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II. REVIEWS AND NOTES

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SOME REMARKS ON TRANSFORMATION OF EMPLOYMENT DISTRIBUTION IN RELATION TO A PATTERN ECONOMY

The paper is devoted to application of some simplified model of mathematical programming to the problem of employment distribution in the process of economic transformation. Taking advantage of some pattern economy renders possible to obtain the minimal stipulated "distance" between the members for every considered pair of economies. Appropriate computational algorithm has been accompanied by suitable numerical example explaining in detail the use of the method. The work enables, in particular, the easy choice of the most rational direction of transformation for the real economy and is addressed to those scientists, who have, aside from sufficient economic knowledge, also some necessary acquirements about practical usage of the quantitative optimization models.

1. INTRODUCTION

The system transformation taking place in Poland sets up an attractive though laborious investigation field for many different science branches. Works dealing with these problems either contain analyses leaning upon the traditional formulas of dynamics and structure indexes (compare *The Economics*... 1994), or propose utilization of highly qualified mathematical material or else use the multidimensional comparative analysis (compare Jajuga et al 1994). To be sure, the simple mathematical models of discrete distributions can be helpful in the exploration of the economic evolution proceeding in developing countries (*The Economics*... 1994). We consider here the model of employment distribution (which, however, can be applied to many other attributes as well) confining ourselves to the case that considered economy is formally separated from other economies. We assume this distribution to be changing in time, and its direction of change is determined by the state of a certain pattern economy.

As will be shown in this paper, we can build up some useful optimization model rendering possible the maximal approach of employment distribution in transformed country to that in the country accepted as a model of transformation. This maximal approach, obviously, depends upon financial conditions of investigated economy, what has been expressed in formulated relationships.

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2. BASIC CONSIDERATIONS

In view of the specific character of distribution we take into account only such points $x = (x_1, x_2, \ldots, x_n)$ of n-dimensional Euclidean space (Halmos 1958), which satisfy the system of relationships $x_j \geqslant 0$, $j = 1, 2, \ldots, n$; $x_1 + x_2 + \ldots + x_n = 1$, or, which is the same, belong to the so-called simplex set. We call these points the *variables of distribution*. Obviously, they play the role of the most important indicators in modern economics.

Both economies, the examined one and the pattern one, consist of n sectors, which means that if j is an arbitrary number from the set $\{1, 2, \ldots, n\}$, then j-th sectors have identical economic significance in these economies. Let T be the assumed period of transformation. If employment in the j-th sector of transformed economy is A_j at the initial moment t_0, then the fractions

$$\alpha_j = \frac{A_j}{\sum_{j=1}^{n} A_j}, \quad j = 1, 2, \ldots, n, \tag{1}$$

express the values of the variables of distribution at this moment. Denoting by ω_j corresponding coordinates of distribution vector of the pattern economy (treated as a constant vector) we can define the strategy of transformation quantitatively as minimization of the distance between the variable point $x = (x_1, x_2, \ldots, x_n)$ and the constant point $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ in considered subset of space. We shall designate this strategy briefly by the symbol $\alpha \to \omega$.

The assumption of separation mentioned above means that considered process of transformation is going on exclusively by displacement of workers from one sector to another within the economy which is to be transformed; so the total number of employment remains constant during the period of transformation. In other words, if we designate the employment in the j-th sector of considered economy at the moment $t = t_0 + T$ by $A_j + \Delta A_j$, where $j = 1, 2, \ldots, n$, then

$$\sum_{j=1}^{n} \Delta A_j = 0. \tag{2}$$

But let us note that the same model can be easily accommodated to such a transformation, which produces unemployment too — it is sufficient for this purpose to consider unemployment as one of the sectors of a transformed economy.

