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STABLE CARMA PROCESSES AS A TOOL FOR STOCHASTIC VOLATILITY MODELLING

1. Introduction

In modern mathematical finance continuous time models play a crucial role because they allow handling unequally spaced data and even high frequency data, which are realistic for liquid data. The probably most famous example is the co-called Black-Scholes model, which is build out of Brownian motion and models the logarithm of an asset price by the solution to the arithmetic Brownian motion (see [5]). The asset pricing model implies that the aggregate returns are normal and independly distributed. But the assumption is unsatisfactory for many observed data. One approach is to replace the Brownian motion in Black-Scholes model by a heavier tailed Lévy process. This will allow returns to the heavy-tailed and skewed and take into account jumps. However, the returns will be independent and stationary, since every Lévy process has stationary independent increments. This approach was for example proposed by Brockwell and Marquardt in [2], where second order Lévy-driven CARMA (continuous time ARMA) processes are reviewed. Gaussian CARMA processes are special cases in which the driving Lévy process is Brownian motion. The use of more general Lévy processes permits the specification of CARMA processes with a wide variety of marginal distributions, which may be asymmetric and heavier tailed than Gaussian. Non-negative CARMA processes are of special interest, partly because of the introduction by Barndorff-Nielsen and Shephard [1] of non-negative Ornstein-Uhlenbeck process as model for stochastic volatility [2].

Because many studies have shown that heavy-tailed distributions allow for modelling different kinds of phenomena when the assumption of normality for the observations seems not to be reasonable, we extend the definition of the second order Lévy-driven CARMA processes considered in Brockwell and Marquardt [2]. We propose to replace the second order process with the symmetric α-stable Lévy motion. The role that α-

-stable Lévy motion plays among stable processes is similar to the role that Brownian motion plays among Gaussian processes. Moreover the α-stable (stable) distributions have found many practical applications, for instance in finance [7], physics [3], electrical engineering [12]. The importance of this class of distributions is strongly supported by the limit theorems which indicate that the stable distribution is the only possible limiting distribution for the normed sum of independent and identically distributed random variables [10].

On one hand the α-stable CARMA processes are an extension of second order Lévy-driven CARMA models, on the other hand they are extension of the ARMA time series models with α-stable innovations described in Nowicka [8]. The discrete ARMA models with innovations from the stable distributions are a special case of considered in Nowicka-Zagrajek and Wyłomańska [10] PARMA models with α-stable innovations as well as ARMA models with time-varying coefficienets and α-stable innovations presented in Nowicka-Zagrajek and Wyłomańska [10]. Additionally, the α-stable CARMA processes have similar asymptotic behaviour like the discrete models.

2. CARMA processes

Definition 1. Second-order Lévy-driven CARMA process.

A second-order Lévy-driven continuous ARMA(p,q) process is defined in terms of the following state-space representation of the formal equation

$$a(D)Y(t) = b(D)L(t), \qquad t \in R, \tag{1}$$

in which D denotes differentiation with respect to t, $\{L(t)\}$ is the background driving Lévy process defined in Brockwell and Marquardt [2],

$$a(z) = z^p + a_1 z^{p-1} + \ldots + a_p,$$

$$b(z) = b_0 + b_1 z + \ldots + b_{p-1} z^{p-1},$$

and the coefficients b_j satisfy $b_q \neq 0$ and $b_j = 0$ $q<j<p$.

The form of the bounded solution of equation (1) as well as the main properties of it are presented in Brockwell and Marquardt [2].

Definition 1 can be extended to the continuous processes with symmetric α-stable Lévy process (see [13]). For simplicity in this paper we study the special case of such models, i.e. the symmetric α-stable CARMA(1,1) processes:

$$DY(t) + aY(t) = bDL^*(t), \qquad t \in R, \tag{2}$$

for non-zero a and b parameters. In equation (2) $\{L^*(t)\}$ is an α-stable Lévy process indexed by R defined in Wyłomańska [13]. In this paper we consider the case $1 < \alpha \leq 2$.

