}(x) = \sum_n p^{\mu}_n \frac{dx^{\nu}_n}{dt} \delta^3(\vec{x} - \vec{x}_n(t)) \qquad (4.10)$$

where $x^0_n(t) = t$. One can notice that this matrix is symmetric:

$$p^{\alpha}_n = E_n \frac{dx^{\alpha}_n}{dt}$$

$$T^{\mu\nu}(x) = \sum_n \frac{p^{\mu}_n p^{\nu}_n}{E_n} \delta^3(\vec{x} - \vec{x}_n(t)) = T^{\nu\mu}(x) \qquad (4.11a)$$

and that it is a tensor,

$$T^{\mu\nu}(x)=\sum_n\int p_n^\mu \frac{dx_n^\nu}{d\tau}\delta^4(x-x_n(\tau))d\tau \qquad (4.11b)$$

One can also find that:

$$\frac{\partial}{\partial x^i}T^{\mu i}(x)=-\sum_n p_n^\mu \frac{dx_n^i}{dt}\frac{\partial}{\partial x_n^i}\delta^3(\vec{x}-\vec{x}_n(t))=-\sum_n p_n^\mu \frac{\partial}{\partial t}\delta^3(\vec{x}-\vec{x}_n(t))=-\frac{\partial T^{\mu 0}(x)}{\partial x^0}$$
$$+\sum_n \frac{dp_n^\mu}{dt}\delta^3(\vec{x}-\vec{x}_n(t)) \qquad (4.12)$$

$$\frac{\partial}{\partial x^\nu}T^{\mu\nu}=F^\mu \qquad (4.13a)$$

where on the right hand one finds *force density*

$$\frac{\partial}{\partial x^\nu}T^{\mu\nu}=F^\mu\equiv\sum_n \frac{d\tau}{dt}f_n^\mu\delta^3(\vec{x}-\vec{x}_n(t)). \qquad (4.13b)$$

If the particles are free, then the energy is conserved,

$$\frac{\partial}{\partial x^\nu}T^{\mu\nu}=0 \quad . \qquad (4.14)$$

If the particles are interacting via electromagnetic interactions, then the *energy of the system: particles plus electromagnetic field* is conserved. In fact

$$\frac{\partial}{\partial x^\nu}T^{\mu\nu}=\sum_n e_n F_\gamma^\mu \frac{dx_n^\gamma}{dt}\delta^3(\vec{x}-\vec{x}_n(t))$$

leads to

$$\frac{\partial}{\partial x^\nu}T^{\mu\nu}=F_\gamma^\mu(x)J^\gamma(x).$$

One can derive from the Maxwell's equations an equation similar to the above one expressing energy conservation:

$$\frac{\partial}{\partial x^\nu}T_{em}^{\mu\nu}=-F_\gamma^\mu(x)J^\gamma(x) \; .$$

Therefore, energy-momentum density tensor of the system, particles plus electromagnetic field,

$$T_{tot}^{\mu\nu}=T^{\mu\nu}+T_{em}^{\mu\nu} \qquad (4.15)$$

is conserved.

One can consider incompressible fluid, which in the rest in some frame of reference. Energy momentum density tensor takes the form characteristic for spherical symmetry:

$$\tilde{T}^{00} = \rho c^2 \qquad \tilde{T}^{ij} = p\delta_{ij} \qquad \tilde{T}^{0i} = 0 \qquad . \qquad (4.16)$$

Where above terms are proper density and pressure, respectively. In the laboratory system the fluid is moving (in a particular spacetime point) with velocity $\vec{v}$

$$x^{\alpha} = \Lambda^{\alpha}_{\beta}(\vec{v})\tilde{x}^{\beta} \qquad \Lambda^{\alpha}_{\beta}(\vec{v}): \qquad \Lambda^0_0 = \gamma \quad \Lambda^0_i = \gamma v_i \quad \Lambda^i_j = \delta_{ij} + v_i v_j \frac{\gamma - 1}{v^2} \qquad . (4.17)$$

In this new frame of reference, energy-momentum density tensor of ideal fluid does take the form:

$$T^{00} = \frac{\rho c^2 + pv^2/c^2}{1 - v^2/c^2} \qquad \tilde{T}^{ij} = p\delta_{ij} + \left(\rho c^2 + p\right)\frac{v_i v_j}{1 - v^2/c^2} \qquad \tilde{T}^{0i} = \left(\rho c^2 + p\right)\frac{1}{1 - v^2/c^2}$$

and it might be given in an apparently covariant form:

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + \left(\rho c^2 + p\right)u^{\alpha}u^{\beta} \qquad u^{\alpha} = \frac{dx^{\alpha}}{d\tau} \qquad . \qquad (4.18)$$

This last expression can be generalized into the fully covariant, generally relativistic equation substituting Minkovski metric tensor by its curved spacetime counterpart:

$$T^{\mu\nu} = pg^{\mu\nu} + \left(\rho c^2 + p\right)u^{\alpha}u^{\beta} . \qquad (4.19)$$

Or by inserting the term g in the energy-momentum density tensor of the system of particles

$$T^{\mu\nu}(x) = g^{-1/2}\sum_n m_n \int \frac{dx_n^{\mu}}{d\tau}\frac{dx_n^{\nu}}{d\tau}\delta^4\left(x - x_n(\tau)\right)d\tau \qquad (4.20)$$

Corresponding to the determinant of the metric tensor:

$$g = Det[g] \qquad \sqrt{g}d^4x = inv .$$

5. Central gravitational field – Schwarzschild's solution

Let us consider a case of a static, isotropic gravitational field, i.e. the case a central gravitational field. Assuming that the field source is an object of mass M, the gravitational potential and the field strength are defined as:

$$\varphi(r) = -\frac{GM}{r} \quad , \qquad (5.1a)$$

$$\vec{E}(r) = -\frac{GM}{r^2}\frac{\vec{r}}{r} \qquad ,(5.1b)$$

respectively. When the gravitational field is weak, it should be expected that, this description would suffice (at least in some sense). Usually we are really dealing with weak gravitational field and if so what kind of criteria should we use in order to distinguish weak from strong fields?

Using (5.1) we would derive the so-called "escape velocity":

$$v_{es}^2 = \frac{2GM}{r} \qquad (5.2)$$

and equating v_u^2 / c^2 to 1 one can claim that gravitational field is weak when „escape velocity" is small:

$$\frac{v_{es}^2}{c^2} = \frac{2GM}{c^2 r} \equiv \left|\frac{2\varphi}{c^2}\right| << 1 \qquad (5.3)$$

When the gravitational potential, expressed in dimensionless variables, is small, then the field may be truly considered weak. The escape velocity of Earth is: $v_{es} = \sqrt{\frac{2GM_Z}{R_Z}} \approx 1.1 \times 10^4 \frac{m}{s}$

and hence the gravitational potential (5.3) is small. Indeed the Earth's gravitational field is weak. Gravitational field of the Sun is only slightly stronger than the Earth's one. Strong gravitational field creates an object for which an escape velocity is comparable to the speed of light. The body for which the escape velocity is greater than the speed of light are called black holes. On the basis of (5.3) we may determine the critical radius of the body of mass M, i.e., when it becomes a "black hole":

$$R_c = \frac{2GM}{c^2} \; .$$

Earth would become a source of such a strong gravitational field, a black hole, if its entire mass would have to be squeezed to the size smaller than the dimensions of a matchbox. These type of considerations are limited since they were conducted in terms of classical approach. While the criterion of weak gravitational field is still qualitative, the expression for critical radius turns out to be rigorous! We shall return to this issue later in this chapter.

5.1. Central static gravitational field – relativistic case

Let us consider relativistic description of the case of static, time independent, isotropic (r dependent only in spherical coordinates) gravitational field (see [2]). What is the general form of a metric tensor? Non-diagonal terms $dx^i dt$ are vanishing since its sign changes under $t \to -t$ conversion,

$$d\tau^2 = g_{00}dt^2 + \sum_{ij} g_{ij} dx^i dx^j \quad . \qquad (5.4)$$

Since the space is isotropic it may be filled with spherical surfaces. On each such a surface with defined radius r, elemental length l may be written in a usual form using spherical coordinates:

$$dl^2 = r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (5.4a)$$

Finally metric tensor may be put into diagonal form and its linear element becomes:

$$ds^2 = g_{00}(r)dt^2 + g_{11}(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \ (5.5)$$

More formal derivation is as follow. „Static and isotropic" means that we may always choose such a set of coordinates $x^1, x^2, x^3, x^0 \equiv t$, that invariant self time $d\tau^2 = g_{\alpha\beta}dx^\alpha dx^\beta$, does not depend on t. Moreover self time is a function of spatial coordinates only via rotational invariant quantities in space, $\vec{x} \circ d\vec{x}, x^2, dx^2$. In most general form,

$$ds^2 = F(r)dt^2 - 2E(r)dt\vec{x} \circ d\vec{x} - D(r)(\vec{x} \circ d\vec{x})^2 - C(r)(d\vec{x})^2 \ . \ (5.6)$$

By choosing spherical coordinates:

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

we obtain

$$ds^2 = F(r)dt^2 - 2E(r)dt \cdot r \cdot dr - D(r)(r \cdot dr)^2 - C(r)\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \cdot d\varphi^2\right) \qquad .(5.7)$$

Since the initial moment is arbitrarily, we may choose a new variable t'

$t' = t + \Phi(r)$,

So we eliminate nondiagonal terms $dt \cdot dr$:

$$\frac{d\Phi(r)}{dr} = -\frac{rE(r)}{F(r)}.$$

Then self time may be transformed into diagonal form $ds^2 = F(r)dt'^2 - G(r)dr^2 - C(r)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \cdot d\varphi^2\right)$, (5.8)

where,

$$G(r) = r^2\left(D(r) + \frac{E^2(r)}{F(r)}\right).$$

Since we a have freedom of defining variable r , thus:

$$r'^2 = C(r)r^2$$

Finally we obtain line element in the form written above (cf. 5.6):

$$ds^2 = B(r')dt'^2 - A(r')dr'^2 - r'^2\left(d\theta^2 + \sin^2\theta \cdot d\varphi^2\right) \tag{5.9}$$

where,

$$B(r') \equiv F(r)$$
$$A(r') \equiv \left(1 + \frac{G(r)}{C(r)}\right)\left(1 + \frac{r}{C(r)}\frac{dC(r)}{dr}\right)^{-2} .$$

Last form of eq. (5.6) is widely used, so we will write it without a 'prime' sign:

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2\left(d\theta^2 + \sin^2\theta \cdot d\varphi^2\right) \tag{5.10}$$

Thus the metric tensor in this case may be written in diagonal form:

$$g_{tt} = B(r) \qquad g_{rr} = -A(r) \qquad g_{\theta\theta} = -r^2 \qquad g_{\varphi\varphi} = -r^2 \sin^2\theta \tag{5.11a}$$

with two unknown functions A, B that will have to be derived from field equations in our case. Inverted tensor, with indices at the top, may be written straightforwardly:

$$g^{tt} = B^{-1}(r) \qquad g^{rr} = -A^{-1}(r) \qquad g^{\theta\theta} = -r^{-2} \qquad g^{\varphi\varphi} = -\left(r^2 \sin^2\theta\right)^{-1} \tag{5.11b}$$

Using derives expression for metric tensor (5.8), affine indices may be written in more general form:

$$\Gamma^{\alpha}_{\beta\delta} = g^{\alpha\gamma}\Gamma_{\gamma\beta\delta} = \frac{1}{2} g^{\alpha\gamma}\left(\frac{\partial g_{\gamma\beta}}{\partial x^{\delta}} + \frac{\partial g_{\gamma\delta}}{\partial x^{\beta}} - \frac{\partial g_{\beta\delta}}{\partial x^{\gamma}}\right) , \qquad (5.11c)$$

$$\Gamma^{t}_{tr} = \Gamma^{t}_{tr} = \frac{B'(r)}{2B(r)} .$$

$$\begin{aligned}
&\Gamma^{r}_{tt} = \frac{B'(r)}{2A(r)} \qquad \Gamma^{r}_{rr} = \frac{A'(r)}{2A(r)} \qquad \Gamma^{r}_{\theta\theta} = -\frac{r}{A(r)} \qquad \Gamma^{r}_{\varphi\varphi} = -\frac{r\sin^2\theta}{A(r)} \\
&\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \qquad \Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta \\
&\Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r} \qquad \Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta
\end{aligned} \qquad . \qquad (5.11d)$$

Remaining indices vanish.

