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THE COEFFICIENT OF DEPENDENCE FOR CONSUMPTION CURVE[1]

1. INTRODUCTION

The small value of the linear dependence coefficient does not rule out that the given set of points on the plane can be approximated by a curve (as a criterion of fit we shall use the generalized method of least squares.) There are several measures of the curvilinear dependence (see: Kendall 1948); one of the best is the correlation quotient of Pearson η (see: Cramer 1975). Its use, however, only makes sense when the data are numerous and can be presented in the form of a correlation table. But the Pearson's quotient does not allow us to state the actual shape of dependence. This problem is currently being researched. Among the latest contributions, we can mention Hellwig's quotient of stochastic dependence (see: Hellwig 1965) and the Bukietyńska's measure of nonlinear dependence (see: Bukietyńska 1992). In this article we shall present the results of research on the coefficient of dependence for the consumption curve (Tömquist).

2. BASIC FORMULAE AND SYMBOLS

Further on by $[x]$ we shall denote

$$[x] = \sum_{i \in \mathcal{P}} x_i ,$$

where $\mathcal{P}$ is a certain set of indices.

Having the regression lines $y = m_1 x + n_1$, $x = m_2 y + n_2$, we can determine the coefficient of the linear correlation r between variables x and y

[1] This paper was published firstly in: Prace Naukowe AE [RW of WAE] 1993, No 663.

$$r^2 = m_1 m_2.$$

Cosine k of the angle between the regression lines we shall call (after Antoniewicz) the coefficient of the rectilinear dependence of the set of points on the plane (see: Antoniewicz 1988). Obviously

$$k = \frac{m_1 + m_2}{\sqrt{1 + m_1^2}\ \sqrt{1 + m_2^2}}.$$

We can see that $k^2 = 1$, when there is a linear dependence between the variables, and if $k = 0$, there is no such dependence. Set of points on the plane $\mathcal{A} = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ will be denoted by $\mathcal{A} = \begin{pmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{pmatrix}$.

3. COEFFICIENT OF DEPENDENCE FOR THE CONSUMPTION CURVE

The important feature of the market is the so-called consumption curve presenting the dependence between the sum spent by the consumer and the income of the consumer within a certain time limit. Because of the 'saturation phenomenon' we use Tömquist functions for the approximation of the curve. All these functions belong to the family of curves described by the algebraic equation of the second degree in two variables:

$$ax^2 - xy + bx - dy + c = 0, \quad a, b, c, d \in \mathcal{R}.$$

According to their shape, we can define four types of Törnquist functions.

3.1. Tömquist functions of zero type are given by the equation

$$xy - ax + by + c = 0.$$

Analogously to the line of regression, we call the Törnquist function given by the equation $a_1xy - x + b_1y + c_1 = 0$ and approximating the given set $\mathcal{A}$ of points best in the class of all curves with the same equation, the curve of regression of x on y. Similarly, the curve given by the equation $a_2xy - y + b_2x + c_2 = 0$ which approximates best the given set of points in the class of all curves with the same equation, is called the curve of regression of y on x (see: Antoniewicz 1988). It turns out that the parameters of the regression curves minimize the functions

$$F(a_1, b_1, c_1, \mathcal{A}) = \sum_{\mathcal{A}} (a_1xy - x + b_1y + c_1)^2$$

and

$$G(a_2, b_2, c_2, \mathcal{A}) = \sum_{\mathcal{A}} (a_2 xy - y + b_2 x + c_2)^2.$$

To find those minima, one calculates the partial derivatives of the first order and equate them to zero. Therefore, the stationary points satisfy the following systems of equations.

$$\begin{aligned} [x^2y^2]\ a_1 + [xy^2]\, b_1 + [xy]\, c_1 &= [x^2y] \\ [xy^2]\ a_1 + [y^2]\, b_1 + [y]\, c_1 &= [xy] \\ [xy]\, a_1 + [y]\, b_1 + [1]\, c_1 &= [x] \end{aligned} \tag{1}$$

and

$$\begin{aligned} [x^2y^2]\ a_2 + [x^2y]\, b_2 + [xy]\, c_2 &= [xy^2] \\ [x^2y]\ a_2 + [x^2]\, b_2 + [x]\, c_2 &= [xy] \\ [xy]\, a_2 + [x]\, b_2 + [1]\, c_2 &= [y]. \end{aligned} \tag{2}$$