The process $\{Y(t)\}$ is given by the following equation (see [13]):

$$Y(t) = b \int_{-\infty}^{t} \exp(-s(t-u))dL^*(u) \tag{3}$$

and for $a > 0$ and $b = 1$ it is called an α-stable Ornstein-Uhlenbeck process (see Example 3.6.3 in [11]).

Moreover for $a > 0$ and $b \neq 0$ the stochastic process $\{Y(t)\}$ is stationary (see [13]).

3. Measures of dependence of symmetric α-stable CARMA(1,1) processes

For considered CARMA processes in case $\alpha < 2$ the covariance is not defined and thus other measures of dependence have to be used. The most popular measures are the covariation and the codifference defined for α-stable random variables.

Definition 2

Let X and Y be jointly symmetric α-stable. The covariation $CV(X,Y)$ defined for $1 < \alpha \leq 2$ is the real number

$$CV(X,Y) = \int_{S_2} s_1 s_2^{<\alpha-1>} \Gamma(ds_1, ds_2), \tag{4}$$

where Γ is the spectral measure of the random vector (X,Y), and the signed power is given by $z^{<p>} = |z|^{p-1} \bar{z}$.

Definition 3

Let X and Y be jointly symmetric α-stable. The codifference $CD(X,Y)$ defined for $0 < \alpha \leq 2$ is given by

$$CD(X,Y) = \ln E\exp(i(X - Y)) - \ln E\exp(iX) - \ln E\exp(-iY). \tag{5}$$

Properties of the considered measures of dependence one can find in [11]. Let us only mention here that, in contrast to the codifference, the covariation is not symmetric in its arguments. Moreover, when $\alpha = 2$ both measures reduce to the covariance, namely

$$Cov(X,Y) = CD(X,Y) = 2CV(X,Y).$$

Using results obtained in [13] we present the measures of dependence given in Definition 2 and 3 for considered symmetric α-stable CARMA(1,1) processes:

Proposition 1

Let $\{Y(t)\}$ be a solution of CARMA(1,1) equation (2) with $a > 0$. Moreover let us assume $1 < \alpha \leq 2$. In this case we have the following formulas

$$CV(Y(t+h), Y(t)) = \frac{|b|^\alpha \exp(-ah)}{a\alpha}, \qquad h > 0$$

$$CD(Y(t+h), Y(t)) = \frac{|b|^\alpha}{a\alpha}\left(1 + \exp(-a\alpha h) - |1 - \exp(-ah)|^\alpha\right), \qquad h > 0.$$

As we see the above formulas are not dependent on t, which is not surprising because of the stationarity of $\{Y(t)\}$.

Theorem 1 (see Theorem 4.1. in 13])

If $\{Y(t)\}$ is the solution of equation (2) and $a>0$, then for $1 < \alpha \leq 2$ and each $t \in R$ the following formula holds:

$$\lim_{h\to\infty}\frac{CD(Y(t+h),Y(t))}{CV(Y(t+h),Y(t))}=\alpha. \qquad (6)$$

The above result can be used for example to the estimation of the unknown index of stability α for considered processes described by CARMA(1,1) equation (2). Results similar to this obtained in Theorem 1 we can observe also for discrete stationary ARMA models (see [8]) as well as for periodic ARMA models with stable innovations (see [10]) and ARMA models with time-varying coefficients and symmetric α-stable structure (see [9]).

4. Examples

In order to illustrate our theoretical results presented in the previous Section let us consider symmetric α-stable CARMA(1,1) process:

$$DY(t)+0.5Y(t)=DL^*(t), \qquad t\in R$$

for $1<\alpha\leq 2$.

We first want to demonstrate how the α parameter influences the behaviour of the process $\{Y(t)\}$, so we plot 1000 realizations of the considered models for $\alpha=2$, $\alpha=1{\cdot}7$ and $\alpha=1{\cdot}4$, see Figure 1. It is easy to notice that the smaller α we take, the greater values of the processes can appear (property of heavy-tailed distributions). Next, let us illustrate the asymptotic relation between the covariation and the codifference that is studied in previous Section. On Figure 2 we present the function $\frac{CD(Y(t+h),Y(t))}{\alpha CV(Y(t+h),Y(t))}$ for $h\in[0,25]$ and $\alpha=1.7$ and $\alpha=1.4$. According to the theoretical results, the quotient tends to 1 as h increases. Because the covariation is not symmetric we illustrate also the relation $\frac{CD(Y(t),Y(t+h))}{\alpha CV(Y(t),Y(t+h))}$. As we see the last quotient tends to 0 as h increases.