Denote by ξ_{ij} the number of workers that are removed from the i-th sector to the j-th one during the period of transformation. The algebraic increment of employment in the j-th sector can then be expressed by the formula

$$\Delta A_j = \sum_{j=1}^{n} (\xi_{ij} - \xi_{ji}). \tag{3}$$

As we see, ΔA_j can be the positive or the negative number. In view of (3) the formula (2) can be now rewritten in the form

$$\sum_{j=1}^{n} \sum_{i=1}^{n} (\xi_{ij} - \xi_{ji}) = 0, \tag{4}$$

and the variables of distribution x_j at the moment t take the form

$$x_j = \frac{A_j + \Delta A_j}{\sum_{j=1}^{n} A_j}, \quad j = 1, 2, \ldots, n. \tag{5}$$

Obviously $\sum_{j=1}^{n} x_j = 1$. As to the inequalities $x_j \geqslant 0, j = 1, 2, \ldots, n$, we accept them as a simple consequence of a fact that employment in no sector can be lowered below zero in the process of transformation. We can now formulate the purpose of transformation in the form of mathematical programming problem (Arrow et al 1958; Varaiya 1967):

$$F(x) = \sum_{j=1}^{n} |x_j - \omega_j| \to \min. \tag{6}$$

If we completed this minimization problem merely with simplex conditions imposed on x, we would obtain only the trivial solution $x = \omega$, which corresponds with the case of ideal transformation. The possibility of obtaining non-trival solutions applicable in practice depends on our acquirements in establishing some additional restraints. Using the variables ξ_{ij} defined above and taking into account the fonds R_i assigned for transformation of the corresponding sectors, where $i = 1, 2, \ldots, n$, we can set up the system of inequalities

$$\left.\begin{array}{l} a_{11}\xi_{11} + a_{12}\xi_{12} + \ldots + a_{1n}\xi_{1n} \leqslant R_1, \\ a_{21}\xi_{21} + a_{22}\xi_{22} + \ldots + a_{2n}\xi_{2n} \leqslant R_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}\xi_{n1} + a_{n2}\xi_{n2} + \ldots + a_{nn}\xi_{nn} \leqslant R_n, \end{array}\right\} \tag{7}$$

or, which is the same, the system of equations

$$\left.\begin{array}{l} a_{11}\xi_{11}+a_{12}\xi_{12}+\ldots+a_{1n}\xi_{1n}+\eta_1 = R_1, \\ a_{21}\xi_{21}+a_{22}\xi_{22}+\ldots+a_{2n}\xi_{2n}+\eta_2 = R_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\xi_{n1}+a_{n2}\xi_{n2}+\ldots+a_{nn}\xi_{nn}+\eta_n = R_n, \end{array}\right\} \tag{8}$$

The coefficient a_{ij} occuring in these relationships has the significance of the cost charging the economy because of displacing the single worker from the i-th sector to the j-th sector of this economy. Solving (8) with respect to ξ_{ij}: $i, j = 1, 2, \ldots, n$ and reckoning with (3) we could determine the variables of distribution from (5).

But there exists another way to set up the system of restraints for the problem (6) that renders possible a significant reduction of the number of variables. Let us define the symbol z_{ij} as follows:

$$z_{ij} = \xi_{ij}-\xi_{ji}; \qquad i, j = 1, 2, \ldots, n. \tag{9}$$

We shall consider this symbol as a resulting increment of employment in the j-th sector of the transformed economy caused by its worker-exchange with the i-th sector. Obviously the sum $\sum_{i=1}^{n} z_{ij}$ expresses similar increment caused in the j-th sector by its worker-exchange with all sectors of economy whereas the sum $\sum_{j=1}^{n} z_{ij}$ is the total increment of employment caused in all sectors by their worker-exchange with the j-th sector of economy. In this connection we can introduce the new variable

$$z_j = \frac{1}{n}\sum_{i=1}^{n} z_{ij}; \qquad i, j = 1, 2, \ldots, n, \tag{10}$$

meaning the average (algebraic) increment of employment caused in the j-th sector of transformed economy by its worker-exchange with an arbitrary sector of this economy. Now we can set up the system of inequalities

$$\left.\begin{array}{l} a_{11}z_1+a_{12}z_2+\ldots+a_{1n}z_n \leqslant R_1, \\ a_{21}z_1+a_{22}z_2+\ldots+a_{2n}z_n \leqslant R_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}z_1+a_{n2}z_2+\ldots+a_{nn}z_n \leqslant R_n, \end{array}\right\} \tag{11}$$

or its equational representation:

$$\left.\begin{array}{l} a_{11}z_1+a_{12}z_2+\ldots+a_{1n}z_n+y_1 = R_1, \\ a_{21}z_1+a_{22}z_2+\ldots+a_{2n}z_n+y_2 = R_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}z_1+a_{n2}z_2+\ldots+a_{nn}z_n+y_n = R_n, \end{array}\right\} \tag{12}$$