5.2. Schwarzschild's metric

Let us assume that the origin of a central, isotropic gravitational field is central body of mass M. Einstein field equation in empty space is:

$$R_{\mu\nu} = 0 \qquad (5.12)$$

where only non-vanishing elements of Ricci tensor are diagonal ones:

$$\begin{aligned}
R_{tt} &= -\frac{B''}{2A} + \frac{1}{4}\frac{B'}{A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\left(\frac{B'}{A}\right) \\
R_{rr} &= \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{1}{r}\left(\frac{A'}{A}\right) \\
R_{\theta\theta} &= -1 + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} \qquad . \qquad (5.12a) \\
R_{\varphi\varphi} &= \sin^2\theta \cdot R_{\theta\theta} \\
R_{\mu\nu} &= 0 \qquad \mu \neq \nu
\end{aligned}$$

Above equations may be put into differential equation of functions A and B. As it turns out the simplest formulation is:

$$\frac{R_{rr}}{A(r)} + \frac{R_{tt}}{B(r)} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \qquad (5.13)$$

yielding final result:

$$\frac{A'}{A} + \frac{B'}{B} = 0 \Rightarrow A(r)B(r) = const \quad . \qquad (5.14)$$

It is easy to prove (Problem 1) that the constant in equation (5.14) is equal to 1,

$$B(r) = A^{-1}(r). \qquad (5.14a)$$

Then,

$$R_{\theta\theta} = -1 + B'(r)r + B(r) = 0 \Rightarrow \frac{d}{dr}(B(r)r) = 1 \quad . \qquad (5.15)$$

Hence B is of following form:

$$B(r) = 1 + \frac{\alpha}{r} \quad . \qquad (5.16)$$

Knowing B we may derive A:

$$A(r) = \left(1 + \frac{\alpha}{r}\right)^{-1}$$

Obviously, (Problem 2), $\alpha = -2GM$ and a final solution of Schwarzschild may be presented as:

$$\begin{aligned} B(r) &= 1 - \frac{2GM}{r} \\ A(r) &= \left(1 - \frac{2GM}{r}\right)^{-1} \end{aligned} \quad . \qquad (5.17)$$

Schwarzschild’s metric (derived first by K. Schwarzschild in 1916)

takes following form:

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta \cdot d\varphi^2\right) \quad . \qquad (5.18)$$

This solution in empty space has singularity in:

$$R_S = 2GM \quad , \qquad (5.19)$$

known as Schwarzschild’s radius or critical radius where „time” and „radial” parts of metric tensor vanish or become infinity respectively. The appearance of singularity is due to choice of variable and with a different choice does not appear.

5.3. Motion in Schwarzschid spacetime – conservation laws

Let us consider a free fall a body in considered above central, static field (see[3]). The appropriate equation of motion is:

$$\frac{du^{\alpha}}{d\tau}+\Gamma^{\alpha}_{\beta\delta}u^{\beta}u^{\delta}=0 \qquad u^{\alpha}\equiv\frac{dx^{\alpha}}{d\tau}$$
$$\Gamma^{\alpha}_{\beta\delta}=g^{\alpha\gamma}\Gamma_{\gamma\beta\delta}=\frac{1}{2}g^{\alpha\gamma}\left(\frac{\partial g_{\gamma\beta}}{\partial x^{\delta}}+\frac{\partial g_{\gamma\delta}}{\partial x^{\beta}}-\frac{\partial g_{\beta\delta}}{\partial x^{\gamma}}\right) \tag{5.20}$$
$$d\tau^2=g_{\gamma\beta}dx^{\gamma}dx^{\beta}$$

will be written explicitly using eq. (5.8c, d) for affine indices. Then

$$\frac{d^2t}{d\tau^2}+\frac{B'(r)}{B(r)}\frac{dt}{d\tau}\frac{dr}{d\tau}=0$$
$$\frac{d^2r}{d\tau^2}+\frac{A'(r)}{2A(r)}\left(\frac{dr}{d\tau}\right)^2+\frac{B'(r)}{2A(r)}\left(\frac{dt}{d\tau}\right)^2-\frac{r}{A(r)}\left(\frac{d\theta}{d\tau}\right)^2-\frac{r\sin^2\theta}{A(r)}\left(\frac{d\varphi}{d\tau}\right)^2=0$$

(5.21a)

$$\frac{d^2\theta}{d\tau^2}+\frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau}-\sin\theta\cos\theta\left(\frac{d\varphi}{d\tau}\right)^2=0$$
$$\frac{d^2\varphi}{d\tau^2}+\frac{2}{r}\frac{d\varphi}{d\tau}\frac{dr}{d\tau}+2\cot\theta\frac{d\theta}{d\tau}\frac{d\varphi}{d\tau}=0 \quad . \tag{5.21b}$$

Considering motion in equatorial plane, $\theta=\frac{\pi}{2}$, we get two motion constants:

$$\frac{d}{d\tau}\left(\ln\frac{dt}{d\tau}+\ln B(r)\right)=0\Rightarrow\frac{dt}{d\tau}=\frac{a}{B(r)},$$
$$\frac{d}{d\tau}\left(\ln\frac{d\varphi}{d\tau}+2\ln r\right)=0\Rightarrow r^2\frac{d\varphi}{d\tau}=J \quad . \tag{5.21b}$$

Radial equation we reduce to the first order equation

$$\frac{d^2r}{d\tau^2}+\frac{A'(r)}{2A(r)}\left(\frac{dr}{d\tau}\right)^2+\frac{B'(r)}{2A(r)}\left(\frac{a}{B(r)}\right)^2-\frac{r}{A(r)}\left(\frac{J}{r^2}\right)^2=0\bigg|A(r)\frac{dr}{d\tau}$$

$$\frac{d}{d\tau}\left\{A(r)\left(\frac{dr}{d\tau}\right)^2-\frac{a^2}{B(r)}+\frac{J^2}{r^2}\right\}=0 \qquad (5.21c)$$

$$A(r)\left(\frac{dr}{d\tau}\right)^2-\frac{a^2}{B(r)}+\frac{J^2}{r^2}=const\equiv b$$

Integral constant b is not independent of two latter motion constants since they are combined with each other by interval:

$$1=B(r)\left(\frac{dt}{d\tau}\right)^2-A(r)\left(\frac{dr}{d\tau}\right)^2-r^2\left(\frac{d\varphi}{d\tau}\right)^2=B(r)\left(\frac{a}{B(r)}\right)^2-A(r)\left(\frac{dr}{d\tau}\right)^2-r^2\left(\frac{J}{r^2}\right)^2 \quad (5.21d)$$

We may see:

$b=1$. (5.21e)

Finally solution of equation of motion in our case takes following form:

$$B(r)\frac{dt}{d\tau}=a$$

$$r^2\frac{d\varphi}{d\tau}=J \qquad . \qquad (5.21f)$$

$$A(r)\left(\frac{dr}{d\tau}\right)^2+\frac{J^2}{r^2}-\frac{a^2}{B(r)}=-1$$

First of eqs. (5.21f) is a relativistic analogy of a classical energy conservation law. Second one illustrates a momentum conservation law. Radial equation:

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2+\left(\frac{J^2}{2r^2}-\frac{GM}{r}+\frac{J^2GM}{r^3}\right)=\frac{1}{2}\left(a^2-1\right) \qquad (5.21g)$$

has the form of classical equation of motion with additional component $\frac{J^2GM}{r^3}$ responsible of interesting relativistic effects. An important aspect of relativistic equations should be pointed out, conservation laws and the radial equation of motion are expressed in terms of a specific, per unit mass, quantities. This form has a deeper meaning: the equations (5.21) have their direct generalization to the case of massless objects - photons. Since photons are on the surface of the light cone so the interval is not a good parameter of the photon geodesic. For massless objects equations of motion of objects (5.21f) have the following form:

$$B(r)\frac{dt}{d\sigma}=\varepsilon$$

$$r^2\frac{d\varphi}{d\sigma}=\lambda \qquad (5.22)$$

$$A(r)\left(\frac{dr}{d\sigma}\right)^2+\frac{\lambda^2}{r^2}-\frac{\varepsilon^2}{B(r)}=0$$

where ε and λ denote conserved energy and conserved momentum of photon respectively.

5.4. Solution of the equation of motion.

3.4.1. Radial free fall

Let us solve radial free fall on center case (for some reasons we will apply general notation used earlier):

$$u^\alpha=\frac{dx^\alpha}{d\tau},\quad u^2=u^3=0\,.$$

And so,

$$d\tau^2=g_{00}\left(dx^0\right)^2+g_{11}\left(dx^1\right)^2$$
$$1=g_{00}\left(u^0\right)^2+g_{11}\left(u^1\right)^2 \,.$$

Zero component of equation of motion takes very Simple form (Problem 3):

$$\frac{du^0}{d\tau}=-\Gamma^0_{\beta\delta}u^\beta u^\delta=-g^{00}\left(\frac{\partial g_{00}}{\partial x^1}\right)u^0u^1 \qquad (5.23)$$

thus

$$g_{00}\frac{du^0}{d\tau}+\left(\frac{\partial g_{00}}{\partial x^1}\right)u^0u^1=\frac{d}{d\tau}\left(g_{00}u^0\right)0\Rightarrow g_{00}u^0=a \qquad (5.23a)$$

Radial component of equation of motion:

$$\frac{du^1}{d\tau}=-\Gamma^1_{\beta\delta}u^\beta u^\delta$$

takes form:

$$\frac{du^1}{d\tau}=(-1)g^{11}\left[\frac{1}{2}\left(\frac{\partial g_{11}}{\partial x^1}\right)\left(u^1\right)^2+\frac{1}{2}\left(-\frac{\partial g_{00}}{\partial x^1}\right)\left(u^0\right)^2\right]\qquad u^0=\frac{a}{g_{00}} \qquad (5.24a)$$

From where we find

$$\frac{d}{d\tau}\left\{g_{11}\left[\left(u^1\right)^2-a^2\right]\right\}=0\Leftrightarrow g_{11}\left[\left(u^1\right)^2-a^2\right]=const\equiv b \qquad (5.24b)$$

$$\left(u^1\right)^2-a^2=\frac{b}{g_{11}}\equiv -bg_{00}$$
$$\left(u^1\right)^2=a^2+bg_{00}\Leftrightarrow b=1 .$$

Equation (5.18b) is identical with classical equation of motion, the Newton equation, where time variable is substituted with self time. Thus the solution is same as in classical case, where it reflects conservation of energy law:

$$\left(u^1\right)^2+bg_{00}=a^2 \quad . \qquad (5.25)$$

Classical solution (5.24a) is connected with solution (5.23a) via interval (see (5.22a)). Finally:

$$\begin{aligned}&\left(u^1\right)^2+bg_{00}=a^2 \qquad g_{00}\left(u^0\right)^2+g_{11}\left(u^1\right)^2=1\Rightarrow b=1\\&\left(u^1\right)^2=a^2-g_{00}\end{aligned} \qquad (5.25)$$

and the connection with classical case is limited. Hence relativistic case of free fall (5.23), (5.25) has solution (5.23a), (5.15a) containing one integral constant. This parameter takes only positive values, $a\in(0,\infty)$:

a) $a=1$ represents the situation of free fall from infinity (escape problem),
b) $0<a<1$ represents problem of free fall from finite distant from the origin,
c) $a>1$ represents situation when the body is thrown in infinity towards the origin (inverting we may consider a case of vertical throw with velocity greater than escape velocity).

