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# COPULA APPROACH IN TWO-STOCK PORTFOLIO ANALYSIS

## 1. Portfolio analysis – some introductory remarks

Portfolio theory proposed by Harry Markowitz [Markowitz 1952] and extended by James Tobin to include risk-free instruments [Tobin 1958], was the main breakthrough in modern finance in general and in risk management in particular. It is worth to notice very important fact, not mentioned often in theoretical studies, that the idea of reducing risk by considering correlation of returns (or prices) is also the main driving force of the use of derivative instruments in risk management. In fact, taking one position on spot market and the opposite position on futures (or forward) market is nothing else that taking opportunity of perfect hedge by using the value of correlation equal to minus 1.

Classical approach in portfolio theory is based on considering two-criteria decision problem, that is maximizing the level of return (understood as expected return – expected value of the distribution of returns) and minimizing level of risk (understood as standard deviation of the distribution of returns). Then the random variable is considered, being the linear combination of returns of individual financial instruments (e.g. stocks). This approach leads to the strong dependence of the solution on the correlation of returns. This sometimes is criticized as the approach being limited to the linear dependencies between returns.

Since the birth of modern portfolio theory many different extensions, modifications and alternative approaches were proposed. For example, as risk measure one may adopt the other measures, different from standard deviation of returns.

In this paper, we propose the approach, where:

- instead of two criterions, the combined risk-return criterion is considered;

- the general notion of dependence, free from the drawbacks of correlation coefficient, is assumed.

For simplicity, we consider here two-stock case (two financial instruments).

Let us use the following notations:

$X$ – return on the first stock, being the random variable,

$Y$ – return on the second stock, being the random variable,

$F$ – cumulative distribution function of $X$,

$G$ – cumulative distribution function of $Y$.

We consider only the continuous distributions of $X$ and $Y$.

As the portfolio criterion we propose here to take the following criterion:

$$P(X \le x, Y \le y). \tag{1}$$

This means that as risk measure one takes the probability that return of EACH stock is below some given level (this level could be different for both stocks, but it is reasonable to assume $x = y$). Clearly, it means that the higher probability, the higher risk.

First of all, it should be noticed, that the criterion given by (1) is in fact risk-return criterion, understood as the probability of falling (risk driven notion) below some level (return driven notion). Secondly, this is the criterion which goes beyond Markowitz approach, since:

$$X \le x \text{ and } Y \le y \Rightarrow \alpha X + (1-\alpha)Y \le \alpha x + (1-\alpha)y.$$

But the opposite implication is not true.

The rationale behind the criterion given by (1) is particularly transparent, when the investor considers the possibility of closing just one position on the market (selling one stock) holding the other position opened (holding the other stock). Thus she (he) holds the positions in different stocks for different periods and she (he) is interested in considering two stocks separately.

## 2. Copula approach – analyzing the multivariate distribution

The classical analysis of multivariate distribution (including bivariate distribution of returns) is based on the covariance matrix, or some generalization, called scatter matrix. Therefore it is assumed that most (or all) information about dependence between the components of the random vector is contained in covariance matrix (through covariances). Since the dependence is the crucial notion in portfolio analysis, it means that in classical approach risk management tools are covariance-driven (correlation-driven).

We propose here an alternative approach to analyze the multivariate distribution (of returns), where instead of analyzing jointly individual risk parameters and dependence parameters, given in the covariance matrix, the analysis is performed separately for individual risk parameters (through the analysis of univariate distributions) and for dependence parameters. This approach is based on the so-called copula analysis.

The main idea of copula analysis lies in the decomposition of the multivariate distribution into two components. The first component – it is marginal distributions. The second component is the function linking these marginal distributions to get a multivariate distribution. This function reflects the structure of the dependence between the components of the random vector. Therefore the analysis of multivariate distribution function is conducted by „separating" univariate distribution from the dependence.

This idea is reflected in Sklar theorem [Sklar 1959], given as:

$$H(x_1,...,x_m) = C(H_1(x_1),...,H_m(x_m)), \tag{2}$$

where:

$H$ – the multivariate distribution function,

$H_i$ – the distribution function of the $i$-th marginal distribution,

$C$ – copula function.

Since we consider here the bivariate distribution (two-stock portfolio), we get the special version of the theorem given by (2):

$$H(x,y) = C(F(x),G(y)). \tag{3}$$

So in this case copula function is simply the distribution function of the bivariate uniform distributions. The bivariate distribution function is given as the function of the univariate (marginal) distribution functions. This function is called copula function and it reflects the dependence between the univariate components.

As we can see, the information about the dependence contained in the bivariate distribution is „retained" in copula function. The advantage of copula analysis lies in the fact that it is suited for many possible distributions (of returns).