It is not difficult to see that the minima are at the stationary points. By analogy to the coefficient of the rectilinear dependence k, as the coefficient of dependence for the Törnquist curves of the zero type k_0, we shall understand the cosine of the angle at the intersection point of the regression curves, which lies closer (in the sense of Euclidean distance) to the gravity centre of the set of points $\mathcal{A}$. Obviously, the curves can intersect at more than one point. The remaining cosines can be treated as local coefficients of dependence. Hence the question 'should one not analyse all the points of intersection?' The numerical examples indicated however that we would not obtain more information, but merely complicate the matter. The more sensible and simple approach is to concentrate on the point which lies closer to the gravity centre of the set $\mathcal{A}$. It turns out that the coefficient k_0 is calculated according to formula:

$$k_0 = \frac{m_1 + m_2}{\sqrt{1 + m_1^2}\ \sqrt{1 + m_2^2}},$$

where m_1 and m_2 are the directional coefficients of tangent lines to regression curves calculated at the point of intersection lying closer to the point

$$\left(\frac{[x]}{[1]}, \frac{[y]}{[1]} \right).$$

3.2. The Törnquist functions of first type are described by the equation:

$$xy - ax + by = 0.$$

Proceeding as in the previous case, the coefficient of dependence k_1 for the Törnquist curves of the first type, will be taken as the cosine of the angle calculated at the intersection point of the regression curves situated closer to the centre of gravity of the set of points $\mathcal{A}$. Obviously k_1 is calculated according to the same formula as in the case of k_0. To find the parameters of the regression curves, we have to solve the reduced systems of equations (1) and (2):

$$\begin{aligned} [x^2y^2]\ a_1 + [xy^2]\, b_1 &= [x^2y\,] \\ [xy^2]\ a_1 + [y^2]\, b_1 &= [xy\,] \end{aligned} \tag{3}$$

and

$$\begin{aligned} [x^2y^2]\ a_2 + [x^2y\,]\, b_2 &= [xy^2] \\ [x^2y\,]\ a_2 + [x^2]\, b_2 &= [xy\,]. \end{aligned} \tag{4}$$

3.3. The Törnquist functions of second type are defined by the equation analogous with one in zero type (the parameters differ only in signs). To calculate the coefficient k_2 we can use the systems of equations (1), (2).

3.4. The Törnquist functions of third type are defined by the equation $xy - ax^2 + by + cy = 0$. This time the systems of equations analogous with (1), (2), (3), (4), allowing to find parameters of the regression curves, are:

$$\begin{aligned} [x^2y^2]\ a_1 - [x^3y\,]\, b_1 + [xy^2]\, c_1 &= [x^2y\,] \\ -\ [x^3y\,]\, a_1 + [x^4]\, b_1 - [x^2y]\, c_1 &= -\,[x^3] \\ [xy^2]\ a_1 - [x^2y\,]\, b_1 + [y^2]\, c_1 &= [xy\,], \end{aligned} \tag{5}$$

$$\begin{aligned} [x^2y^2]\ a_2 - [x^3y\,]\, b_2 + [x^2y\,]\, c_2 &= [xy^2] \\ -\ [x^3y\,]\, a_2 + [x^4]\, b_2 - [x^3]\, c_2 &= -\,[x^2y\,] \\ [x^2y\,]\ a_2 - [x^3]\, b_2 + [x^2]\, c_2 &= [xy\,]. \end{aligned} \tag{6}$$

We proceed as in the first case, and we can calculate the coefficient k_3:

$$k_3 = \frac{m_1 + m_2}{\sqrt{1 + m_1^2}\ \sqrt{1 + m_2^2}},$$

where m_1, m_2 are the directional coefficients of tangents calculated at the intersection point of the regression curves lying closer to the point

$$\left(\frac{[x]}{[1]}, \frac{[y]}{[1]}\right).$$

Because these systems of equations are linear, we shall always obtain exact solutions. Looking for the intersection points comes down to solving quadratic equations.

4. NUMERICAL EXAMPLES

We shall give now examples illustrating changes of coefficients k_0, k_3. For the purpose of comparison we shall include the value of linear correlation coefficient r. Numerical data is supplemented by the graphs of the regression curves.