How the free fall is seen by different observers?

The distant observer, called an inertial one (I), states that velocity of free falling body towards the center is a following function of position:

$$\upsilon_I^2=\frac{1}{c^2}\left(\frac{dr}{dt}\right)^2\frac{\left(u^1\right)^2}{\left(u^0\right)^2}=\left(g_{00}\right)^2\left(1-g_{00}\right)=\left(1-\frac{R_S}{r}\right)^2\left(\frac{R_S}{r}\right) . \qquad (5.26)$$

He states that initially velocity grows but then reduces! (Problem 5) The observer falling on the center (IF – in-falling), performs proper-time measurement:

$$\left(u^1\right)^2 = a - g_{00}$$
$$\frac{d\tau}{dr} = -\frac{1}{u^1} = -\frac{1}{\sqrt{a-\left(1-\bar{r}^{-1}\right)}} \quad . \qquad (5.27)$$

Local stationary observer (LSO) positioned in r, which velocity 4-vector (unit! – Problem 7) is as follows (Problem 8):

$$u^0_{LSO} = \frac{1}{\sqrt{g_{00}}} \quad . \qquad (5.28)$$

He measures the velocity of a free falling body the moment it passes him. LSO states that the velocity of the free falling body from infinity is:

$$v^2_{LSO} = 1 - g_{00} \quad . \qquad (5.28)$$

This meets the definition of escape velocity: the expression (5.28) is the same as the classic expression for the escape velocity (see [4])!

5.4.2 Radial fall of a photon

Let us examine some aspects of motion of massless particles: (see (5.23)).

$$B(r)\frac{dt}{d\sigma} = \varepsilon$$
$$r^2\frac{d\varphi}{d\sigma} = \lambda$$
$$A(r)\left(\frac{dr}{d\sigma}\right)^2 + \frac{\lambda^2}{r^2} - \frac{\varepsilon^2}{B(r)} = 0 \quad .$$

Let us consider radial fall: fall of photon towards center, $\lambda = 0$.

Then a vector

$$\kappa^\alpha = \frac{dx^\alpha}{d\sigma}$$

is proportional to a photon momentum vector and it has a nonvanishing components:

$$\tilde{\kappa} = \left(\kappa^t, \kappa^r, 0, 0\right).$$

Time component, frequency (!), we find from conservation laws:

$$\kappa^t = \frac{\varepsilon}{B(r)},$$

The radial component remains constant $\kappa^r = -\varepsilon$ which follows eq. (5.23).

vector $\tilde{\kappa}$, which may be conveniently named „wave vector" of photon, Has in this case unusual simple form

$$\tilde{\kappa} = \left(\frac{\varepsilon}{B(r)}, -\varepsilon, 0, 0 \right),$$

Way of interpreting the radial free fall of photon towards center is related with measurement moment. LSO observer measures frequency of photon as::

$$\omega_{LSO} = \frac{\varepsilon}{\sqrt{B(r)}}$$

where $\omega_I = \varepsilon$ is a photon frequency send from distant, inertial observer positioned in infinity. This is well known fact of gravitational Shift towards Violet. And vice versa photon send by LSO will be measured by inertial observer as shifted towards red:

$$\frac{\omega_I(\infty)}{\omega_{LSO}(r)} = \sqrt{B(r)}.$$

Problems:

1. Prove that the constant in eq. (5.11) is equal to 1.
2. Prove that the constant in eq. (5.12) equals -2GM.
3. Using equation of motion (5.16), symmetry rules and free fall on a center considerations derive:
   a) equation (5.17)
   b) equation (5.18)
4. Prove that choice of integration constant in the solution of the equations of motion (5.17, 5.19) describes situations in paragraphs a- c) (see eq. 5.19a and a comment).
5. From the inertial observer point of view what is the maximal velocity of the object free falling on a „black hole"? When the body will reach a critical surface r=R(S), set by Schwarzschild's radius?
6. Prove that an observer falling onto a "black hole" reaches critical radius in finite time (see 5.21).
7. Prove that a velocity 4-vector of inertial observer in position r, $g_{00}(r) = 1 - \frac{2GM}{c^2 r}$, is a unit vector.

8. Prove that LSO velocity 4-vector has only one nonvanishing component equaled to (5.22).

9. What is the velocity measured by LSO at r, $g_{00}(r) = 1 - \frac{2GM}{c^2 r}$ in the case of a body radially thrown in infinity with initial velocity v_0?

6. Expanding Universe

6.1 Nebula in Andromeda

The night sky paradox remained unresolved until late twenties of the twentieth century. The breakthrough came thanks to the research of Edwin Hubble (see [1], [3]). The astronomer studied objects called nebulae mainly trying to determine their distance from Earth. It should be emphasized that at the beginning of the twentieth century people were aware of the existence of our galaxy, the Milky Way as the only galaxy. It was known that the Milky Way resembled a disk consisting of a vast number of stars forming clusters of more or less irregular configurations. The size of the Milky Way that time was estimated on a few thousands light years. A little was known about the outer areas and nothing or almost nothing about nebulae. Due to the characteristic shape of the nebula, a flattened ellipsoid, probably resembling the Milky Way viewed from the "long distance" (several times greater than its diameter), it was speculated that instead of gas clouds in our galaxy these distant galaxies are like our own! Surprisingly, this hypothesis, first denounced by the philosopher (of Konigsberg) I. Kant, was to be proven correct. However other arguments of Kant, such as hierarchical universe were false. The hierarchical universe concept assumed that a structure of the solar system, is repeated at subsequent stages from the infinitely small distances up to a distance infinitely large. In such a world, the stars congregated in clusters, seen as galaxies followed by a clusters of galaxies that at the next level would form metagalaxies and so on. Henceforth Kant suggested that the nebulae may be distant galaxies, which according to the distance scale tend to coalesce into clusters. On the way to verify this or other hypotheses on the nature of the Milky Way was the lack of methods or tools for measuring very long distances, as it turned out, of the order of hundreds of thousands of light years. However Hubble was able to find such a tool: the painstaking observations of the Andromeda nebula led to the identification of a flashing "dot”. In the attached photo there was a mark "VAR", written by the astronomer. “VAR” was the abbreviation of the "Variable" (star) word, which meant that the regular flashing "dot" is a variable star, so-called “Cepheid”. Knowing the relationship between Cepheid’s absolute brightness and a period of change and comparing it with apparent brightness and measuring period of change one can determine the distance between us and a Cepheid belonging to the "cloud". Hubble calculated that the distance from the nebula in Andromeda was more than eight hundred thousands light-years exceeding the distance limits for the Galaxy. At the same time, it became clear why the nebula resembled the shape of flattened disks: they were simply galaxies similar to our Galaxy but at distance from it by hundreds of thousands of light years and perhaps millions or tens of millions of light years. Of course, finding a variable star in one of the nebulae was not a universal guideline in the studies of a very distant objects in the Universe. But it was the crucial step that allowed the construction of other tools to conduct such measurements. By determining the distances to other galaxies nebulae, Hubble was able to build the statistics of galaxies and use other measurement techniques. By determining luminescence of the brightest stars in distant galaxies he was able to determine their distances. In short, in the twenties of 20th century, Edwin Hubble was ready to calculate the distances separating us from the spread of nebulae in the Universe. Recent data show that Nebulae, which are essentially huge clusters of stars similar to the Milky Way, contain from one hundred to eight hundred billion stars. Milky Way is probably one of nothing distinctive galaxies, probably like the sun is just a star, not anything in particular is standing out among the eight hundred billion stars in the Milky Way. Interestingly, the number of galaxies that are currently in our "vision area" is not less than the number of stars in the Galaxy.

6.2 Hubble's discovery - an expanding universe

Edwin Hubble was the first scientist that realized that nebulae are in fact distant galaxies. But he was not the first one to examine their movement. The simplest approach would be to assume that the nebulae-galaxies in the Universe, scattered over vast distances, are in mutual, chaotic motion. Hubble said, however, that beyond any doubt, the motion of distant galaxies is not chaotic. He exercised at the same time the method previously developed and used by Vesto Slipher.

Slipher at the beginning of the twentieth century undertook research of the properties of nebulae and came to the conclusion that, basically, they are moving away from our Galaxy. Although this result was sensational and enthusiastically adopted by Slipher, his announcement, was quickly forgotten. The reason was a failure to understand the essence of the discovery. The main obstacle in the correct reading of the Slipher's result, who suggested that statistically nebulae are moving away from us, was unfamiliarity with the nature of nebulae. Perhaps above all, a case that has sometimes played a more prominent role was a chance. Vesto Slipher announced the results of their research in 1915, somewhat earlier, before Albert Einstein introduced the formulation of general theory of relativity (1916r.). What was the Slipher's research?

6.3 The spectrum line of hydrogen atom

Every atom emits a light in a universal way that is characteristic for a given element. Hydrogen atom, the simplest one, consists of a positively charged proton and negatively charged electron, both equal values, $|e| = 1.6 \cdot 10^{-19} C$, resulting in its electrically neutrality. The atoms of every other element are also electrically neutral because they consists of the same number of protons confined within the nuclei, and the number of electrons forming a sort of "cloud" screening the nucleus. As the mass of the proton is much greater than the mass of the electron, almost the entire mass of the atom is concentrated in its nucleus, which occupies the center of the atom. In comparison with the size of the atom, $1A = 10^{-10} m$ the atomic nucleus is the tens of thousands of times smaller and has a size of the order $10^{-15} m$. Electrons surrounding the atomic nucleus can be arranged only at certain, discrete energy states. Electronic states may be found by solving stationary Schrödinger equation and although in general this not a trivial task, a hydrogen atom is one of the exactly solvable problems in quantum mechanics. Energy states are described by a simple formula:

$$E_n = -\frac{1}{n^2} 13.6 eV \qquad (6.1)$$

The lowest energy state, in this case corresponding to $E_1 = -13.6 eV$, is called ground state; higher energy states are called, excited energy states. The transitions between n-th and m-th energy state (6.1) are accompanied by emission or absorption of a photon carrying an energy $E_{nm} = h\nu_{nm}$ corresponding to such an energy transfer:

$$h\nu_{nm} = h\nu_0 \left(\frac{1}{n^2} - \frac{1}{m^2} \right) \qquad (6.2)$$

In this sense one can observe emission (bright) or absorption (dark) linear spectrum of hydrogen atom. This linear spectrum is arranged in a form of series of lines:

$$\frac{1}{\lambda_{nm}} = \frac{1}{\lambda_0}\left(\frac{1}{n^2} - \frac{1}{m^2}\right) \quad \lambda_0 = \frac{hc}{13.6eV} \approx 100nm$$

known as

a) Balmer series, $n = 1 \quad m > 1$, $\frac{4}{3}\lambda_0 \approx 133nm > \lambda_{1m} > \lambda_0 \approx 100nm$

b) Lymann series, $n = 2 \quad m > 2$, $\frac{36}{5}\lambda_0 \approx 720nm > \lambda_{2m} > 4\lambda_0 \approx 400nm$

c) Paschen series, $n = 3 \quad m > 3$, $\frac{144}{7}\lambda_0 \approx 2050nm > \lambda_{3m} > 9\lambda_0 \approx 900nm$

etc. The second series lay entirely in a visible part of electromagnetic spectrum.