Of course, one can get very many possible and suitable distributions, depending on the choice of marginal distributions and the choice of copula function. The classical approach of bivariate normal distribution can be put in the framework of copula analysis, by assuming univariate normal distribution as marginal distribution and choosing the so-called normal (Gaussian) copula, given through distribution functions of univariate and bivariate normal distribution as ($\rho$ stands for correlation coefficient):

$$C(u_1,u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left(-\frac{x^2-2\rho xy+y^2}{2(1-\rho^2)}\right)dxdy. \tag{4}$$

By applying normal copula to non-normal univariate marginal distributions we get different distributions. On the other hand, we can apply also other than normal copula to univariate normal distributions. Of course, non-normal univariate distributions and non-normal copula function can be also combined.

There are very many possible copula functions, which were explored in theoretical studies and used in empirical examples. Except for normal (Gaussian) copula, one often studies the following copulas:

- Gumbel copula, given as:

$$C(u_1,u_2)=\exp(-((-\ln u_1)^\theta+(-\ln u_2)^\theta)^{1/\theta}),\quad \theta\in[1;\infty); \tag{5}$$

- Clayton copula, given as:

$$C(u_1,u_2)=\max((u^{-\theta}+v^{-\theta}-1)^{-1/\theta};0),\quad \theta\in[-1;\infty),\theta\neq 0; \tag{6}$$

- Ali-Mikhail-Haq copula, given as:

$$C(u_1,u_2)=\frac{u_1u_2}{1-\theta(1-u_1)(1-u_2)},\quad \theta\in[-1;1]; \tag{7}$$

- Frank copula, given as:

$$C(u_1,u_2)=-\frac{1}{\theta}\ln\left(1+\frac{(e^{-\theta u_1}-1)(e^{-\theta u_2}-1)}{(e^{-\theta}-1)}\right),\quad \theta\neq 0; \tag{8}$$

- Farlie-Gumbel-Morgenstern copula, given as:

$$C(u_1,u_2)=u_1u_2+\theta u_1u_2(1-u_1)(1-u_2),\quad \theta\in[-1;1]. \tag{9}$$

All copula functions presented above are one-parameter function, denoted by $\theta$. This parameter can be interpreted as the dependence parameter, where the dependence is understood in more general sense than correlation (suitable in the case of bivariate normal distribution).

This is strictly related to the importance of copula analysis, which comes from its usefulness of modeling the dependence, not necessarily linear dependence. This is possible because of some properties of the copula function. The most important are the following properties (presented here for the bivariate case):

- for independent variables we have:

$$C(u_1,u_2)=C^{\neg}(u_1,u_2)=u_1u_2, \tag{10}$$

- the lower limit for copula function is:

$$C^{-}(u_1,u_2)=\max(u_1+u_2-1;0), \tag{11}$$

- the upper limit for copula function is:

$$C^{+}(u_1,u_2)=\min(u_1,u_2). \tag{12}$$

The lower and upper limits of the copula function have important consequences for the modeling of dependence. Suppose that we have two variables, $X$ and $Y$, and there exists monotonic function (not necessarily a linear one), which links these two variables. We have the so-called total positive dependence between $X$ and $Y$, when $Y = T(X)$ and $T$ is the increasing function. Similarly, we have the so-called total negative dependence between $X$ and $Y$, when $Y = T(X)$ and $T$ is the decreasing function. Then:

- in the case of total positive dependence the following relation holds:

$$C(u_1,u_2) = C^+(u_1,u_2) = \min(u_1,u_2), \tag{13}$$

- in the case of total negative dependence the following relation holds:

$$C(u_1,u_2) = C^-(u_1,u_2) = \max(u_1 + u_2 - 1;0). \tag{14}$$

This leads to the natural ordering of the multivariate distributions with respect to the strength and the direction of the dependence. It is given by the following order:

$$C_1(u_1,u_2) \le C_2(u_1,u_2) \Rightarrow C_1 \prec C_2$$

and then we have:

$$C^- \prec C^{\perp} \prec C^+.$$

The presented considerations are valid for any type of the dependence, including classical linear dependence.

The detailed description of copula functions is given in Nelsen [1999] and Joe [1997].

## 3. Copula approach in portfolio theory

Now we are ready to present the proposal of the use of copula approach in two--stock portfolio theory, when risk of portfolio is understood as the probability that returns on each of two stocks fall below given level. Here risk criterion given in (1) is linked with decomposition of bivariate distribution given in (3). Therefore we get the following risk criterion:

$$P(X \le x, Y \le y) = C(F(x), G(y)). \tag{15}$$

As one can see from (15), risk of a portfolio is a function of the individual risk of each stock (understood as falling below some level of return) and the copula function "linking" these two individual risks. Therefore copula function, reflecting dependence, plays here similar role as correlation of returns in classical approach.

