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ON HARTWICK'S RULE[1]

Abstract

In this paper two mathematical versions of the Hartwick's rule are described. The first is connected with optimal control models, and price variables are used. The other one is formulated without price variables in control models without optimization. It is shown that the first formulation can be used also in general control models.

The Hartwick's rule is connected with the following model of the development of renewable and non-renewable capitals described by the well known system of differential equations (briefly by the system), (see [1-4]):

$$k' = f(k, r) - \delta k - c, \quad (1)$$
$$s' = -r,$$

where k is a renewable capital, s is a non-renewable capital, c denotes the consumption, δ denotes the constant depreciation rate of the augmentable capital, r denotes the rate of extraction of non-renewable capital, $f(k, r)$ denotes the production function (differentiable of second order).

We use designations $k' = dk/dt$, $F_z = \partial F/\partial z$ and suppose that population and technology are constant over time.

The system (1) is a control system for variables $(k, s)\in V$ with the control parameters $(c, r)\in U$, where V and U are open sets in R^2.

The tangent space $TV\cong V\times R^2$ is the space of change rates (k, s, k', s') of capitals. Its dual space (cotangent space) $T^*V \cong V\times R^2$ is the space of evaluations (k, s, ξ, ψ) of this change rates, i.e. of the linear maps $\xi k' + \psi s'$.

The system (1) is autonomous for constants $c = c_0$, $r = r_0$ and thus determines:

1. Hamiltonian $H(k, s, \xi, \psi, c, r) = \xi k' + \psi s' = \xi(f(k, r) - \delta k - c) + \psi(-r)$ of the system (1). It is a function on T^*V for any $(r, c)\in U$, i.e. on $T^*V\times U$. It expresses the evaluation of change rates of capitals.

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2. The flow Φ_t on V, that is a local one-parameter group of diffeomorphism on V. Its cotangent prolongation $T^*\Phi_t$ determines for $c = c_0$, $r = r_0$ the flow of the following system

$$k' = f(k, r) - \delta k - c = H_\xi, \qquad \xi' = -H_k = \xi(\delta - f_k), \tag{2}$$
$$s' = -r = H_\psi, \qquad \psi' = -H_s = 0, \text{ i.e. } \psi = \psi_0.$$

The system (1) is a subsystem of the system (2), i.e. every solution of (2) gives the solution of (1). The system in the second column of (2) is well known from the Pontryagin's maximum principle. Readers are referred to [5; 6] for details in theory of natural tangent and cotangent prolongations.

Now we introduce the Hartwick's rule.

Hartwick's rule: Invest the rent from the exhaustible resource used at each date in the net accumulation of the produced capital good.

We discuss two mathematical versions of this rule used in literature.

a. $k' = f_r\, r$. (3)

This version (see for example [1]), says that the change rate of renewable capital is equal to the part of the increment of the production function corresponding to $\Delta r = r$, if we neglect the infinitesimal values of the second and higher order.

b. $\xi k' = \psi r$, i.e. $H(k, s, \xi, \psi, c, r) = \xi k' + \psi s' = \xi(f(k, r) - \delta k - c) + \psi(-r) = 0$. (4)

It can be interpreted as follows. The prices of the increment of the renewable capital and of the flow of exhaustible resource per time unit are equal (see [1; 4]). This b.-version is in literature connected with optimal control problems. Our approach shows that it can be used in any control problem.

We will compare these two versions and will find conditions when they are strictly equivalent.

Definition 1

A curve $\rho(t) = (r(t), c(t))$ in U is called a control path. Solution of the systems (1) and (2) for $r = r(t)$, $c = c(t)$ are called solutions corresponding to the control path $\rho(t)$. A control path $\rho(t)$ is said to be a 1Hart-path if together with a corresponding solution of the system (1) satisfy the relation (3). A control path $\rho(t)$ is called a 2Hart-path or a 2Hart-0-path if together with a corresponding solution of (2) satisfy $H(t) = H(k(t), s(t), \xi(t), \psi(t), \rho(t)) \equiv H/\rho(t) = \text{const}$ or $H/\rho(t) = 0$, i.e. if the relation (4) is satisfied. A control path is called equitable if $c(t) = c_0$ is constant.

If a 1Hart-path is equitable then calculating the derivative (3) with respect to t we get

$$df_r/dt = f_k f_r - \delta f_r. \tag{5}$$

Definition 2

A control path is said to be competitive if, together with a corresponding solution of (1), it satisfies the relation (5).

Evidently a 1Hart-path is equitable iff it is competitive. Authors in [1] proved that if $\rho(t)$ is equitable and competitive, then it is a 1Hart-path and $\delta = 0$.

By a direct computation we get that a control path $\rho(t)$ is a 2Hart-path iff, together with a corresponding solution of (2), it satisfies the relations

$$H_r r' + H_c c' = 0, \quad \text{i.e. } (\xi f_r - \psi_0)r' - \xi c' = 0. \tag{6}$$

Evidently every constant control path $\rho(t) = (r_0, c_0)$ is a 2Hart-path.