Such a series of absorption (dark) lines if observed in a spectra of light incoming from distant objects may be shifted towards smaller, blueshift or larger, redshift wavelengths. The meaning of such a shift is that a source of the light is approaching or receding, respectively, from us.

Vesto Slipher and then Edwin Hubble found that spectra of the light coming from the nebulae, turned out to be statistically redshifted. However, Hubble was able to estimate the distance from the nebulae, hence identifying them as distant galaxies, receding from our own galaxy. Next, he was also able to find a statistical relation between recession speed v and the distance r: these two features turned out to be proportional each other,

$$v = Hr \qquad (6.3)$$

Coefficient proportionality, H is called Hubble's constant and its current value is estimated to be

$$H = 50 - 100\frac{km/s}{Mpsc}$$

In the large enough scale, of the order of magnitude, 300 mln l.y. the universe is homogeneous and isotric. In that scale it is expanding according to the Hubble's law (6.3). There are two interesting features associated with this law. First, Hubble's constant gives the age of universe,

$$t_0 = \frac{2}{3} H^{-1} \quad . \quad (6.4)$$

Second, its value enable us to predict whether the expansion would be stopped and contraction would follow, or not and expansion would continue “forever” (see below).

7 Dynamical properties of expanding Universe

7.1 Isotropic and homogenous

Following cosmological principle we may assume that in sufficiently large scale the Universe is isotropic and homogenous (see [1], [5]). We consider properties of that Universe and so all the conclusions apply to such a medium. The answer to the question in what scale the cosmological principle is satisfied brings the analysis of large scale matter distribution in the Universe. All the experiments we are able to perform clearly show that space is neither homogenous nor isotropic. We will take a closer look at the hierarchical structure of celestial objects:

- 10^{13} - The Solar System with characteristic planet distribution (distance scale 5 billion km),
- 10^{21} – a galaxy, the Milky Way, a cluster of 100 billion or, according to recent estimates, even up to 800 billion starts. It has disk like shape with a radius of the order of forty thousand light years and thickness of one thousand light years,
- 10^{22} – galaxies, with size comparable to the Milky Way, the distances between galaxies are of order of hundreds of thousands light years,
- Galaxies have a tendency to form groups of higher size order so called *clusters of galaxies* with a size of the order of millions of light years, 10^6 ly.,
- 10^{23} - group of galaxies to which the Milky Way belongs. It is dominated by a Virgo galaxy and thus it is called the Virgo Cluster.
- 10^{24} - Clusters of galaxies merge into larger clusters, so-called *super clusters of galaxies*, of the size of the order of tens and even hundreds millions of light years. Super-cluster to which we belong is called the Virgo super cluster, because it is dominated by the Virgo-cluster and has the size of the order of a hundred million light years. In the Virgo super cluster there are 2 500 large galaxies. Meta-galaxies are the products of heterogeneous, large-scale distances, inaccessible to ideas even of the early twentieth century scientists. However they seem to be at the top of the hierarchical structures in the Universe. There is no higher-order celestial formations. Super clusters of galaxies, roughly speaking, are spread uniformly (randomly) across the Universe so they are not clustering into larger structures.

So we may assume that at this level, the distance scale of 300 million light years , the cosmological principle is satisfied. Today, one can say that the cosmological principle states:

The universe is isotropic and homogeneous in the distance scale $3 \cdot 10^8 ly$.

7.2 Robertson-Walker's metric

Let us consider isotropic and homogeneous space. It turns out that from the geometrical point of view only three kinds of isotropic and homogeneous spaces are possible: closed, open and flat. The geometry of the closed space may be represented by a two-dimensional sphere in usual three-dimensional space. The geometry of the open space cannot be illustrated in this way, since two dimensional surface which meets the condition of isotropy and homogeneity of open space, cannot be presented in our three-dimensional world. In local terms, however, a saddle surface may illustrate it. A characteristic feature of a closed surface is that if we cut out a disk on the surface and then span it on the plane we will observe a surface deficiency: the surface of a circle of radius *r* turns out to be less than πr^2. The circumference of the circle turns out to be less than $2\pi r$. In the case of open geometry the opposite statement turns out to be true: the circumference of a circle of radius r is greater than $2\pi r$ and the surface of the disk is greater than πr^2 ; cutting out a disk and spanning it on the plane we will have "additional tabs". So the question appears: how geometry of a 3-dimensional closed, open and flat space may be described in analytic way?

We will present a derivation of the spatial part of a metric tensor in the case of two-dimensional geometry of the closed space (see [3], [5]); the transition to three-dimensional space and the other two types of geometry is a natural extension of derived metric. Let us consider a plane, two-dimensional homogeneous and isotropic space. Let us fill the entire space with circles centered at point O, the origin of the cylindrical system of coordinates. One of the coordinates is an angle $\varphi \in \langle 0, 2\pi)$ with the corresponding linear element $r d\varphi$, the other one is variable *r*. If the space is flat then corresponding linear element is *dr* and then line element is:

$$ds^2 = dr^2 + r^2 d\varphi^2 \qquad (7.1)$$

However, if may occur that the space is not flat. The line element along radial direction is not dr . If the two-dimensional space is the area of the sphere of radius *R* then:

$$ds^2 = R^2 d\chi^2 + r^2 d\varphi^2 , \qquad (7.2)$$

Then the line element expressed via variables (r, φ), has following form:

$$r = R\sin\chi \qquad dr = R\cos\chi d\chi$$

$$R^2 d\chi^2 = \frac{1}{\cos^2\chi} dr = \frac{1}{1-\frac{r^2}{R^2}} dr^2 \quad . (7.3)$$

$$ds^2 = \frac{1}{1-\frac{r^2}{R^2}} dr^2 + r^2 d\varphi^2$$

Such a structure can be extended to the case of three-dimensional space. Let us fill the space with sets of points with a fixed distance from any chosen point O, in our case the spheres with center O and radius r . For a single sphere we chose variables θ, φ and a line element,

$$dl^2 = r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right). \qquad (7.4a)$$

In the flat space variable r is related to an elemental length dr . If space is not flat but curved in a manner similar to the curved two dimensional space, described above, we may use the analogy of the surface of a sphere of radius *R* in four-dimensional space. As above, in the two-dimensional case, moving radially from the point r to the point $r + dr$ we cover distance $d\tilde{l}$:

$$d\tilde{l}^2 = \frac{1}{1-\frac{r^2}{R^2}} dr^2 \qquad (7.4b)$$

Line element in three-dimensional, *closed* space has the form:

$$ds^2 = \frac{1}{1-\frac{r^2}{R^2}} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 \qquad (7.5a)$$

or substituting r with variable, σ :

$$ds^2 = R^2\left(\frac{1}{1-\sigma^2}d\sigma^2 + \sigma^2\left[d\theta^2 + \sin^2\theta d\varphi^2\right]\right) \qquad (7.5b)$$
$$\sigma = \frac{r}{R} \in \langle 0,1\rangle$$

In the case of the *open* space, part of the angular line element remains the same due to the isotropic nature of space, however a radial part changes its from:

$$ds^2 = \frac{1}{1+\frac{r^2}{R^2}}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 . \qquad (7.6a)$$

Omitting the derivation in this case, we may notice that the substitution:

$$r = R\sinh\chi \equiv R\sigma \qquad dl = Rd\chi .$$

leads to (7.6a). Spatial part of the metric can be written as previously (cf. 7.5b):

$$ds^2 = R^2\left(\frac{1}{1+\sigma^2}d\sigma^2 + \sigma^2\left[d\theta^2 + \sin^2\theta d\varphi^2\right]\right) . \qquad (7.6b)$$
$$\sigma = \frac{r}{R} \in \langle 0,\infty\rangle$$

In the flat space case, the linear element, expressed in spherical coordinates has a well-known form:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 , \qquad (7.7a)$$

or, in analogy to (7.5a, 7.6a)

$$ds^2 = R^2\left(d\sigma^2 + \sigma^2\left[d\theta^2 + \sin^2\theta d\varphi^2\right]\right) . \qquad (7.7b)$$

Geometries of isotropic and homogeneous spaces, closed (7.5), open (7.6) and flat (7.7) can be described using a single metric, called *Robertson-Walker metric:*

$$ds^2 = \frac{1}{1-k\frac{r^2}{a^2}}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2 , \qquad (7.8a)$$

or

$$ds^2 = a^2\left(\frac{1}{1-k\sigma^2}d\sigma^2 + \sigma^2\left[d\theta^2 + \sin^2\theta d\varphi^2\right]\right) \qquad (7.8b).$$

where the parameter k takes three values

$$k = \begin{cases} +1 \\ -1 \\ 0 \end{cases} \qquad (7.9)$$

respectively for the closed, *open and flat space.* In the case of a closed space a parameter may be regarded as *the radius of the Universe,* however in other cases it cannot be interpreted in this way. In all three cases, $a(t)$ is the parameter of particular importance: we use it to measure distances in the Universe. The evolution of the Universe is a result of the change of *a* as a function of time, which means that changing the distance between points ("superclusters of galaxies") in the space takes place in a uniform manner throughout the space. Simply, the configuration of galaxies remain constant, "escape of the galaxies" has no literal sense: they *do not run* away from each other. Galaxies are moving away due to the expansion of space (see Problem 1).

7.3 Friedmann's equations

The properties of the evolving Universe, isotropic and homogeneous, describe the Einstein equation (4.8). Since the geometry of space, (7.8) contains only one time-dependent parameter, a(t) these equations should be reduced to a set of two differential equations. Actually, it turns out that in the case of isotropic and homogeneous spaces we obtain two differential equations for this parameter:

$$\frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G}{3}\rho \ , \qquad (7.10a)$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + kc^2}{a^2} = -\frac{8\pi G}{c^2}p \ , \qquad (7.10b)$$

where $\dot{a}$ is a derivative of a over time. Components of energy-momentum tensor on the right side of equality (7.10) , the density and pressure of "incompressible fluid", correspond to matter and radiation, filling the space. To solve the above equations, one would know the relationship between p and ρ, i.e. the equation of state. The equation of state has different character for different types of energy, leading to distinct types of expansions, of the Universe dominated by matter (now) and a universe dominated by radiation (early stage of expanding Universe – see below).

Equations (7.10) have two important features. First, it suggests that the Universe expansion decelerates

$$\frac{\ddot{a}}{a} = -\frac{8\pi G}{3c^2}\left(3p + \rho c^2\right) < 0 \quad (7.11)$$

On the one hand it complies with the naive expectation that the expansion should be slowing down by gravitational interactions. This kind of impression is deeply associated with the classical interpretation of the gravitational forces rather than gravity. The expanding isotropic and homogeneous Universe has no center of mass, resulting in slowing the "runaway" matter. Slowdown in the expansion (7.11) should be rather regarded as a relativistic description of gravitational interactions.