Additionally we can take advantage of the equation $\sum_{j=1}^{n} z_{ij} = 0$ following (2) and (3) in view of (9). The capital R_i in (11) or (12) preserve its meaning from (7) or (8), but the coefficient a_{ij} in the latter two systems is related to different variable than in the former two, so, in (11) or (12) it expresses the cost charging the economy because of the unitary increment of employment in the j-th sector of economy caused by its worker-exchange with the i-th sector. As we see, the system (12) has only $2n$ variables – less by $n(n-1)$ than (8). And although solutions of the system (12) differ from those of (8), one can, nevertheless, compute the variables of distribution x_j using the formula (5), because

$$\Delta A_j = \sum_{j=1}^{n} z_{ij} = n \cdot z_j, \quad j = 1, 2, \ldots, n, \tag{13}$$

as it follows from (10) in view of (9) and (3).

3. COMPUTATIONAL METHOD

Numerical realization of the strategy $\alpha \to \omega$ in concrete practical problems of transformation can be significantly simplified in comparison with general programming processes (Halmos 1958). First of all we should notice that the modules in expression (6) of the function F can be eliminated for the reason that both vectors, and are constant, so all their coordinates are the known numbers. Thus, the problem $(x) \to \min$ becomes a linear one. Secondly, our design is to achieve the maximal approach to pattern distribution, no matter what immediate values real distribution has taken on; in other words, we shall be able to choose the simplest one from all possible traces of iteration process, i.e. to take into account only such solutions which belong to the straight line section with equation

$$x = \alpha + (\omega - \alpha)\, t, \quad 0 \leqslant t \leqslant 1, \tag{14}$$

(and, of course, fulfil the simplex conditions). We shall call it the *guide section*. The intersection point of this section with one of the hyperplanes delimiting the polyhedron (11) – let us record this hyperplanes in form of

$$H_0 = \left\{ z : \sum_{j=1}^{n} a_{i_0 j} z_j = R_{i_0} \right\}, \tag{14'}$$

– is the solution we are looking for. Passage from the variable $z = (z_1, z_2, \ldots, z_n)$ to the variable of distribution $x = (x_1, x_2, \ldots, x_n)$ is determined by the relationships (5) and (13). Let $I \subset \{1, 2, \ldots, n\}$ be the index-set of those hyperplanes delimiting the polyhedron (11), which have the common points with the guide section. If $x^1, x^2, \ldots, x^k$ are these points, then the point in demand is that from among them, which maximizes the function F. Let us denote such a point by x^0_{opt}. We have therefore

$$x^0_{\text{opt}} = \arg\max F(x^i), \quad i \in l. \tag{14''}$$

There are generally three situations possible:

a. The set of solution of the system (11) includes the vector and does not include the vector ω; then the solution of the optimum problem set forth certainly exists, i.e. one can find among the hyperplanes bounding the polyhedron (11) such a hyperplane H_0, whose intersection with the section (14) minimizes the function (6) subject to (11).

b. Both sectors, α and ω, belong to the set of solutions of the system (11). Then exists only trivial solution $x_{\text{opt}} = \omega$ corresponding with $F(x) = 0$.

c. Both sectors, α and ω, lay outside the set of the solutions of the system (11); then the case of trivial solution also takes place, but the last has the absolute meaning.

The following example explains the application of a model.
The transformed country: Poland
The pattern economy: Spain

Division into sectors:		I	II	III
		agriculture	*industry*	*service & trade*
constant vectors:	α	0.276	0.355	0.369
	ω	0.112	0.312	0.576

$$\sum_{j=1}^{n} A_j = 12 \text{ m (workers; m } - \text{ millions).}$$

Fonds for transformation:	R_1	R_2	R_3
	60 m	80 m	20 m ($)

The matrix of unitary costs (in dollars) as defined for (11) or (12):

$$\begin{bmatrix} 40 & 80 & 150 \\ 30 & 50 & 140 \\ 20 & 20 & 120 \end{bmatrix}$$