As an application to stochastic volatility modelling Berndorff-Nielsen and Shephard [1] introduced a model for asset-pricing in which the logarithm of an asset price is the solution of the stochastic differential equation:

$$DY(t)=\mu+\beta\sigma^2(t)+\sigma(t)DW(t),$$

where $\{\sigma^2(t)\}$, the instantaneous volatility, is a non-negative Lévy-driven Ornstein-Uhlenbeck process, $\{W(t)\}$ is standard Brownian-motion and μ and β are constants (see [2]).

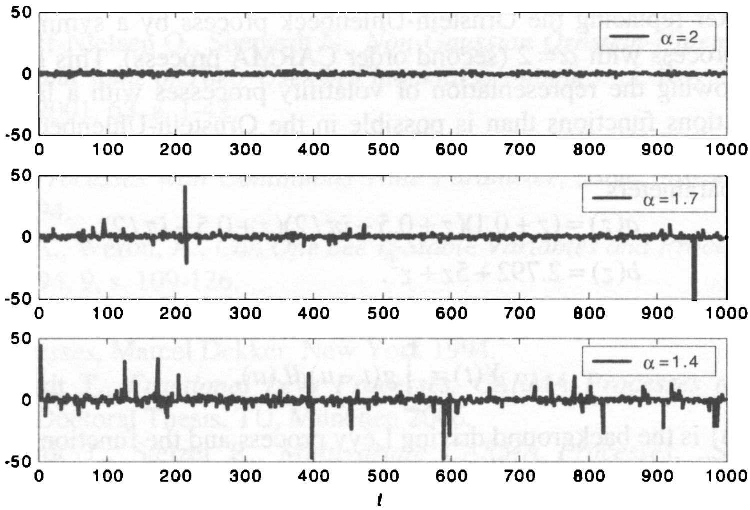

Fig. 1. The realizations of symmetric α-stable CARMA(1,1) process with $\alpha = 2$ (top panel), $\alpha = 1.7$ (middle panel) and $\alpha = 1.4$ (bottom panel)
Source: own calculations.

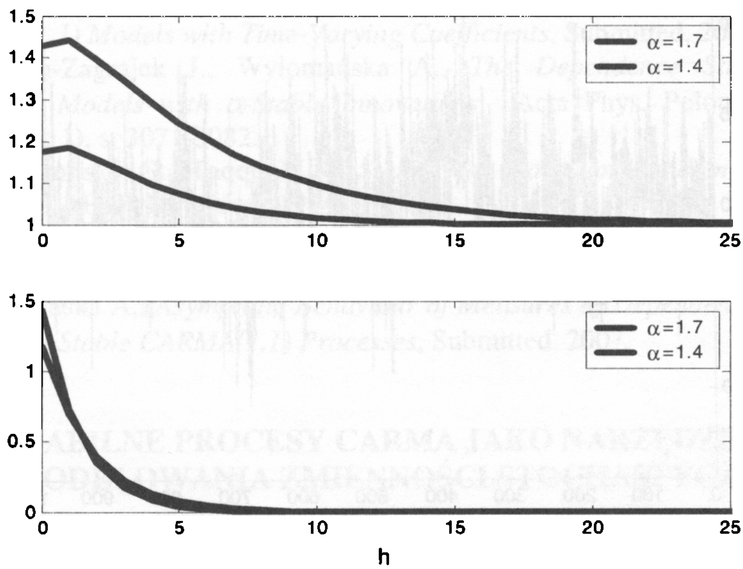

Fig. 2. The plots of the functions $\dfrac{CD(Y(t+h),Y(t))}{\alpha CV(Y(t+h),Y(t))}$ (top panel) and $\dfrac{CD(Y(t),Y(t+h))}{\alpha CV(Y(t),Y(t+h))}$ (bottom panel) vs $h \in [0,25]$ for $\alpha = 1.7$ (solid line) and $\alpha = 1.4$ (dotted line)
Source: own calculations.

