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I. ARTICLES

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PRE-ORDERS IN ECONOMICS AND MATHEMATICS

Preferences are fundamental for economics and we can find them by suitable experiments. Measuring the utility is rather a hard problem. The main thesis of this paper says that we can measure quality if we have defined relation of preference. Occasionally we have showed that the principle of mathematical induction is an obvious lemma about preference relation. There is also given some general fixed point theorem in a form of simple lemma. At the end we are giving some representation theorem. Every order in a finite set is isomorphic with the relation of divisibility in a set of natural numbers.

1. Theory of preferences is a foundation of economics, because theory of economical equilibrium can be reduced to theory of preferences. Preferences are special two-argument relations which are called pre-orders. Relation P defined in the set X is called pre-order if it is reflective and transitive. Any order, that is reflexive, antisymmetric and transitive relation, is of course pre-order. Naturally identity function I from set X into X is a relation of preference. Also the full relation $X \times X$ is a relation of preference. This relation is the multifunction assigning to every point of the set X set X as its meaning.

Relations of preference are typical in economics and we can easily find them by suitable public opinion polls. Measuring the utility is a difficult problem (Luce 1996). In economics we do not have strict definitions, so we cannot measure phenomena described by economic notions. Level of living is a very good example of that kind of notion, particularly poverty, other are social prestige, utility of goods, quality of education, and economical development of a region. All these notions we may define by one term – preference. Even risk is a preference (Rybicki, Smoluk 1996). For very large class of preferences there are theorems on their numerical representation. So we have scales of utility. It means that if in the set X exists preference, then there exists numerical function which is increasing and accommodated with the relation of preference. Such a function is just a numerical representation of the relation of preference, that is this function measures the phenomenon defined by the relation. In special cases these functions are even continuous.

Evidently every numerical function uniquely defines preference relation of a special kind. The set of all numerical functions is imbedded in the set of

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preferences, and set of preferences is imbedded into the family of multifunctions. Our main thesis is the proposition that preference is an origin notion and the measure results from it. We can measure quality if we have defined a relation of preference. Of course not every relation of preference has a numerical representation. For example, a lexicographic order in set R^2 (Cartesian product of the set R of real numbers) cannot be measured in the sense stated above. Pre-order P in the set X is numerically realizable if and only if there exists a function $f : X \to R$, such that for every member of X is assigned a number – measure of quality that this member has – with the following property: for every a, $b \in X$ statement aPb is equivalent to the statement $f(a) \leq f(b)$. The most important relation of preferences is the relation of implication in a set of some sentences. Occasionally we show that principle of mathematical induction is an obvious lemma about preference relation. In the last we state and prove some general lemma on existing of a fixed point. That lemma is usually implicitly main part of almost all proofs of the theorems on the existence of fixed point. It is stated here because the lemma is universal and uses preferences. We can see that not only principle of mathematical induction, but also some theorems on existence of a fixed point follow from a simple lemma concerning the relation of preference. At the end we are giving some representation theorem. Every order in a finite set is isomorphic with the relation of divisibility in a set of natural numbers. This at first glance unusual statement is quite simple and obvious on second thought.

2. The symbol $F(X, X)$ represents the family of all functions f with arguments and values in a set X. Let further the symbol $P(X)$ represent the family of all preferences in the set X. If f is an element of the set $F(X, X)$, than symbol $P(f)$ represents the relation of preference such that $xP(f)y$ if and only if exists natural number n such that $y = f^n(x)$, where f^0 is the identity function and $f^{n+1} := f \cdot f^n$ (composition of functions).

LEMMA. Relation $P(f)$ is a preference relation in the set X. It is called pre-order generated by the function f

Proof is obvious. If pre-order is generated by a function then for every element there is a consequence – the successor immediately following this element.

THEOREM. The transformation which associates the preference relation $P(f) \in P(X)$ with each function f from the set $F(X, X)$ is an imbedding.

Proof is simple and is omitted. On the basis of this theorem we can treat members of the set $F(X, X)$ as preference relations. We can even say that the set $F(X, X)$ is the subset of $P(X)$.

3. Every function whose domain is the set $N = \{0,1,...\}$ of natural numbers is called a sequence. Sequence f is called increasing if the range of it is an ordered (pre-ordered) set and if $n \leq k$, then $f_n \leq f_k$, where $f = (f_0, f_1, ...)$.

If the range is naturally ordered two-element set $\{0, 1\}$, that is $0 \leq 1$, then we can easily put down the family of all increasing sequences: (1, 1, 1,...), (0,1,1,...), (0, 0, 1, ...), ... and of course the zero sequence (0, 0, 0,...). In that case every increasing sequence is a stationary sequence or a sequence with finite number of zeros at the beginning and then with all terms equal one. This statement is just the principle of mathematical induction. If the increasing sequence has a term equal 1, then all its terms, excluding perhaps a finite number, are evidently also equal 1.

LEMMA. (Principle of mathematical induction). If a function $f : N \to Y$ mapping the set N of natural numbers into an ordered set Y is increasing, and the term f_0 is a maximal element of the set Y, then the function f is constant function.

Relation of implication is a key for understanding the principle of mathematical induction. Let I be a relation in some set S of statements defined as follows: pIq if and only if $p, q \in S$ and p implies q. Relation I is a pre-order in the set S. Every pre-order relation generates an order relation. A relation E defined by the condition: pEq if and only if pIq, and qIp is an equivalence relation in the set S. Family of the equivalence classes S/E is, if the set S is non-empty and closed with respect of negation (p in S implies that non p is in S), the naturally ordered two points set $\{0, 1\}$. The element 0 represents equivalence class of all false sentences and 1 represents equivalence class of all true sentences. The element 1 is the unique maximal element of the set $\{0, 1\}$ of equivalence classes. We can now express the principle of mathematical induction in the form stated above in the lemma. It is equivalent with classical formulation (Smoluk 1987).

