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Inwestycje finansowe i ubezpieczenia – tendencje światowe a polski rynek

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# SELECTED PROBLEMS OF ACTUARIAL DEMOGRAPHY

## 1. Introduction

Actuaries and demographers need mortality tables for many of their calculations. The table used in any particular instance must be applicable to the lives under consideration. Thus, the actuary must choose a mortality bases suitable for ordinary life-assurance contract, and a different mortality basis appropriate to annuitants when he determines annuity purchase prices.

Before selecting a mortality basis for his calculations, the actuary (or demographer) leads to examine the mortality experience of lives to those in question. Two alternatives are then open to him. He may either:

i. adopt a particular standard table as his mortality basis,
ii. construct a special new table.

It is obviously easier to use an existing table, and a number of statistical tests have been devised to determine whether the lives in a particular mortality experience can be regarded as coming from a population whose mortality rates have already been accurately determined. These tests are not the topic of this article.

The construction of a new table is rather more complicated. The main problem is the adjustment of the observed rates to produce smooth decrement rates which are accurate estimates of the underlying mortality. The adjustment procedure that reduces the random errors in the observed rates as well as smoothing them is known as graduation, and various methods are described in this article.

## 2. Mortality tables

The mortality (life) table is an effective model for summarizing mortality experience. Because mortality experience is different for males and females, the life table is always calculated separately for each sex. It provides a number of useful indices with exerted symbols. We will use these of them:

$l_x$ the number of lives who survive to exact age $x$ out of an original cohort of births ($l_0$),

$d_x$ the number dying between exact age $x$ and age $x + 1$,

$q_x = \dfrac{d_x}{l_x}$ the rates of mortality, or proportion of lives of exact age $x$ who die before attaining exact age $x + 1$, which represents the total effect of the mortality in terms of the proportion who fail to survive the whole year of age $x$ to $x + 1$ without reference to the variation of mortality risk over the course of that year,

$L_x$ the average number alive between $x$ and $x + 1$,

$m_x = \dfrac{d_x}{L_x}$ the central death rate or average mortality rate over the age interval $x$ to $x + 1$ is the average risk to which the population is subjected during its passage through the year of age $x$ to $x + 1$.

In estimating mortality the actuary traditionally employs the life table as a model but he knows that the past experience from which that table has been derived will never be exactly reproduced in the future. Mortality is itself constantly varying: there are fluctuations about an underlying trend. The actual observations from which he life table has been derived are a sample of total experience if only in terms of time, i. e. covering a short period of years. A certain random element of fluctuation will be inherent in the observations. The fluctuations, which we may regard as sampling errors, may be in different directions at different ages. These errors (deviations from the true underlying rates) may be assumed to be random and to fluctuate from age to age both in size and sign.

## 3. Methods of graduation

The adjustment procedure that reduced the random errors in the observed rates as well as smoothing them is known as graduation. We consider graduation as the principles and methods of adjusting a set of observed rates to provide a suitable basis for actuarial and demographic calculations of a practical nature. There are three main types of graduation process:

- The graphical methods, in which a hypothetical curve is drawn, by inspection, through the area bounded by the confidence intervals.
- Nonparametric adjusted moving averages methods, which depend on the principle that the standard error of the weighed mean of two or more independent random errors is less than the sum of the correspondingly weighed individual standard errors.
- Analytical, or curve-fitting methods, which are based on the assumption that the underlying values have a particular mathematical form whose parameters may be estimated from the observed values.

## 4. The graphical method

The graphical method is perhaps the most widely used graduation technique in pension funds, where mortality data are often scanty. An important advantage of the method is that it is easy to make allowance for special features, e. g. discontinuities in retirement rates.

Mortality rates are calculated from the available data and represented by points. A smooth curve, its shape suggested by the lie of the points , is dawn to pass as near to these points as possible while still providing a reasonable progression of rates, and smoothed decrement rates are read from the curve. The progression of the rates may be further improved by examination and adjustment of the first few orders of differences to obtain graduates rates which satisfy prescribed requirements of smoothness and adherence to data. This process is referred to as „hand-polishing“.

## 5. The adjusted moving averages graduation

British actuaries have used the smoothing properties of moving averages to develop graduation formulae [Benjamin, Polard 1993]:

- *Wittstein's formula*

The smoothed value $\hat{q}_x$ at time $x$ is the weighted moving average, defined by

$$\hat{q}_x = \frac{1}{25}\left[5q_x + 4\left(q_{x-1} + q_{x+1}\right) + 3\left(q_{x-2} + q_{x+2}\right) + 2\left(q_{x-3} + q_{x+3}\right) + \left(q_{x-4} + q_{x+4}\right)\right].$$

Smoothing by this formula is adequate as the double smoothing by simple moving average method by formula

$$\frac{1}{5}\left(q_{x-2} + q_{x-1} + q_x + q_{x+1} + q_{x+2}\right).$$

The graphical result of graduation by Wittstein's formula using the *Smoothing* procedure of the statistical analytical system STATGRAPHICS *Centurion* XV shows Figure 1.