Definition 3

A control path is said to be regular if there is a corresponding solution of the system (2) such that $\xi(t) \neq 0$ for $t \geq 0$.

Proposition 1. If a regular 2Hart-path is equitable but not constant then it is competitive.

Proof. By (6) $\xi f_r - \psi_0 = 0$. Calculating the derivative with respect to t we get that the relation (5) is satisfied.

Definition 4

A regular control path is called r-stationary if it together with a corresponding solution of the system (2) satisfies the condition $H_r = 0$, i.e. $\psi_0 = \xi(t) f_r(k(t), r(t))$.

From the relation (6) immediately follows

Proposition 2. If a control path is r-stationary then it is a 2Hart-path iff it is equitable.

Proposition 3. If a control path is r-stationary then it is competitive.

Proof. As it holds $\xi f_r - \psi_0 = 0$ then proof is identical with the proof of Proposition 1.

Theorem 1

Let a control path be r-stationary. Then it is a 2Hart-0-path if it is a 1Hart-path.

Proof. Let $\psi_0 = \xi(t) f_r(k(t), r(t))$. Then $k' = f_r r$ iff $\xi k' = \xi f_r r$, that is iff $\xi k' = \psi_0 r$.

Remark 1. If a control path is r-stationary then it is competitive. If it is also equitable then it is both a 2Hart-path and a 1Hart-path.

Remark 2 (about optimal control models). An optimal control model is determined by the system (1) with the following optimization condition

$$\max \int_0^\infty U(c(t))dt,$$

where $U(c)$ is an utility function and integral is supposed to be convergent.

By the standard procedure (see [7]), the basic system of this model arises from the system (1) adding the equation $p' = U(c)$. Then the cotangent prolongation of the basic system is

$$p' = U(c) = \underline{H}_{\vartheta}, \qquad \vartheta' = -\underline{H}_{p} = 0, \text{ i.e. } \vartheta = \vartheta_0, \tag{7}$$
$$k' = f(k,r) - \delta k - c = H_{\xi} = \underline{H}_{\xi}, \qquad \xi' = -H_k = \xi(\delta - f_k) = -\underline{H}_k,$$
$$s' = -r = H_{\psi} = \underline{H}_{\psi}, \qquad \psi' = -H_s = -\underline{H}_s = 0, \text{ i.e. } \psi = \psi_0,$$

where $\underline{H} = \vartheta U(c) + H$. The system (2) is a subsystem of (7), so every solution of the system (7) gives the solution of the system (2).

If $\rho^*(t) = (r^*(t), c^*(t))$ is an optimal control path, then by the Pontryagin's maximum principle this path together with a corresponding solution of the system (7) satisfy the following equations

$$\underline{H}_c = \vartheta_0 dU/dc = 0, \qquad \underline{H}_r = \xi f_r - \psi_0 = 0.$$

Then every optimal control path is r-stationary. Therefore a regular optimal control path is a 2Hart-0-path iff is a 1Hart-path.

Remark 3. Let $\underline{H}/\rho^*$ or H/ρ^* denote the values of the function $\underline{H}$ or H on the path ρ^* and on a corresponding solution of the system (7). It is easy to see that $\underline{H}/\rho^* = 0$. If ρ^* is also a 2Hart-path, i.e. $H/\rho^* = 0$, then ρ^* is equitable, i.e. is constant and thus $U(c^*(t))$ is constant too. Then integral is divergent. If we take $U(c) = \pi(t)u(c)$, where $\pi(t)$ is the so called discount factor, then also $u(c^*(t))$ is constant but integral is convergent.

Bibliography

[1] Buchholz W., Dasgupta S., Mitra T., "Intemporal equity and Hartwick's rule in exhaustible resource model", *Scandinavian Journal of Economics* 2005, vol. 1007, no. 3, pp. 547-561.

[2] Dixit A., Hammond P., Hoel M., "On Hartwick's rule for regular maximin path of capital accumulation and resource depletion", *Review of Economic Studies* 1980, vol. 47, pp. 551-555.

[3] Hartwick J.M., "International equity and investing of rents from exhaustible resources", *Journal of American Economic Association* 1977, vol. 67, no. 5, pp. 972-974.

[4] Jurca P., "On sustainability constraint in models with non-renewable resource", [in:] *Proceedings of conference "Mathematical methods in economics and industry 2007"*, Slovak Republic, to appear.

[5] Kolar I., Michor P.W., Slovak J., *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin 1993.

[6] Nijmeijer H., Schaft A., *Nonlinear Dynamical System*, Springer-Verlag, Berlin 1990.

[7] Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mischenko E.F., *The Mathematical Theory of Optimal Processes*, Moscow, 1983.

O REGULE HARTWICKA

Streszczenie

W artykule rozważane są dwie matematyczne wersje reguły Hartwicka. Jedna z nich związana jest z modelami optymalnego sterowania, gdzie zmienne cenowe są wykorzystywane. Druga zaś jest sformułowana bez tych zmiennych. Wykazano, że pierwsza wersja może być stosowana w ogólnych modelach sterowania.