If the sequence (p_n) of sentences fulfils the conditions:

1° p_0 is true,

2° for every $n \in N$ the statement p_n implies p_{n+1} is true, then every sentence p_n is true.

First condition means that the equivalence class p_0/E is equal 1, and the second condition means that the sequence (p_n/E) of equivalence classes is increasing. From educational practise we know that above lemma on increasing sequences is quite well accepted, but the principle of mathematical induction in classical formulation is rather hard for students.

4. Sequence (x_n) whose terms are members of the set X is called trajectory of the point x_0 relative to a function $f \in F(X, X)$ if $x_{n+1} = f(x_n)$ for every subscript $n \in N$. From the definition of the pre-order $P(f)$ generated by the function f it follows that trajectory is an increasing sequence relative to pre-order $P(f)$.

Trajectory (x_n) is a cycle if there exists a natural number m such that $1 \leq m$ and $x_0 = f^m(x_0)$. Minimal number m with this property is called the cycle length. Subset A of the set X is called invariant relative to function $f \in F(X, X)$, if image $f(A)$ of the set A is a subset of A. Of course the set $\{x_0, x_1, ...\}$, that is the range of the trajectory (x_n) relative to the function f, is invariant relative to f.

Pre-order $P(f)$, as any pre-order, is naturally an oriented graph for which X is the set of the vertices and the set of arrows is $\bigcup_{n=0}^{\infty} f^n$. Maximal connected (connected in the sense of graph) subset of the set X is called an orbit of the functions f. Orbits are invariant sets. If the set $\{a\}$ is invariant relative to function f, then the element a of the X is called fixed point of the function f. The set of all fixed points of function f is denoted by Fix(f). There is an important question: for which function f the set Fix(f) is non-empty? This question concerns existence and this is the reason why it is so significant in mathematics and applications. Under what condition the equation

$$f(x) = x,$$

in not contradictory? Many problems concerning existence of resolutions of some equations can be reduced to the question whether the set Fix(f) is non-empty. If x_0 is a fixed point of the function f, then its trajectory (x_n) is a cycle of length 1. If trajectory (x_n) is a cycle of length m and a subset A of the set $\{x_0, x_1,...\}$ has m points, then $A = \{x_0, x_1,...\}$ and $f(A) = A$. It means that if Y is the family of equivalence classes given by the equivalence relation $E(f)$ generated by the pre-order $P(f)$, then the set A is a member of the family Y and this class A is a fixed point of the mapping

$$F: \quad Y \to Y$$

which is defined by the equation

$$F(x/P(f)) = f(x)/P(f),$$

where $x/P(f)$ is the equivalence class of the member $x \in X$, that is

$$x/P(f) = \{y \in X \mid xP(f)y\}.$$

THEOREM. There exists a cycle of the function $f \in F(X, X)$ if and only if the set Fix(F) is non-empty, where the transformation F was defined earlier.

Conclusion. If set X is finite, then every function $f \in F(X, X)$ has a cycle. That means that function F generated by f has a fixed point.

LEMMA. If a function $f \in F(X, X)$, X is a compact Hausdorff's space, and f is continuous at point $a \in X$ which is unique point of accumulation of the trajectory (x_n), then $a \in$ Fix(f).

Proof. Because space X is compact and the sequence (x_n) has unique accumulation point $a \in X$, then $a = \lim(x_n)$. Function f is continuous at this point, so we have $f(a) = \lim(f(x_n))$. From definition of the trajectory (x_n) it follows that $x_{n+1} = f(x_n)$, so we receive the conclusion: $f(a) = \lim(f(x_n)) =$

$= \lim(x_n) = a$. This simple lemma is an example of general theorem concerning fixed points.

5. If P is a pre-order in the X, then there is a unique multifunction $h : X \to 2^X$ such that $h(a) := \{x \in X | \; xPa\}$. Relation P is an order if and only if the function h is an imbedding. In the sequel the empty set is excluded from the range of multifunction. If $f : X \to 2^X$ is a multifunction, then by definition f^0 is trivial (identical) multifunction such that $f^0(x) = \{x\}$, for each $x \in X$. For every natural number n multifunction f^{n+1} is defined by induction: $f^{n+1}(x) = = \bigcup\{f(y) \,|\, y \in f^n(x)\}$. Obviously every function $f \in F(X, X)$ may be treated as multifunction if we put $H(x) = \{h(x)\}$. If f is a multifunction, then f generates unique pre-order in X in the same way as ordinary function: namely xPy if and only if there is such $n \in N$ that $y \in f^n(x)$.

THEOREM. For every pre-order P in the set X there is a multifunction f such that

$$P = P(f).$$

For proof it is enough to see that if P is a pre-order, then multifunction $f(a) = \{x \in X | \; aPx\}$ generates a pre-order P.

Conclusion. The set of functions $F(X, X)$ is imbedded in the set of all pre-orders $P(X)$, and the set $P(X)$ of all pre-orders is imbedded in the set $M(X, X)$ of all multifunctions, so we have

$$F(X, X) \subset P(X) \subset M(X, X).$$

6. THEOREM. Every order in the finite set is isomorphic with divisibility in the subset of natural numbers (Smoluk 1996).

In a finite set there is not any other order than divisibility. It is not so strange. Divisions are ideal and relation of divisibility is reduced to the relation of inclusion of ideals. At the end we receive the universal order – relation of inclusion in the family of sets.

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