- *Spenser's 15-term and 21-term formulae*:

These weighted moving averages methods developer for actuarial work. The smoothed value at time x by *Spenser's 15-term formula* is given by

$$\dot{q}_x^{S(15)} = \frac{1}{320}\Big[74q_x + 67\left(q_{x-1} + q_{x+1}\right) + 46\left(q_{x-2} + q_{x+2}\right) + 21\left(q_{x-3} + q_{x+3}\right) + 3\left(q_{x-4} + q_{x+4}\right) - \\ -5\left(q_{x-5} + q_{x+5}\right) - 6\left(q_{x-6} + q_{x+6}\right) - 3\left(q_{x-7} + q_{x+7}\right)\Big].$$

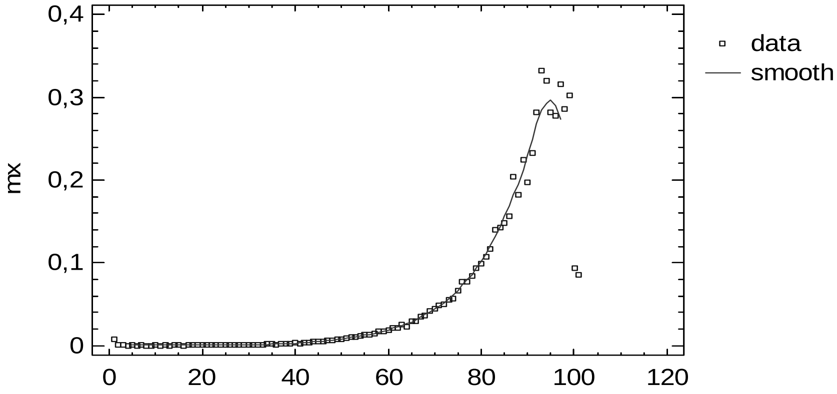


Fig. 1. Witstein's method of graduation of Slovak national mortality data of men 2006
Source: own calculations.

- *Spenser's 21-term* is given by:

$$\hat{q}_x^{S(21)} = \frac{1}{350}[60q_x + 57(q_{x-1} + q_{x+1}) + 47(q_{x-2} + q_{x+2}) + 33(q_{x-3} + q_{x+3}) + 18(q_{x-4} + q_{x+4}) + \\ + 6(q_{x-5} + q_{x+5}) - 2(q_{x-6} + q_{x+6}) - 5(q_{x-7} + q_{x+7}) - 5(q_{x-8} + q_{x+8}) - 3(q_{x-9} + q_{x+9}) - \\ -(q_{x-10} + q_{x+10})].$$

The graphical result of graduation by Spenser's 21-term formula using the *Smoothing* procedure of the statistical analytical system STATGRAPHICS Centurion XV has shown Figure 2.

Disadvantages of above weighted moving averages methods are that they do not provide values at the beginning and end of the life table.

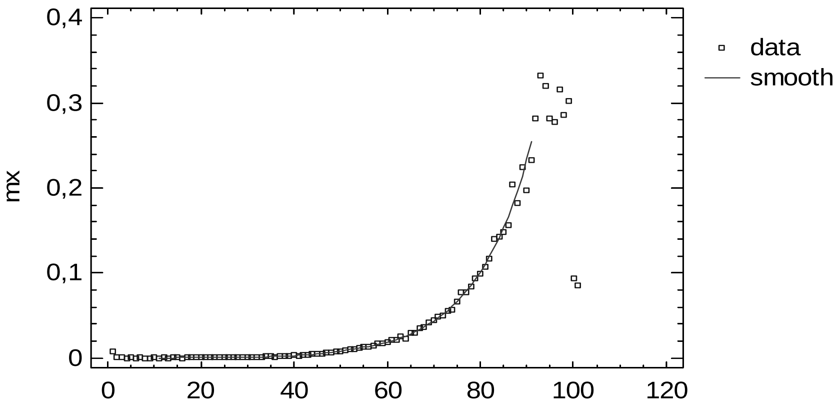


Fig. 2. Spenser's 21-term graduation of Slovak national mortality data of men 2006
Source: own calculations.

## 6. Curve-fitting methods of graduation

These methods are known also as methods of graduation by mathematical formula. Successful use of these methods requires some prior knowledge or reasonable hypothesis about the general shape and trend of the set of observed values to be graduated [Benjamin, Polard 1993].