Inequalities (11):

$$\left.\begin{aligned} 40z_1 + 80z_2 + 150z_3 &\leqslant 60 \text{ m}, \\ 30z_1 + 50z_2 + 140z_3 &\leqslant 80 \text{ m}, \\ 20z_1 + 20z_2 + 120z_3 &\leqslant 20 \text{ m}, \end{aligned}\right\} \tag{15}$$

$$A_1 = \alpha_1 \sum_{j=1}^{3} A_j = 0.276 \cdot 12 \text{ m} = 3.312 \text{ m},$$

$$A_2 = \alpha_2 \sum_{j=1}^{3} A_j = 0.355 \cdot 12 \text{ m} = 4.60 \text{ m}.$$

$$A_3 = \alpha_3 \sum_{j=1}^{3} A_j = 0.369 \cdot 12 \text{ m} = 4.428 \text{ m}.$$

Hence, according to (5) and (13) we obtain for $\Delta A_j = 3 z_j, j = 1, 2, 3$:

$$x_1 = \frac{3.312 \text{ m} + 3 z_1}{12 \text{ m}} = 0.276 + \frac{z_1}{4 \text{ m}} \Rightarrow z_1 = 4 \text{ m} (x_1 - 0.276).$$

$$x_1 = \frac{4.260 \text{ m} + 3 z_2}{12 \text{ m}} = 0.355 + \frac{z_2}{4 \text{ m}} \Rightarrow z_2 = 4 \text{ m} (x_1 - 0.355).$$

$$x_1 = \frac{4.428 \text{ m} + 3 z_3}{12 \text{ m}} = 0.369 + \frac{z_3}{4 \text{ m}} \Rightarrow z_3 = 4 \text{ m} (x_1 - 0.369).$$

and by substitution in (15) we receive:

$$\left.\begin{aligned} 8(x_1 - 0.276) + 16(x_2 - 0.355) + 30(x_3 - 0.369) &\leqslant 3, \\ 3(x_1 - 0.276) + 5(x_2 - 0.355) + 14(x_3 - 0.369) &\leqslant 2, \\ 4(x_1 - 0.276) + 4(x_2 - 0.355) + 24(x_3 - 0.369) &\leqslant 1. \end{aligned}\right\} \tag{16}$$

Additionally $x_1 + x_2 + x_2 = 1$.

Denoting by S the set of solutions of (16) we see that obviously

$$\alpha = (0.276; 0.355; 0.369) \in S \quad \text{and} \quad \omega = (0.112; 0.312; 0.576) \notin S.$$

This is, therefore, the case of existing non-trivial solution of the optimization problem (6) with restraints (16).

The equations of boundary planes:

$$\left.\begin{aligned} 8x_1 + 16x_2 + 30x_3 &= 21.958; \\ 3x_1 + 5x_2 + 14x_3 &= 9.769; \\ 4x_1 + 4x_2 + 24x_3 &= 12.380. \end{aligned}\right\} \tag{17}$$

The equations of the guide section (14):

$$\begin{aligned} x_1 &= 0.276 - 0.164t, \\ x_2 &= 0.355 - 0.043t, \quad 0 \leqslant t \leqslant l. \\ x_3 &= 0.369 + 0.207t, \end{aligned} \tag{18}$$

From (17) and (18) we obtain three solutions:

	x_1	x_2	x_3
x_1	0.159	0.324	0.516
x_2	0.126	0.316	0.558
x_3	0.236	0.345	0.419

According to formula (14') we have here:

$$x^0_{opt} = (0.236; 0.345; 0.419) \quad \text{and} \quad F(x^0_{opt}) = 0.314.$$

Substituting x^0_{opt} in relationships (16) we see that only this point belongs to polyhedron (11), so it is really a unique solution.

4. CONCLUSIONS

1° The model of distribution proposed here, though presented in an example of employment, can be exploited as an effective tool of investigation connected with a wider class of economic features, like the parameters of space distributions, or those of distribution of national income and others.

2° The coordinates of constant vectors α and ω used in the example is real data taken from the yearbook (*Rocznik statystyki*... 1994). Similarly the value of employment $A_1+A_2+A_3$. However the fonds R_i and coefficients a_{ij} utilized here are only reasonably estimated numbers which may not be equal to the real values from practical processes, though may be near to them. Thus, the example reported here has primarily an explanatory character.

3° It is proper to pay attention that the pattern-idea in the quantitative form presented here enables, on the one hand, the simplification of the numerical processes of programming, and on the other, deliver one more prediction method applicable in some class of practical problems if properly combined with the restraint system (Varaiya 1967).

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