Secondly, equation (7.10) is a criterion for determining the value of the parameter k for our universe. So: is the Universe closed, open or flat? Using (7.10), we obtain the following equation:

$$kc^2 = a^2 \frac{8\pi G}{3}\left(\rho - \rho_c\right), \qquad (7.12a)$$

$$\rho_c = \frac{3H^2}{8\pi G} \qquad (7.12b)$$

where H is the Hubble constant, $H = \frac{\dot{a}}{a}$. Since the left-hand side of equation (7.12a) is a constant, the right-hand side must also remain constant. Sign of the right side depends on the relationship between the density of matter and radiation. The expression (7.12 b), having a density dimension, should be regarded as a *critical density value;* it is enough to measure the

Hubble constant value to determine the value of the critical density of matter. As is apparent from equation (7.12) universe is:

a. closed $k = 1$ when $\rho > \rho_c$,
b. open $k = -1$ when $\rho < \rho_c$,
c. flat $k = 0$ when $\rho = \rho_c$.

If we take the value of the Hubble constant:

$$H = \frac{\dot{a}}{a} = 50 - 100 \frac{(km/s)}{Mpsc} ,$$

then the critical density is of the order of:

$$\rho_c \propto 10^{-25} \frac{kg}{m^3} . \quad (7.13)$$

The observed density of matter, called *luminous matter*, is of the order of

$$\rho_{lm} \propto 10^{-27} \frac{kg}{m^3} . \quad (7.13)$$

roughly corresponding to the average of one proton per cubic meter (Problem2). The density of radiation is less than the density of matter by a factor of 1100.

Thus, it seems that the Universe *is open,* because the dominant form of energy, matter has a density equal to 3% - 5% of critical density.

Equations (7.10) bring yet another valuable piece of information. The first equation can be written as follows:

$$\frac{d}{dt}\left(\dot{a}^2 + kc^2\right) = \frac{8\pi G}{3c^2}\frac{d}{dt}\left(a^2 \rho c^2\right) . \qquad (7.14a)$$

The second one multiplied by a factor $a\dot{a}$ yields

$$2\dot{a}\ddot{a} = -\frac{8\pi G}{3c^2} a\dot{a}\left(3p + \rho c^2\right) . \qquad (7.14b)$$

Since the left sides of equations (7.14) are equal, we obtain the following relationship:

$$\frac{d}{dt}\left(a^2\rho c^2\right) = -a\dot{a}\left(3p+\rho c^2\right) = -\frac{1}{2}\left(3p+\rho c^2\right)\frac{d}{dt}\left(a^2\right).$$

and finally we find:

$$3\frac{1}{a}\frac{d}{dt}(a) + \frac{1}{p+\rho c^2}\frac{d}{dt}\left(\rho c^2\right) = 0 \qquad (7.15)$$

Equation (7.15) we may write in the form of first order differential equation

$$a^3\frac{d}{dt}\left(\rho c^2\right) + \left(p+\rho c^2\right)\frac{d}{dt}\left(a^3\right) = 0 \qquad (7.16a)$$

and because the elements of the equation has (energy / time) dimension and has the form similar to the principle of conservation of energy, it can be regarded as an *energy conservation law:*

$$\frac{d}{dt}\left(a^3\rho c^2\right) = -p\frac{d}{dt}\left(a^3\right) \ . \qquad (7.16b)$$

Rewriting equation (7.16) in the integral form

$$3\int\frac{\dot{a}}{a}dt + \int\frac{\dot{\rho}c^2}{p+\rho c^2}dt = const \ , \qquad (7.16c)$$

we find that the relationship between the density and pressure, equation of state, will actually allow us to solve the problem of dynamical properties of the expanding Universe.

7.4 Matter domination vs. radiation domination

7.4.1 Matter domiantion

If matter is the dominant form of energy as a diffused "gas", in which the role of single molecules takes galaxies superclusters, it can be assumed that the pressure of such a "gas" vanishes

$$p = 0 . \qquad (7.17)$$

Then the relation (7.16b) simply states that the density of matter is inversely proportional to the volume (amount of matter is conserved)

$$\rho \propto a^{-3}. \qquad (7.17b)$$

In such a world dominated by the matter the first Friedmann equation implies that if you omit kc^2 factor (true for very large values of a), a distance scale factor, in this case, the radius of the Universe, depends on time as

$$a(t) \propto t^{\frac{2}{3}}. \ (7.18)$$

7.4.2 Radiation domination

In the case of radiation domination, the gas equation of state (photon) is the equation of state of radiation being in equilibrium with matter i.e. blackbody radiation,

$$p = \frac{1}{3}\rho c^2. \ (7.19)$$

In this case, we obtain the characteristic for electromagnetic radiation relationship:

$$\rho_r \propto a^{-4}. \ (7.19b)$$

As with the dominance of matter case, we find an analytical description time dependence of the Universe size:

$$a(t) \propto t^{\frac{1}{2}}. \ (7.20)$$

7.4.3 The density of matter and density of radiation

Estimation of luminous matter density leads to value of the order one proton per one cubic meter

$$\rho_m(t_0) \propto 10^{-27}\frac{kg}{m^3} \quad . \qquad (7.21)$$

Estimating the density of radiation concerns only the relic radiation, CMB, with a temperature $T_{CMB} = 2.73K$ and leads to value:

$$\rho_r(t_0) \propto 10^{-30}\frac{kg}{m^3} . \qquad (7.21b)$$

The density of matter is about three orders of magnitude greater than the density of radiation, namely:

$$\frac{\rho_m(t_0)}{\rho_r(t_0)} = 1100 \quad . \qquad (7.22)$$

Today the Universe is dominated by matter. What's more, according to the equation (7.19), and in all future expansion of the Universe, the dominant form of energy will be matter. However, in the distant past there was a moment when the density of matter and radiation densities were equal. One can quite accurately determine that moment. As the ratio of the density of matter and radiation density is proportional to the gauge factor, in other words, the size

$$\frac{\rho_m(t)}{\rho_r(t)} \propto a(t) \qquad (7.21)$$

This ratio, which current, $t = t_0$, value is 1100, was equal to one in the Universe of 1,100 times less, $t = t_s$

$$1 = \frac{\rho_m(t_s)}{\rho_r(t_s)} = Aa(t_s) = (1100)^{-1}\frac{\rho_m(t_0)}{\rho_r(t_0)} \Rightarrow \frac{a(t_0)}{1100} = a(t_s) \quad (7.22)$$

If we assume that the current age of the Universe is about 10 billion years, i.e.

$$t_0 = 10^{18} s \quad . \qquad (7.23a)$$

This separation of radiation and matter took place at the moment

$$t_s = 10^{13} s \qquad (7.23b)$$

i.e. 300 000 years from the time that can be taken as the beginning of the expansion.

Problems

1. Consider the isotropic and homogeneous, closed two-dimensional space. Let the radius of the Universe at a certain moment would be $R(t_0)$. Define a straight line in this space? How do you measure the distance between points in this space? If we assume that the radius of this space is a linear function of time, $R(t) = At$, How to change the distance between the points of this space? Calculate the value of the constant Hubble'a this case.
2. What is the density of matter in the earth conditions? (assume the mass of Earth, $M_E = 6 \cdot 10^{24} kg$ and radius $R_E = 6.4 \cdot 10^6 m$)
3. What is the value of the density of matter in the Galaxy (assume the mass of the galaxy as equivalent to a 100 billion stars, $M_{MW} = 10^{11} M_S = 10^{11} \cdot 10^{30} kg$ and the radius and thickness as appropriate, $R_{MW} = 50000 ly$, $d_{MW} = 1000 ly$)
4. Calculate the density of matter in the Galaxy in the number protons per one cubic meter.
5. The available observations of the Universe is about 10 million groups of galaxies (like our own cluster, Virgo). Assuming size of the order of 10 billion light years, what is the corresponding the density of matter, expressed in "protons / cubic meter" in the Universe?
6. Derive formula (6.18) (6.20) determining the distance dependence of gauge factor $a(t)$ in the matter domination and radiation domination phases.
7. Take the inverse of the Hubble constant $\tau_H = H^{-1}$, for the age of the Universe. Express this value in *s*. The real age of the Universe is $t_0 = \frac{2}{3} H^{-1}$ - derive this value.
8. Rewrite Robertson-Walker metric (6.8) using a variable χ (See (6.2)) instead of the variable r.

8. Thermodynamic properties of the expanding universe

8.1 Relic radiation

Relic radiation, known as Cosmic Microwave Background, was discovered in 1965 by radio astronomers Penzias and Wilson [1], [3]. It is highly isotropic and homogeneous; it was shown that the relic radiation does not exhibit heterogeneity up to accuracy one ten-thousandth part. In 1993, the COBE experiment (Cosmic Background Explorer) [3], proved that in homogeneities appear at the level of one hundred-thousandth part. The history of the cosmic background radiation starts in the end of the forties of the twentieth century, when George Gamov, together with colleagues, Alpher and Herman, began studies on the early phase of expansion. In contrast to the present dominance of matter in the early period the space was dominated by the radiation, being in thermal equilibrium with matter. The thermal equilibrium of radiation and matter resulted in energy exchange: high-energy photons decay rapidly together with the birth of particle - antiparticle pairs and annihilation of such pairs occurs with the emission of photons. The Universe expansion resulted in the temperature decrease, which, in turn, led to lowering the photon energy. Gradually, the photon’s energy was insufficient to produce heavier particles, and also lighter particle-antiparticle pairs, and finally the emergence of such a pairs was completed. The other pairs annihilated and radiation and the matter remained in the form of excess particles. For some time radiation prevented the condensation of matter. Finally energy of photons has become insufficient to prevent of protons and electrons to be create bound states, and the first atoms where formed. In this way the radiation was decoupled from matter. Although the process was extended in time, a matter and radiation separated when the radiation temperature reached a 3000 K. From that moment, radiation and matter evolved independently in the sense that that the matter density decreased slower than the density of radiation. In followed expansion these two forms of energy never again reached a state of thermodynamic equilibrium. In the world of matter, condensation processes were taking place resulting in formation of structures of an increasing complexity: gradually nuclei are being formed (light atoms!), followed by the first atoms, then clouds of matter, protostars, stars, inside of which nuclei of heavier elements (but lighter than iron) were formed, aggregations stars, etc. The radiation in an expanding universe was cooling down in such a way that its temperature remained inversely proportional to its size.

Gamov studied the process of separating the radiation and matter, and proposed a hypothesis of a transition from radiation domination to matter domination phase, associated with the release of radiation. As a result of an expansion the temperature lowered down by 10^8 (eight orders of magnitude) and should be about 10 Kelvin degree. More detailed considerations lead to the estimation of the temperature of the radiation released at the time of separation:

$$T_0 = 5K$$

In 1965, a team composed of Dicke, Peebles, Roll and Wilkinson [2] undertook a study to examine Gamow hypothesis. But before the actual research had begun, the Gamow cosmic background radiation was discovered. Although the discovery was accidental, and its theoretical essence was interpreted by Dick, the discovery has brought its authors a Nobel Prize. Indeed, the cosmic background radiation, in addition to the discovery of the Hubble, is a key argument in the debate over the expansion of the Universe.