Much of the analysis of Berndorff-Nielsen and Shephard can however be carried out after replacing the Ornstein-Uhlenbeck process by a symmetric α-stable CARMA process with $\alpha = 2$ (second order CARMA process). This has the advantage of allowing the representation of volatility processes with a larger range of autocorrelations functions than is possible in the Ornstein-Uhlenbeck framework. Brockwell and Marquardt in [2] propose the take CARMA(3,2) process with the following parameters

$$a(z) = (z + 0.1)(z + 0.5 - i\pi/2)(z + 0.5 + i\pi/2),$$

$$b(z) = 2.792 + 5z + z^2.$$

In this case the solution process $\{Y(t)\}$ has the following form

$$Y(t) = \int_{-\infty}^{\infty} g(t-u)dL(u),$$

where $\{L(t)\}$ is the background driving Lévy process and the function

$$g(t) = 0.8762\exp(-0.1t) + \left(0.1238\cos\frac{\pi t}{2} + 2.5780\sin\frac{\pi t}{2}\right)\exp(-0.5t), \qquad t \geq 0.$$

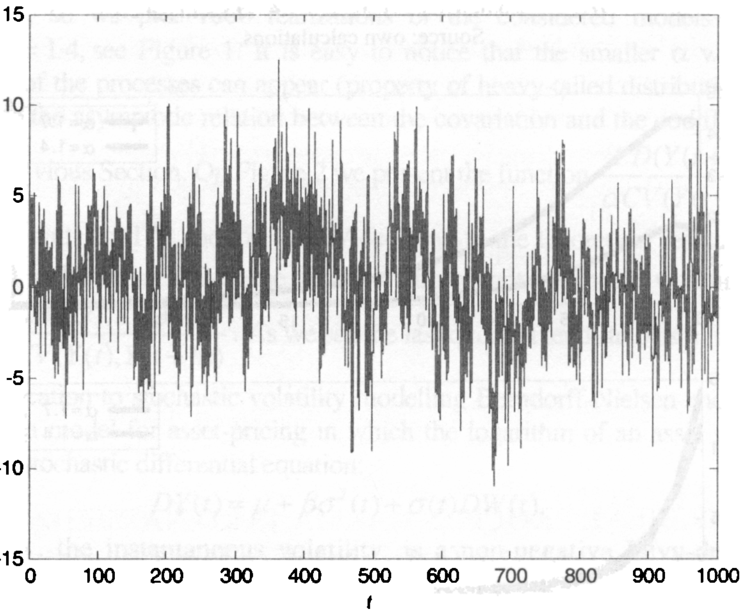

Fig. 3. The realizations of CARMA(3,2) process
Source: own calculations.

On Figure 3 we present the realizations of the process $\{Y(t)\}$.

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STABILNE PROCESY CARMA JAKO NARZĘDZIE DO MODELOWANIA ZMIENNOŚCI STOCHASTYCZNEJ

Streszczenie

W artykule zbadano nową klasę procesów stochastycznych wykorzystywanych w matematyce finansowej, a mianowicie procesy CARMA (ciągłe modele ARMA) z symetrycznymi stabilnymi innowacjami, które są naturalnym rozszerzeniem rozpatrywanych w [2] procesów CARMA o skończonych drugich momentach (zob.

również [6]). Są one także rozszerzeniem opisanych w [8] modeli ARMA z symetrycznymi α-stabilnymi innowacjami. Dla rozpatrywanych modeli funkcja kowariancji nie jest zdefiniowana, dlatego też rozpatruje się inne miary zależności.

W artykule podano postać rozwiązania rozpatrywanych modeli ciągłych, a także przestudiowano kodyferencję i kowariację – dwie najpopularniejsze miary zależności zdefiniowane dla symetrycznych procesów α-stabilnych. Pokazano także, iż rozpatrywane miary są asymptotycznie proporcjonalne ze współczynnikiemi proporcjonalności równym α. Otrzymane rezultaty są analogiczne z wynikami uzyskanymi w przypadku dyskretnych modeli rozpatrywanych w [8; 10]. Rozpatrywane procesy zastosowano do modelowania zmienności stochastycznej.