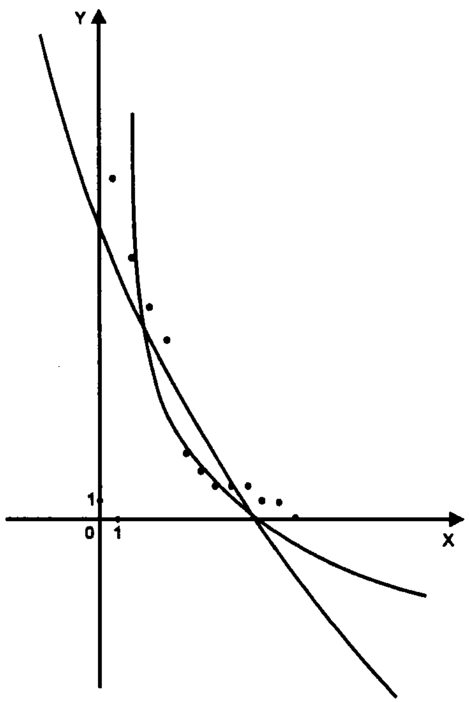
EXAMPLE 1

Set of points $\mathcal{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 21 & 16 & 13 & 11 & 4 & 3 & 2 & 2 & 2 & 1 & 1 & 0 \end{pmatrix}$

$k_0 = 0{,}949871655$, $r = -0{,}899373949$.

Vertical asymptotes: $x = -2{,}684857023$, $x = -31{,}67555113$.

Horizontal asymptotes: $y = -8{,}43964992$, $y = -58{,}28715626$.



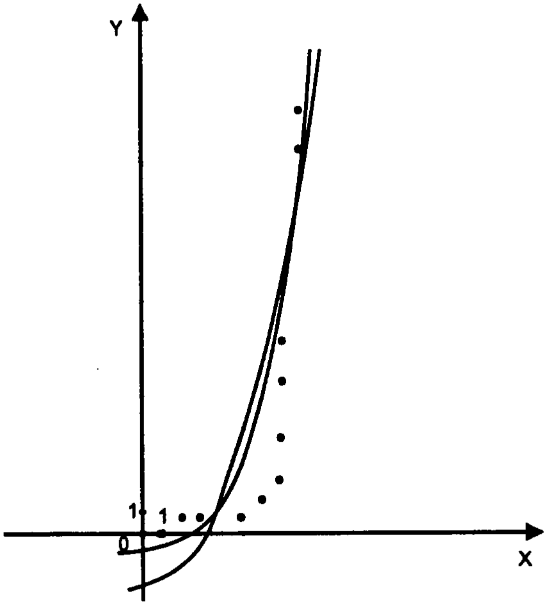
EXAMPLE 2

Set of points $\mathcal{A} = \begin{pmatrix} 0 & 1 & 2 & 3 & 5 & 6 & 7 & 7 & 7 & 7 & 8 & 8 \\ 0 & 0 & 1 & 1 & 1 & 2 & 3 & 5 & 8 & 10 & 20 & 22 \end{pmatrix}$

$k_0 = 0{,}958201832$, $r = 0{,}705389702$.

Vertical asymptotes: $x = 8{,}921484125$, $x = 8{,}341581904$.

Horizontal asymptotes: $y = -3{,}253996856$, $y = -0{,}922574042$.



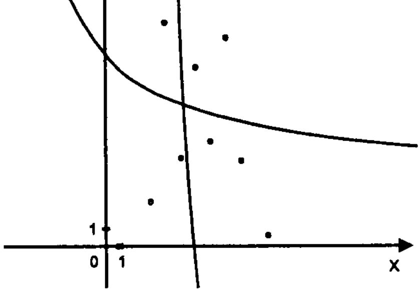
EXAMPLE 3

Set of points $\mathcal{A} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 20 & 0 & 18 & 3 & 15 & 6 & 12 & 7 & 14 & 6 & 18 & 1 \end{pmatrix}$

$k_0 = 0{,}372583089$, $r = -0{,}168511827$.

Vertical asymptotes: $x = -16{,}47781254$, $x = -7{,}30120542$.

Horizontal asymptotes: $y = -232{,}7431923$, $y = 5{,}848554869$.



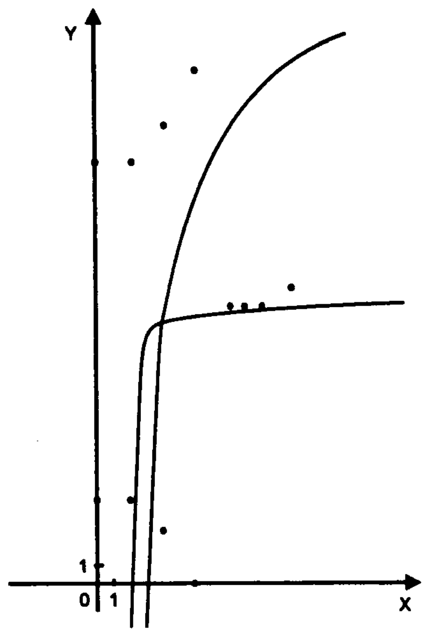
EXAMPLE 4

Set of points $\mathcal{A} = \begin{pmatrix} 0 & 0 & 2 & 2 & 4 & 4 & 6 & 6 & 8 & 9 & 10 & 12 \\ 25 & 5 & 25 & 5 & 27 & 3 & 30 & 0 & 16 & 16 & 16 & 17 \end{pmatrix}$

$k_0 = 0{,}181034856$, $r = 0{,}056107944$.