8.2 The evolution of thermal radiation

The temperature of the cosmic radiation, presently measured with high accuracy, is

$$T_{CMB} = 2.726K \quad (8.1)$$

and corresponds to radiation emitted by a black body with certain temperature T. What's more it's the closest to the ideal example of the blackbody radiation distributions. The radiation comes from all directions and as emphasized above, it is extremely homogeneous and isotropic. At the moment of release, that is, at the moment of matter-radiation separation, the temperature was 1100 degrees higher:

$$T_s = 3000K \quad (8.2)$$

Starting from that moment on, the whole evolution was dominated by matter, and this phase can, in contrast to the former one, is called the cold. Interaction between radiation and matter is negligible in macroscopic terms and radiation can be treated in such a universe as expanding adiabatically. Also in the hot phase, when matter and radiation were in equilibrium, the gas can is regarded as the gas of hot, i.e., relativistic particles,

$$pc >> mc^2 \Rightarrow E = \sqrt{m^2c^4 + p^2c^2} \approx pc \qquad (8.3)$$

and also be treated as adiabatic expansion. Photon's equation of state:

$$p = \frac{1}{3}u \equiv \frac{1}{3}\rho c^2 \qquad (8.4)$$

where p denotes the pressure and ρ energy density, shows that in the process of adiabatic expansion:

$$dU = d(uV) = -pdV \qquad (8.5a)$$

$$dU = d(uV) = -\frac{u}{3}dV \Rightarrow uV^{4/3} = const \qquad (8.5b)$$

internal energy density is inversely proportional to the volume's (4 / 3) power

$$u_r \propto R^{-4}. \qquad (8.6)$$

This well-known result is worth comment because it relates to the relativistic particles. In contrast to the case of classical particles, where the energy density is inversely proportional to the volume

$$\rho_m c^2 = u_m \propto R^{-3} \qquad (8.7)$$

energy density of relativistic particles do behave as above (8.6). This is because the energy of relativistic particles, (8.3) is inversely proportional to the wavelength associated with it:

$$E \approx pc = \frac{hc}{\lambda} \qquad (8.8)$$

Therefore, the energy density of radiation is proportional to the inverse fourth power (characteristic) length, i.e. the scale factor:

$$u_r \propto a^{-4}. \qquad (8.9)$$

Statistical physics provides, in addition to the equation of state (8.4), all of the information about the properties of photon gas in equilibrium, i.e. the properties of blackbody radiation. Spectral distribution function:

$$u_r(\nu)d\nu = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right)-1}d\nu \qquad (8.10)$$

provides the total energy density of radiation u_r:

$$u_r = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{\exp\left(\frac{h\nu}{k_B T}\right)-1}d\nu = \frac{\pi^2}{15}\frac{(k_B T)^4}{(\hbar c)^3} \equiv AT^4 \qquad (8.11a)$$

and the density of radiation

$$n_r = \int_0^\infty \frac{u(\nu)}{h\nu}d\nu = 1.202\frac{2}{\pi^2}\frac{(k_B T)^3}{(\hbar c)^3} \equiv BT^3 \ . \qquad (8.11b)$$

The energy density of a blackbody radiation (8.11) is therefore proportional to the fourth power of its temperature *(the Stefan-Boltzmann law)*

$$u_r \propto T^{-4}. \qquad (8.12)$$

Therefore radiation evolves so that its temperature, both during the radiation domination phase as well as during matter domination phase, is inversely proportional to the size of (8.9), (8.12) that is inversely proportional to the distance scale factor:

$T \propto a^{-1}$. (8.13)

The statement (8.13) concludes the earlier statement about the moment radiation release: the temperature was 1100 times greater, hence, the Universe was 1100 times smaller. The value of the expression (8.11a) is now $T_0 = 2.726$, about

$u_r \propto 10^{-30} kg/m^3$. (8.14)

and is almost 1100 times less than the density of (energy) matter.

It is worth to take a closer look at another parameter. The density of the CMB photons (8.11b), is

$$n_r = 4\cdot 10^8 \frac{1}{m^3}. \qquad (8.15)$$

Bright (luminous) matter density is equivalent to one proton per 2 m^3 yielding a conclusion that the ratio of the number of photons and the number of protons is of the order

$$\frac{N_f}{N_p} \propto 10^9. \qquad (8.16)$$

This value may be regarded as a constant. On the other hand if (8.16) is the *entropy of the Universe* it would have to be considered as the constant volume entropy.

For a full picture of the photon gas thermodynamics let us derive the expression for the entropy. Because

$$\begin{gathered} dS(T,V) = \frac{1}{T}\left[d\left(u(T)V\right) + p(T)dV\right] \Rightarrow \frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 S}{\partial V \partial T} \\ \frac{\partial S}{\partial T} = \frac{V}{T}\frac{\partial u}{\partial T} \qquad \frac{\partial S}{\partial V} = \frac{1}{T}[u+p] \Rightarrow \frac{\partial}{\partial V}\left[\frac{V}{T}\frac{du}{dT}\right] = \frac{\partial}{\partial T}\left[\frac{1}{T}[u+p]\right] \\ -\frac{1}{T^2}[u+p] + \frac{1}{T}\frac{dp}{dT} = 0 \Rightarrow \frac{dp}{dT} = \frac{1}{T}[u+p] \end{gathered} \qquad (8.17)$$

and following *Einstein's equations for isotropic and homogeneous medium,* i.e. the expanding Universe (see previous section)

$$\begin{gathered} \frac{d}{dt}\left[(u+p)R^3\right] = R^3\frac{dp}{dt} \qquad \frac{dp}{dT} = \frac{1}{T}[u+p] \\ \Rightarrow R^3\frac{dp}{dt} = R^3\frac{dT}{dt}\frac{dp}{dT} = \frac{dT}{dt}\left\{\frac{R^3}{T}[u+p]\right\} \\ \frac{d}{dt}\left[(u+p)R^3\right] - \frac{dT}{dt}\left\{\frac{R^3}{T}[u+p]\right\} = 0 \Leftrightarrow \frac{d}{dt}\left[\frac{R^3}{T}(u+p)\right] = 0 \end{gathered} \qquad (8.18)$$

we get conservation law, which combining with the expression

$$dS(T,V)=\frac{1}{T}\left[d([u+p]V)-Vdp\right]=\frac{1}{T}d([u+p]V)-\frac{V}{T^2}[u+p]dT \quad (8.18a)$$

leads to the conclusion that the photon gas entropy is preserved.

Entropy density in equilibrium is equal to:

$$S(T,V)=\frac{V}{k_BT}[u+p]=\frac{4V}{3k_BT}u_r \Rightarrow s=\frac{S}{V}=\frac{4u_r}{3k_BT}=2.888\cdot 10^9\frac{1}{m^3} \quad (8.19).$$

However, the ratio of the number of photons and the number of protons, having same order, remains approximately constant.

8.3 The evolution of the hot Universe

We are now prepared to describe the evolution of the Universe dominated by radiation. Before dividing the history of the Universe on different eras in radiation domination phase, let us determine the moment when matter and radiation, was separated from each other.

Since at present the expansion rate is characterized by:

$$a \propto t^{2/3}$$

a separation occurred at the moment t_s when the scaling parameter $a(t)$ was 1100 times smaller:

$$\frac{a(t_0)}{a(t_S)}=\left[\frac{t_0}{t_S}\right]^{2/3}=1100 \ .$$

Hence in the moment of separation, the age of the Universe may be approximated by:

$$t_S=\frac{t_0}{(1100)^{3/2}} \ .$$

The current age of the Universe may be determined from the Hubble constant value at around 15 billion years:

$$t_0 = \frac{2}{3H} \propto 10^{17} - 10^{18} s$$

Hence at the moment of separation, $t_S \approx 10^{13} s$ the Universe was 300 000 year “old”, and its temperature was 1100 times higher, 3 000 K. The density was a billion times larger than today and its value was

$$\rho(t_S) = 10^9 \rho(t_0) \propto 10^{-18} - 10^{-19} kg/m^3 .$$

How we may define the moment of separation of radiation and matter? This "moment" (in the history of the expansion of the Universe) marks the end of the period of radiation domination. Throughout this period, divided into smaller periods, also referred to as the "eras" (lepton, hadron, etc.) we are dealing with energy in the form of proton-photon plasma (this is only a conventional name). The earlier period is the history of the expansion of hot universe, whose size (inversely proportional to the temperature) changes over time in proportion to the square root:

$$a(t < t_S) \propto t^{1/2}$$

However, the density is proportional to the inverse of the parameter’s fourth power $a(t)$:

$$\rho(t < t_S) = \rho_r(t) \propto a(t)^{-4} \propto t^{-2}$$

and the temperature remained (as during the whole process of expansion) proportional to the inverse of size. Let us then look back over the course of events at a younger, denser and hotter Universe.

Problems

1. Justify the correctness of choosing the quotient of the number of photons and the number of protons (8.16) as measure of the entropy of the Universe.
2. What does it mean that the entropy of the expanding universe is constant? Is it permanent?
3. What is the density of matter in the earth conditions? (assume the mass of Earth $M_E = 6 \cdot 10^{24} kg$, and radius $R_E = 6.4 \cdot 10^6 m$)

9. Big Bang Scenario

The dominance of matter: a "galactic" period

"The Galactic" period was initiated as a result of the separation of matter and radiation. As the expansion rate is, $a \propto t^{2/3}$, the Hubble constant allows us to determine the age of the Universe, $t_0 = \frac{2}{3H} \propto 10^{17} - 10^{18} s$ for about fifteen billion years. The current density of matter is 1100 times greater than the density of radiation, $\frac{\rho_m(t_0)}{\rho_r(t_0)} = 1100$. These values were comparable at the moment of separation, when the Universe was 1100 times smaller, $a(t_S) = \frac{a(t_0)}{1100}$, and was some 300 000 years old.

That time the temperature was 3 000 K. It was the end of the equilibrium of radiation and matter: the pre-radiation was so energetic that it prevented atoms to be formed. The atoms were created only when protons and electrons of atoms of hydrogen and helium were not able to be broken down by the quanta of radiation. When finally atoms were formed and all the protons in a nucleus of helium and hydrogen bounded the electrons. The radiation, in principle, no longer had an effect on matter at least interaction with matter had declined dramatically. We say that the Universe at the moment of separation had become "transparent" to radiation. The matter, however, electrically neutral, as a result of the gravitational attraction underwent to what can be described as cumulative, irreversible process of condensation in an increasingly complex structures. Matter eventually began to bear alone new atoms and then more complex and sophisticated structures. This extremely complex process, involved the formation of protostars, the birth of stars, which ignite the hydrogen and helium, the end of life of stars so that iron could be created, the birth of the next generations of stars, the formation of clusters of stars, galaxies, metagalaxies. The most astonishing effect of this process is obviously present ability to observe it and the ability to ask questions about Big Bang history. The history of the matter creation will be explained in the second part of this study.

Preceding period of galactic age, is the phase of radiation dominance. Radiation dominated since its inception (the "Big Bang") until the moment separation i.e. approximately 300 000 years.

The dominance of radiation: a "photonic" period

At the end of this era, called "a radiant era", we are dealing with a "proton-photon" plasma. Looking back the temperature of the plasma rises proportionally to the inverse of the gauge factor,

$T \propto a^{-1}$, (1)

which in turn varies proportionally to the square root of time:

$a \propto t^{1/2}$ (2).

Energy density varies proportionally to the inverse fourth power $a(t)$,

$\rho(t < t_S) = \rho_r(t) \propto a(t)^{-4} \propto t^{-2}$ (3).