The first important contribution towards finding a "law of mortality"(as it was then called) was made by Benjamin Gompertz (1825), who argues on physiological grounds that the intensity of mortality gained equal proportions of age and would be represented by formula $q_x = B \cdot c^x$. A development of Gompertz's law was subsequently made by Makeham (1860), who adopted the formula $q_x = A + B \cdot c^x$.

With its three constants A, B and c this formula was found to give satisfactory adherence to data for a number of experiences and several standard tables were graduated by its use.

We have used the Gompertz-Makeham formula for graduation of mortality data of the old men above 60 in the year 2006 in Slovak Republic. We have used piecemeal solution by [Fiala 2005].

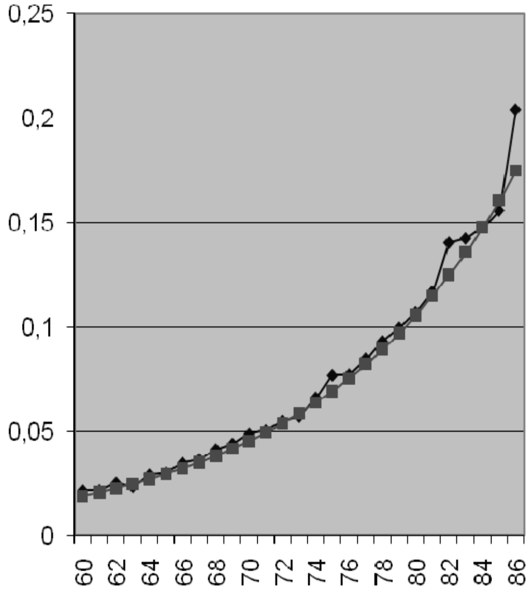


Fig. 3. Gompertz-Makeham graduation of the old age men mortality 2006 in SR
Source: own calculations.

From next three equations we can get estimation of parameters *a*, *b* and *c*.

$$G_1 = \sum_{x_0}^{x_0+K-1} m_x = \sum_{x_0}^{x_0+K-1} \left(a + b \cdot c^{x+0,5}\right)$$

$$G_2 = \sum_{x_0+K}^{x_0+2K-1} m_x = \sum_{x_0+K}^{x_0+2K-1} \left(a + b \cdot c^{x+0,5}\right)$$

$$G_2 = \sum_{x_0+2K}^{x_0+3K-1} m_x = \sum_{x_0+2K}^{x_0+3K-1} \left(a + b \cdot c^{x+0,5}\right)$$

Their solution is in the form:

$$a = \frac{G_1 - b \cdot K_0}{K} \quad b = \frac{G_2 - G_1}{\left(c^K - 1\right) \cdot K_0} \quad c = \left(\frac{G_3 - G_2}{G_2 - G_1}\right)^{\frac{1}{K}}$$

If we put $x_0 = 60$ and $K = 8$, then we have $a = –0{,}001$, $b = 0.0001$ and $c = 1, 0867$. Graduation of mortality data by Gompertz-Makeham formula with these parameters shows Figure 3.

## 7. Graduation by reference to a standard table

When the data are scanty but are known or suspected to come from an experience simile to that for which a standard (graduated) table already exists, it may be possible to use this standard table as a base curve for graduating the new data. This is the typical situation in life insurance companies. Process of graduation in this case describes publication [Sivašová 2008].

Table 1. A mortality experience and graduated rates of mortality

| Age | $D_x$ | $E_x$ | $q_x^s$ | $\hat{q}_x$ |
|---|---|---|---|---|
| 50 | 101 | 42069 | 0,003775 | **0,002603** |
| 51 | 128 | 41172 | 0,004187 | **0,002893** |
| 52 | 116 | 41102 | 0,004620 | **0,003199** |
| 53 | 157 | 41000 | 0,005060 | **0,003510** |
| 54 | 166 | 39647 | 0,005528 | **0,003840** |
| 55 | 150 | 37085 | 0,006063 | **0,004218** |
| 56 | 152 | 35263 | 0,006700 | **0,004668** |
| 57 | 174 | 34314 | 0,007428 | **0,005181** |
| 58 | 200 | 31485 | 0,008239 | **0,005754** |
| 59 | 191 | 28351 | 0,009089 | **0,006354** |
| 60 | 176 | 28037 | 0,009922 | **0,006942** |

Source: own calculations.

The one of mathematical formula that has been suggested for graduation by reference to a standard table is $\hat{q}_x = a \cdot q_x^s + b$. Because of $\hat{q}_x = f(q_x^s)$ and $q_x^s = g(x)$, consequently $\hat{q}_x = f(g(x))$. To estimate the parameters we will use the least of square method that lead to equations

$$\sum_{y \le x} D_y = a \cdot \sum_{y \le x} E_y q_y^s + b \cdot \sum_{y \le x} E_y$$

$$\sum_x \sum_{y \le x} D_y = a \cdot \sum_x \sum_{y \le x} E_y q_y^s + b \cdot \sum_x \sum_{y \le x} E_y$$

We demonstrate this method for fitting formula $\hat{q}_x = a \cdot q_x^s + b$ to the experience women mortality data of the one of life insurance companies in Slovakia [*Graduácia mier...* 2006].