Vertical asymptotes: $x = 2{,}896100172$, $x = 2{,}30867649$.

Horizontal asymptotes: $y = 31{,}95954903$, $y = 16{,}222380363$.



EXAMPLE 5

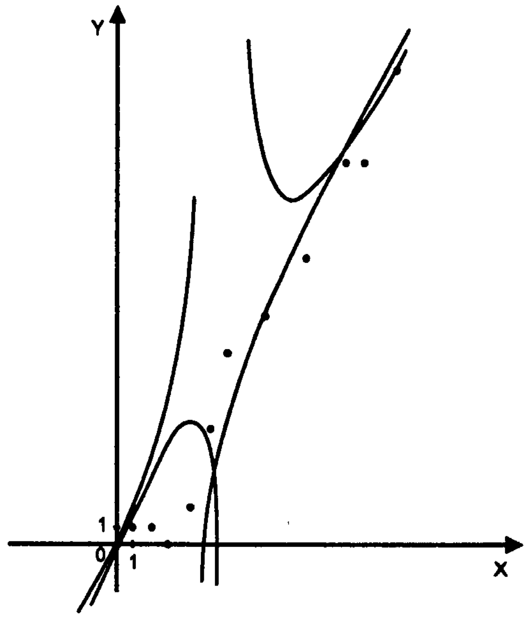
Set of points $\mathcal{A} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 10 & 12 & 13 & 15 \\ 0 & 1 & 1 & 0 & 2 & 6 & 10 & 12 & 15 & 20 & 20 & 25 \end{pmatrix}$

$k_3 = -0{,}9212223$, $r = 0{,}983038491$.

Vertical asymptotes: $x = 6{,}040925567$, $x = 4{,}684725373$.

Diagonal asymptotes: $y = 1{,}546488796x + 0{,}525599917$,

$y = 1{,}64684934x - 0{,}445004317$.



EXAMPLE 6

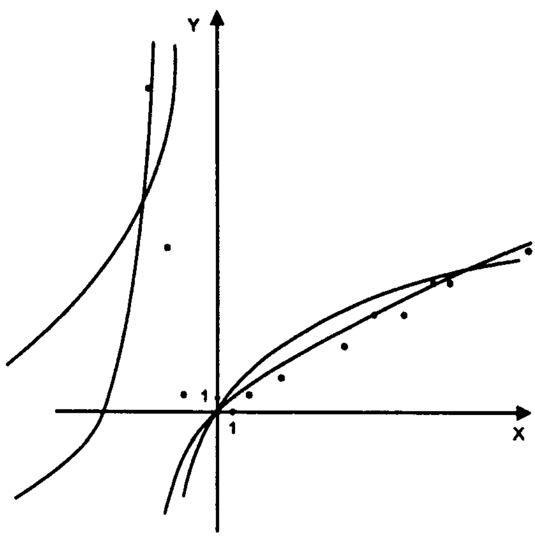
Set of points $\mathcal{A} = \begin{pmatrix} -4 & -3 & 0 & 2 & 4 & 8 & 10 & 12 & 14 & 15 & 20 & -2 \\ 20 & 10 & 0 & 1 & 2 & 4 & 6 & 6 & 8 & 8 & 10 & 1 \end{pmatrix}$

$k_3 = 0{,}9819401 74$, $r = 0{,}026000697$.

Vertical asymptotes: $x = -2{,}560070992$, $x = -3{,}944447506$.

Diagonal asymptotes: $y = 0{,}221346262x + 5{,}698879728$,

$y = 0{,}468184926x + 1{,}092035993$.



5. CONCLUDING REMARKS

We limited ourselves here to coefficients k_0 and k_3, because k_1 is a particular case of k_0 and in the case of k_2 we use the same equations as for k_0 (the only difference in signs). On the basis of examples 1, 2 and 5, we can infer that a strong linear dependence results in a great (as the absolute value) value of the coefficients k_0 and k_3. In the examples 3 and 4, the coefficient k_0 seems slightly more suitable for measuring the dependence of the variables. In the last case, we have a very weak linear dependence, and the value of the coefficient k_3 close to one.

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