With some approximation, this is dynamical and thermal picture of the expanding universe dominated by the radiation. If we assume previously derived parameters such as temperature, age, density, corresponding to the time t = t_s, then presented scenario (1, 2, 3) allows us to investigate the behavior of these parameters over the entire evolution of the Universe. What are the characteristic periods in the era of radiation dominance?

These periods are known as the eras: lepton, hadron, quark, and finally the inflation era of the earliest era of Planck.

9.1 Lepton's era

Leptons are light elementary particles, electrons, tau and muon neutrinos and the associated electronic, muon and tau neutrinos. It is assumed that the characteristic energy corresponds to the mass of the electron,

$m_e = 9 \cdot 10^{-31} kg \qquad E = m_e c^2 = 531 keV$.

Lepton era was the period where the dominant matter, being in equilibrium with the radiation, is consisted of leptons. They were created in a form of lepton-antilepton pairs, by a single quantum of radiation. Thus, the characteristic energy of the radiation in this period was 531 keV which corresponds to the temperature

$$T_L \propto 10^{10} K \, .$$

It was a temperature of the Universe and was of the order of a billion Kelvin.:

$$\frac{a(t_S)}{a(t_L)} = \left[\frac{t_S}{t_L}\right]^{1/2} = 10^6 \, .$$

Approximately one second after the Big Bang:

$$t_L \approx 1s \, .$$

When the dimensions were less than six orders lower than at moment of the density separation

$$\rho(t_L) = \rho(t_S) \cdot 10^{24} \propto 10^8$$

being a thousand times greater than the density of water. Throughout this period, as a result of the turbulent energy exchange between matter and radiation leptons constantly arose and disappeared. Because the radiation contains fewer photons with energy exceeding 1 MeV, at the end of lepton era, the dominant processes become annihilation particle-antiparticle pairs. When the temperature had finally dropped below one billion Kelvin, all creation and annihilation processes were suspended and lepton composition became similar to one observed today.

9.2 Hadron's Era

The characteristic energy in this period was the energy corresponding to a single proton:

$$m_p = 1.3 \cdot 10^{-27} kg \qquad E = m_p c^2 \propto 10^3 MeV \, .$$

Thus, the temperature corresponding to the end of hadron era was of the order of trillion Kelvin:

$$T_H \propto 10^{13} K \, .$$

In relation to the lepton era the temperature were increasing thousand times and the dimensions were decreasing thousand times,

$$\frac{a(t_L)}{a(t_H)} = 10^3 .$$

Thus the age of the Universe with such a conditions may be estimated as a one millionth of a second

$$t_H \propto 10^6 s .$$

The density of matter-energy has a value

$$\rho(t_H) = \rho(t_L) \cdot 10^{12} \propto 10^{20} kg/m^3 .$$

9.3 Unification of electromagnetic and weak interactions

Another characteristic period of the expansion is associated with the so-called a unification of electroweak interactions. Electric and magnetic fields have been unified in a single electromagnetic interaction by Maxwell in 1865. A similar procedure of unifying different forms of long range electromagnetic and short range weak interaction called electroweak, was conducted in the late sixties of the twentieth century by S. Glashow, A. Salam and S. Weinberg. Today's distinction between weak and electromagnetic interactions is the result of phase transition, having the nature of spontaneous symmetry breaking. The effect of a symmetry breaking is the difference in the interactions nature: mediatory bosons in the case of electromagnetic interactions are massless objects and therefore the interaction is a long range one. Mediatory bosons in weak interaction, W and Z particles, are objects with finite mass, hence the interactions are short range. Characteristic energy for the unification of electroweak interactions corresponds to a temperature two orders of magnitude higher than the temperature characteristic of the hadron era

$$T_{E-WU} \propto 10^{15} K .$$

On this basis, one can estimate the age of the Universe

$$\frac{t_H}{t_{E-WU}} = \left(\frac{T_{E-WU}}{T_H}\right)^2 \propto 10^4 \Rightarrow t_{E-WU} \propto 10^{-4} t_L$$

$$t_{E-WU} \approx 10^{-10} s$$

(tenth of one billionth of a second i.e. hundred picoseconds) and the size and energy density in thr expanding Universe

$$\frac{\rho_{E-WU}}{\rho_s} = \left[\frac{a(t_s)}{a(t_{E-WU})}\right]^4 = \left[\frac{t_s}{t_{E-WU}}\right]^2 = 10^{46}$$
$$\rho_{E-WU} \propto 10^{28} kg/m^3$$

It is worth noting that the end electroweak interactions unification era corresponds to the current experimental capabilities. In other words, the theoretical prediction in this energetic region are experimentally verified. Looking further back in the expansion process is burdened with uncertainty.

9.4 Unification of electroweak and strong interactions

It is believed that all interactions are derived from a common pre-interaction. This opinion can not be proven and in opposition to it there are some eminent physicists. It is believed that gravitational interactions can not combined with other three types of interactions. But the belief in merging electromagnetic, weak and strong interactions into a single electro-weak-strong, seems to be feasible. The first part of the job was done and a theory of electroweak interactions successfully passed experimental tests. The second part of the task, the combination of electroweak interactions and strong short range interactions, where mediatory bosons are endowed with mass gluons, remains yet unfilled. However, if you accept that there is a uniform electro-weak-strong interactions, their distinction is also related, as previously in the case of electroweak interactions, with common symmetry breaking. Then the characteristic energy is about twelve orders greater and corresponds to the temperature

$$T_{GU} \propto 10^{27} K.$$

Thus, age of the Universe with strong-weak-electromagnetic interaction, can be assessed on an extremely small fraction of a second:

$$\frac{t_H}{t_{GU}} = \left(\frac{T_{GU}}{T_H}\right)^2 \propto 10^{28} \Rightarrow t_{GU} \propto 10^{-28} t_H \propto 10^{-34}$$

$$t_{GU} \approx 10^{-34} s$$

energy density was then:

$$\frac{\rho_{GU}}{\rho_s} = \left[\frac{a(t_s)}{a(t_{GU})}\right]^4 = \left[\frac{t_s}{t_{GU}}\right]^2 = 10^{104}$$

$$\rho_{GU} \propto 10^{86} kg/m^3$$

Currently we are twelve orders of magnitude from the area, being a negligible part of the history of expansion, it might seem that nothing special had happened before, but this is exactly the opposite. Exactly that period brings very important questions from the point of view not only history but also a Universe. These questions are related with two major periods: the inflation and the Planck eras. These two, however are only briefly mentioned within a table below, giving an essence of the story already presented.

HISTORY OF THE UNIVERSE

	Era	Time	Temperature (K)	Density kg/m^3	Comment	Remnant
	„Now"	15 bln y. $10^{18} s$	2.726	10^{-27}	Life on the Earth	
		5 bln y.			Solar System	**Planetary systems (?)**

Matter domination	Galaxy era	1 bln y. $10^{16} s$			First stars: nuclear fusion (heavier elements)	Galaxies, Supernovae
		300 000 y (10^{13}s)	**3 000**	10^{-18}	**Universe becomes transparent for radiation** **First atoms appear: H, He**	**Cosmic Microwave Background**
	Radiation				Plasma: electrons and protons	
					Nuclear fusion: He(4)	
		$1\ s$	10^{10}	10^{10}	Anti-electrons (positrons) disappearance	
Radiation domination	**Lepton**				Leptons domination	**Neutrino´s radiation**
		$10^{-4} s$	10^{12}	10^{16}	Antiprotons disappearance	
	Hadron				Hadrons appearance: quarks are bound into baryons and mesons	
		10^{-10}	10^{15}	10^{25}		
	Quark				Quarks – anti-quarks	

		10^{-34}	10^{27}	10^{76}		
	Inflation	10^{-35}				**Magnetic monopoles;**
	Great Unification				Broken symmetry: strong-electro-weak	**Asymmetry matter-antimatter?**
		10^{-43}	10^{32}	10^{96}		**Gravitons**
	Planck´s era	**?**	**?**	**?**	Quantum gravity?	**Supersymmetry?** **Superstrings?**

10. Dark Matter and Dark Energy

10.1 Dark Matter

Historical remarks

In 1933, Frank Zwicky studying Coma cluster's properties, found inconsistency in its dynamics. According to these considerations, there should be much more matter there than we could detect ("see"). The difference between the amount of "visible" (luminous) matter and "invisible" matter appeared to be enormous: there should be about 400 more matter there than we could "see". Zwicky's considerations have been ignored for tens of years. In 60's and 70's of the XX century, Vera Rubin investigating galaxy dynamics, noticed inconsistencies similar to those reported by Zwicky but on the galaxy scale. Dynamics of galaxies themselves was different from expected when taking into account luminous matter; there should exist non-luminous (dark) matter, or Newton mechanics is inappropriate tool in a quite non-relativistic regime. Rubin's investigations had began the discussion that lasts until today: we believe that there exists dark matter, i.e. a form of matter that interacts gravitationally but not electromagnetically. Current estimations show that there is much more dark matter than the luminous matter (see below). Key issue is that we still do not know *what* is a dark matter. In fact all of the arguments concerning the existence of the dark matter are of indirect kind and we are searching for the constituents of this, still mysterious form of matter that appears to be common in the Universe.

Evidence and candidates – current status

There is growing awareness of the existence on a very wide range of distances of unknown component of matter in galaxies and clusters of galaxies. First evidence comes from anomalous rotation curves of galaxies. One expects a decrease of a velocity of orbiting body as a function of increasing distance from the galactic centre. One observes, and this a common observation without exceptions, saturation of this quantity (velocity) for perypherial objects ("perypherial stars"). The simplest conjecture is an existence of "dark matter halo" in the galaxy scale, i.e. in the range of $10^5\ psc$.

The other type of evidence may be observed within the range of cluster of galaxies $10^6\ psc$ Massive, invisible objects are responsible for gravitational lensing. This represent relativistic effect: light rays travelling from distant objects are bent due to the presence of supermassive complexes, that remain invisible (do not recorded by means of other methods). The outcome arises as multiplied or distorted image that allows to calculate an undetectable by other means amount of dark matter.

Gravitational lensing is applied in the still other kind of observations: remnants of galaxy collisions containing hot X-rays sources. In such a case, illustrated in the *Bullet cluster* a crucial feature is separation of luminous and dark matter. Galaxy cluster 1E 0657-56, known as the "bullet cluster", is located 3.4 billion light-years away. The bullet cluster's individual galaxies are seen in the optical image data.