Therefore it seems that some results obtained in classical two-stock portfolio analysis may have corresponding results in the proposed approach. In particular, three portfolios can be naturally defined:

- the lowest risk portfolio, obtained for lower limit copula function, given as:

$$P(X \le x, Y \le y) = \max(P(X \le x) + P(Y \le y) - 1;0), \tag{16}$$

- the highest risk portfolio, obtained for upper limit copula function, given as:

$$P(X \le x, Y \le y) = \min(P(X \le x), P(Y \le y)), \tag{17}$$

- portfolio with independent components, given as:

$$P(X \le x, Y \le y) = P(X \le x)P(Y \le y). \tag{18}$$

The portfolios given by (16), (17) and (18) correspond to classical two-stock portfolios, when correlation of returns is equal to –1, +1 and 0, respectively.

Now we present simple numerical example. For simplicity we consider only dependence problem. Therefore we assume two cases:

- the first one (safety-first situation):

$$P(X \le x) = 0.1; P(Y \le y) = 0.1,$$

- the second one (normal situation):

$$P(X \le x) = 0.5; P(Y \le y) = 0.5.$$

In each case we calculate risk of portfolio, defined in (1), by considering copula functions presented above. For comparison, we present also three portfolios given by (16), (17) and (18), as well as portfolios obtained by assuming bivariate normal distribution with different correlation coefficients. The results are summarized in Tab. 1.

**Table 1.** Risk of two-stock portfolios (probabilities) for different copula function

| Copula | $P(X \le x) = 0.1$ $P(Y \le y) = 0.1$ | $P(X \le x) = 0.5$ $P(Y \le y) = 0.5$ |
|---|---|---|
| Lower limit | 0 | 0 |
| Independence | 0.01 | 0.25 |
| Upper limit | 0.1 | 0.5 |
| Normal, correlation: –0.9 | 0 | 0.072 |
| Normal, correlation: –0.5 | 0.0006 | 0.167 |
| Normal, correlation: 0 | 0.01 | 0.25 |
| Normal, correlation: 0.5 | 0.0334 | 0.333 |
| Normal, correlation: 0.9 | 0.0680 | 0.428 |
| Farlie-Gumbel-Morgenstern $\theta = -1$ | 0.0019 | 0.1875 |
| Farlie-Gumbel-Morgenstern $\theta = 0$ | 0.01 | 0.25 |
| Farlie-Gumbel-Morgenstern $\theta = 1$ | 0.0181 | 0.3125 |
| Ali-Mikhail-Haq $\theta = -1$ | 0.0055 | 0.2 |
| Ali-Mikhail-Haq $\theta = 0$ | 0.01 | 0.25 |
| Ali-Mikhail-Haq $\theta = 0.95$ | 0.0434 | 0.3279 |
| Gumbel $\theta = 1$ | 0.01 | 0.25 |
| Gumbel $\theta = 5$ | 0.071 | 0.4510 |
| Clayton $\theta = -1$ | 0 | 0 |
| Clayton $\theta = 1$ | 0.0526 | 0.3333 |
| Clayton $\theta = 10$ | 0.0933 | 0.4665 |
| Frank $\theta = -10$ | 0 | 0.0686 |
| Frank $\theta = -1$ | 0.0064 | 0.2191 |
| Frank $\theta = 1$ | 0.0144 | 0.2809 |
| Frank $\theta = 10$ | 0.051 | 0.3125 |

The analysis of the results given in tab. 1 leads to the following conclusions:

- there is strong relation between dependence parameter of copula and risk (defined in (1));

- in the case of Clayton copula the lowest possible risk, given by lower limit copula is obtained for the lowest value of dependence parameter;
- in the case of Gumbel copula the lowest value of dependence parameter gives the portfolio with independent components, therefore for this copula there is no possibility of negative effect of dependence leading to more reduction of risk;
- in the case of three functions, namely Farlie-Gumbel-Morgenstern copula, Ali-Mikhail-Haq copula and Frank copula, in the safety-first situation we see low values of probabilities, distant from possible upper level, defined by upper limit copula function;
- the results for normal copula show approximately linear dependence of risk (defined in (1)) on correlation coefficient.

## Literature

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## PODEJŚCIE COPULA W ANALIZIE PORTFELA DWUSKŁADNIKOWEGO

### Streszczenie

Artykuł zawiera propozycje zastosowania podejścia mającego u podstaw funkcje połączeń w teorii portfela dwóch składników. Autor przedstawia podstawy teorii funkcji połączeń, a następnie wskazuje ich zastosowanie w analizie portfela dwóch składników, gdzie kryterium tworzenia portfela określone jest jako prawdopodobieństwo osiągnięcia zadanych stóp zwrotu przez każdy składnik portfela.