Table 2. The steps of graduation by reference to a standard table

| $x$ | $D_x$ | $\sum_{y \le x} D_y$ | $\sum_x \sum_{y \le x} D_y$ | $E$ | $\sum_{y \le x} E_y$ | $\sum_x \sum_{y \le x} E_y$ | $E_x q_x^s$ | $\sum_{y \le x} E_y q_y^s$ | $\sum_x \sum_{y \le x} E_y q_y^s$ |
|---|---|---|---|---|---|---|---|---|---|
| 50 | 101 | 101 | 101 | 42069 | 42069 | 42069 | 158,81 | 158,81 | 158,81 |
| 51 | 128 | 229 | 330 | 41172 | 83241 | 125310 | 172,39 | 331,20 | 490,01 |
| 52 | 116 | 345 | 675 | 41102 | 124343 | 249653 | 189,89 | 521,09 | 1011,10 |
| 53 | 157 | 502 | 1177 | 41000 | 165343 | 414996 | 207,46 | 728,55 | 1739,65 |
| 54 | 166 | 668 | 1845 | 39647 | 204990 | 619986 | 219,17 | 947,72 | 2687,36 |
| 55 | 150 | 818 | 2663 | 37085 | 242075 | 862061 | 224,85 | 1172,56 | 3859,93 |
| 56 | 152 | 970 | 3633 | 35263 | 277338 | 1139399 | 236,26 | 1408,83 | 5268,75 |
| 57 | 174 | 1144 | 4777 | 34314 | 311652 | 1451051 | 254,88 | 1663,71 | 6932,46 |
| 58 | 200 | 1344 | 6121 | 31485 | 343137 | 1794188 | 259,40 | 1923,12 | 8855,58 |
| 59 | 191 | 1535 | 7656 | 28351 | 371488 | 2165676 | 257,68 | 2180,80 | 11036,38 |
| 60 | 176 | 1711 | 9367 | 28037 | 399525 | 2565201 | 278,18 | 2458,98 | 13495,36 |

Source: own calculations.

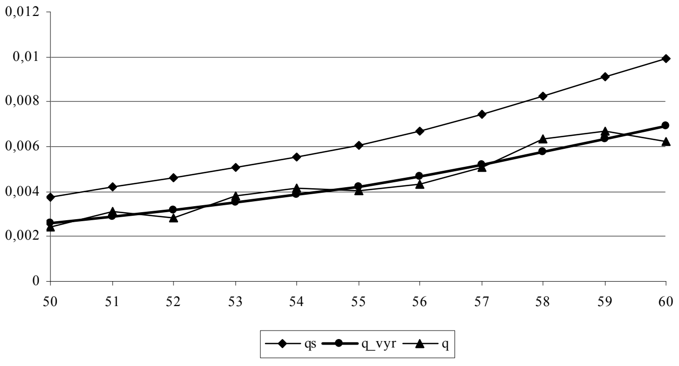


Fig. 4. The result of graduation by reference to a standard table
Source: own calculations.

As a standard table we shall use the Slovak Life Table 2003 (women). Table 2 contains the steps of calculation.

When we solve the simultaneous equations

$$1\,711 = 2\,458{,}981 \cdot a + 399\,525 \cdot b$$

$$9\,367 = 1\,349\,536 \cdot a + 2\,565\,201 \cdot b$$

we obtain graduation function $\hat{q}_x = 0{,}705977 \cdot q_x^s - 6{,}25342 \cdot 10^{-5}$.

Figure 4 shows the graphical result of graduation by reference to a standard table.

## References


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## WYBRANE PROBLEMY DEMOGRAFII AKTUARIALNEJ

### Streszczenie


Konstruowanie nowej tablicy życia w zakładzie ubezpieczeń na życie jest raczej skomplikowanym procesem. Podstawowym problemem jest takie dostosowanie zaobserwowanych stóp śmiertelności, aby uzyskać wartości wygładzone i malejące będące trafnymi estymatorami śmiertelności. Procedura dostosowania, która zarówno redukuje błędy losowe w zaobserwowanych stopach, jak i je wygładza, nazywana jest stopniowaniem.

Artykuł opisuje i prezentuje zastosowanie metody stopniowania opartej na obserwowanych danych śmiertelności z wykorzystaniem Excela i pakietu statystycznego Statgraphics Centurion XV.