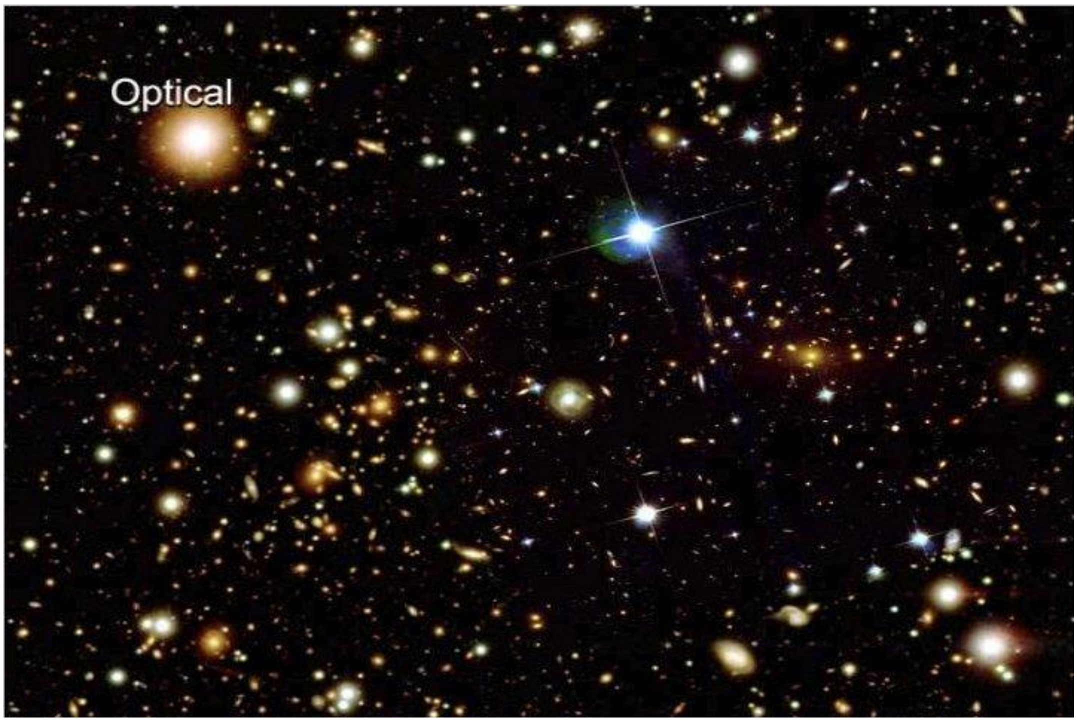

Fig. 3 Bullet cluster matter distribution

Their total mass adds to far less than the mass of the cluster's two clouds of hot x-ray emitting gas shown in red.

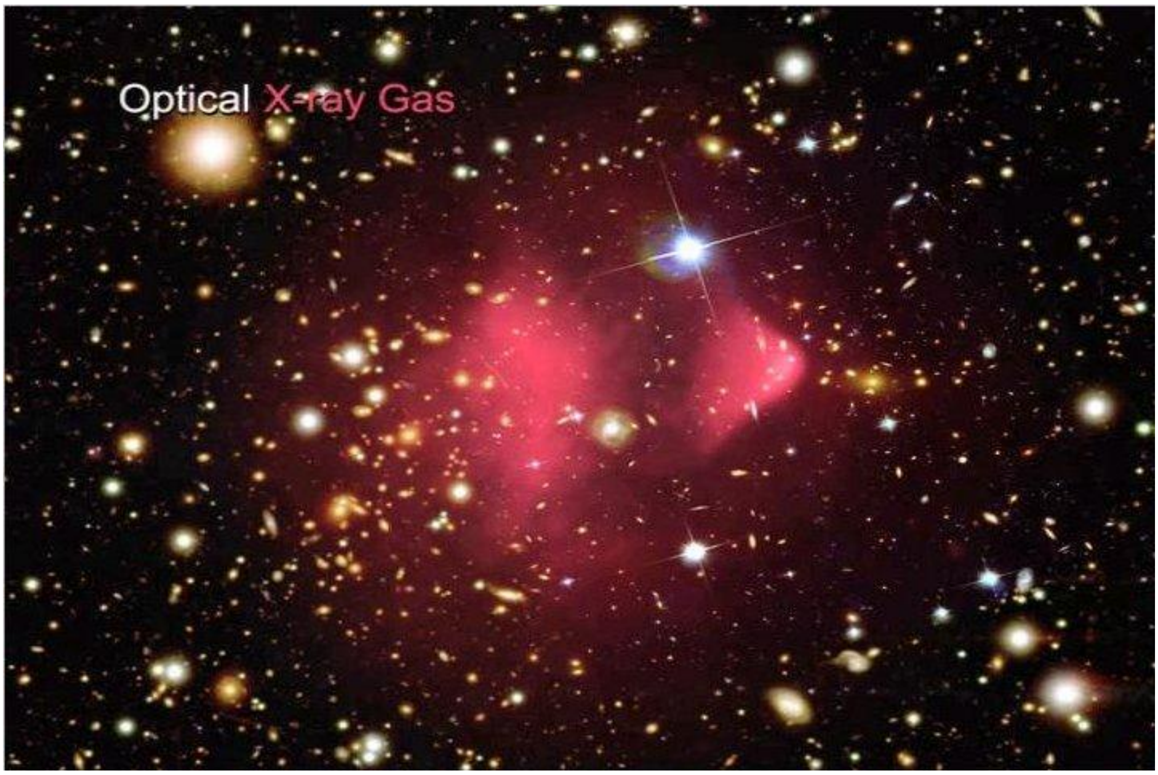

Fig. 4 Hot gas (X-ray source) distribution in Bullet cluster

Representing even more mass than the optical galaxies and x-ray gas combined, the blue hues show the distribution of dark matter in the cluster.

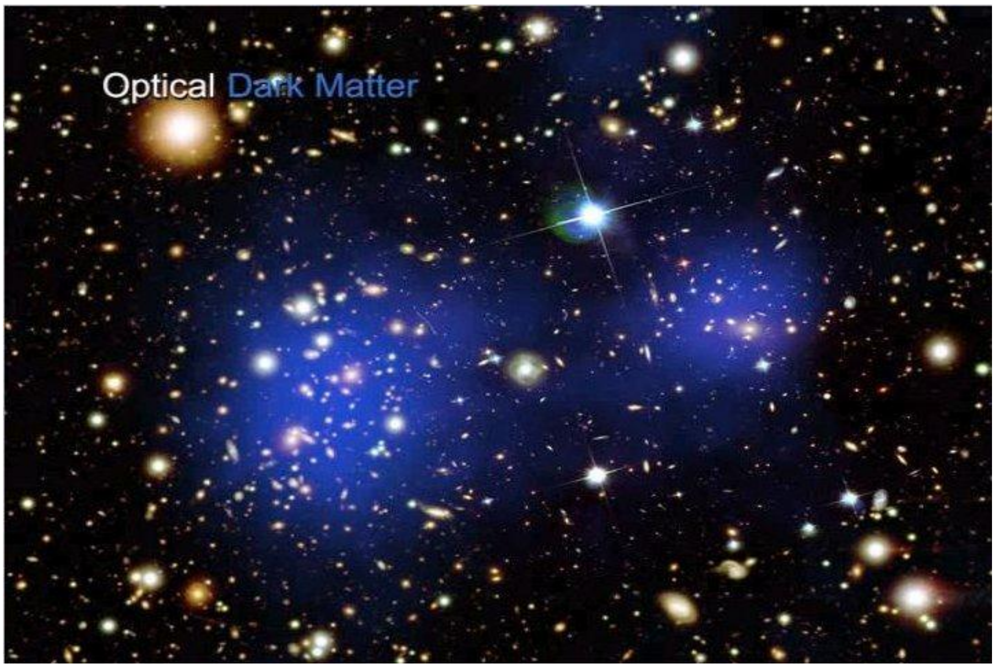

Fig. 5 Bullet cluster dark matter distribution

Otherwise invisible to telescopic views, the dark matter was mapped by observations of gravitational lensing of background galaxies.

The bullet-shaped cloud of gas at the right was distorted during the titanic collision between two clusters that created the larger bullet cluster itself.

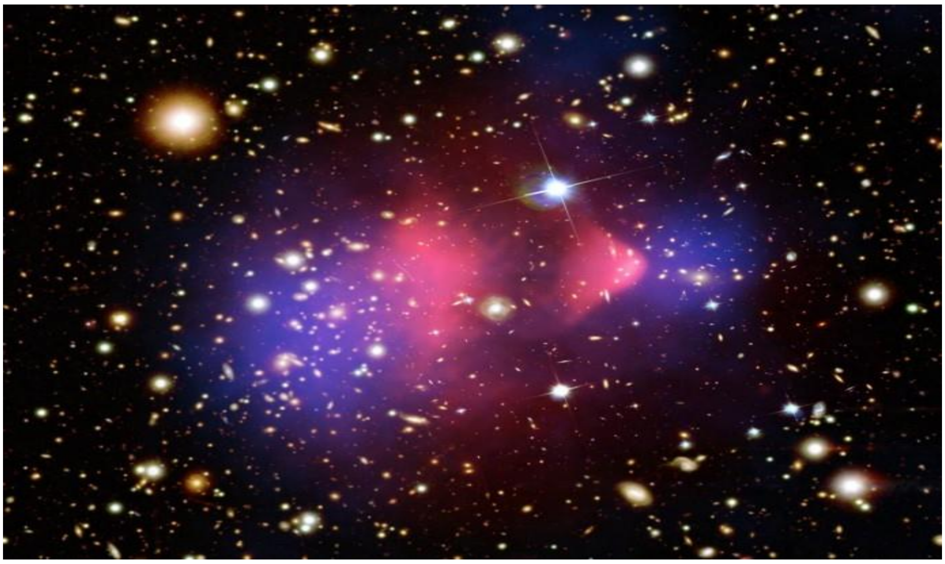

Fig. 6 Bullet cluster: dark and ordinary matter distribution

But the dark matter present has not interacted with the cluster gas except by gravity.

The clear separation of dark matter and gas clouds is considered direct evidence that dark matter exists.

Among candidates for the dark matter constituents, the following three are the best known.

MACHO – Massive Compact Halo Objects, “brown” objects of mass much smaller than solar mass are supposed to be present within the galaxies. Due to their small mass, these baryonic origin objects, could not initiate of the nuclear burning. Search, based on the lensing effect. MACHO may be noticed when crossing the line between an observer and light source, led to the conclusion that some 20 % Galaxy Halo might be non-emitting light objects

Exotic particles, created at a very hot period of the universe evolution. *Axions* are hypotetical light particles, of the mass of fraction of eV created at very early period, so now almost evenly distributed. WIMP – Weakly Interacting Massive Particles, yet exotic, non-baryonic particles. Their expected mass, related to the supersymmetry approach, belongs to the Large Hadron Collider and is supposed to be verified soon.

Despite great efforts, only single experiment performed in Italy (2003, San Gasso) reported a positive signal indicating presence of dark matter particles.

10.2 Dark Energy

and expansion of the Universe

Expansion of the Universe may be regarded in terms of Hubble's law, that means uniform recede of distant objects, with Hubble's constant as a coefficient of proportionality (see former Chapters). This expansion slows down whichever scenario it follows. The deceleration can be tested by applying SN Ia objects. Recent observations of supernovae of this Ia type showed that they are fainter than expected. The data gathered for some 40 SNIa were analysed at the end of XX c. with a final conclusion, that high redshifted objects are half-expected-luminosity indicating that the Universe is rather accelerating in its expansions. In the past decade more distant objects have been scrutinized confirming negative value of deceleration, i.e. signalling acceleration expansion rate at the distances comparable to the size of the Universe.

This may be considered as a "return" of cosmological constant Λ, firstly invented in order to stop expansion, currently being used as a parameter describing acceleration. It is interpreted then as a vacuum energy density accompanied by a negative pressure. Generally accepted term is "Dark Energy".

It appears then, galaxies, clusters of galaxies and the universe as a whole, contains far more matter than that which interacts with electromagnetic radiation – *dark matter component*.

Density of ordinary matter, baryons, and radiation is equivalent to 1 proton per cubic meter of space.

Only 4 % of the total energy density can be seen directly.

22 - 23 % of the total energy density is thought to be composed of the dark matter.

73 - 74 % of the total energy density is thought to consists of dark energy

Baryonic matter being a small fraction of energy of our universe is the only component known to us at the moment and it appears to be one the most challanging questions of the future to resolve still mysterious question: what actually means „Dark Matter” and „Dark Energy”

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