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Michele Mininni, Giuseppe Orlando**, Giovanni Taglialatela****

A GENERALIZED DERIVATION OF THE BLACK-SCHOLES IMPLIED VOLATILITY THROUGH HYPERBOLIC TANGENTS

This article extends the previous research on the notion of a standardized call function and how to obtain an approximate model of the Black-Scholes formula via the hyperbolic tangent. Although the Black-Scholes approach is outdated and suffers from many limitations, it is still widely used to derive the implied volatility of options. This is particularly important for traders because it represents the risk of the underlying, and is the main factor in the option price. The approximation error of the suggested solution was estimated and the results compared with the most common methods available in the literature. A new formula was provided to correct some cases of underestimation of implied volatility. Graphic evidence, stress tests and Monte Carlo analysis confirm the quality of the results obtained. Finally, further literature is provided as to why implied volatility is used in decision making.

Keywords: implied volatility, approximation methods, hyperbolic tangent, the Black-Scholes model

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1. INTRODUCTION

In price options, a common approach is first and second order Taylor expansions. Indeed, given the option price, its delta and range, a financial trader can derive a quadratic approximation that is easily computable even for exotic options (which can be difficult to evaluate) and "more importantly, it is linear in the underlying price change and in its squared value and is thus readily aggregated across instruments, portfolios and business units of the firm" (Estrella, 1995).

* Department of Economics and Finance, Università degli Studi di Bari "Aldo Moro", Bari, Italy.
** Department of Economics and Finance, Università degli Studi di Bari "Aldo Moro", Bari, Italy. ORCID: 0000-0003-2630-5403.
*** Department of Economics and Finance, Università degli Studi di Bari "Aldo Moro", Bari, Italy. ORCID: 0000-0002-5381-8792.

In addition, for the purposes of hedging a portfolio, options are traded in order to obtain a neutral delta strategy. However, the measure has a problem in that it is computed on past performances, and may have little to do with the current level of risk. Within the Black and Scholes (1973) framework, extended by Merton (1973), it is possible to identify a relation between the value of an asset and the option written on it.

This is because traders have to evaluate the risk they run in their portfolio. However, the fact that this assessment is retrospective poses a problem, as it is calculated on past performance and may have little to do with the current level of risk. Within the framework of Black and Scholes (1973), expanded by Merton (1973), it is possible to identify a relationship between the value of an asset and the related option written on it. This has resulted in a connection between the option price and certain factors such as time, rates, dividends and, most importantly, actual volatility; see Hull (2006). Therefore, in financial markets, options are quoted directly in terms of volatility rather than price. Hence, given the importance of determining the implied volatility, the research was focused in two directions: "one, pragmatic, which aims to provide approximations that can be easily calculated in a spreadsheet, and the other, theoretical, which explores the mathematical properties of the implied volatility" (Orlando & Taglialatela, 2017).

Given the structure of the BS formula which cannot be inverted analytically, implied volatility can only be found through numerical approximation methods. Notwithstanding this, in some cases these methods may also fail for computational reasons (Orlando & Taglialatela, 2017).

In this paper the authors extended some considerations and analyses regarding a closed-form formula to approximate the BS formula by means of an appropriate parameterization of the hyperbolic tangent, as shown by Mininni et al. (2021). In particular, this paper presents further literature, with the purpose of calculating implied volatility and its use in decision making, and provides further graphical and numerical evidence. This allows to discover both the call value for any changes in the key variables and derive the implied volatility at once, for all combinations of strike, underlying, time, etc. To obtain this result, the so-called "standardized call function", i.e. a single parameter function representing the general family of calls, was employed.

The structure of the paper is as follows. The first section provides some basic information on both the topic and the literature, and sets out the notations. The second and third sections illustrate some closed-form approximations of the call function, both where the call is deep In-The-Money/Out of the Money, and in the case of At-The-Money. The fourth section deals with the derivation of the approximations of the implied volatility in the cases mentioned above. In the fifth section, some numerical simulations are illustrated and these results are compared with those provided by the literature. The sixth section illustrates the importance of implied volatility and stresses how its approximations are useful in the decision-making process, and the last section concludes.

2. BACKGROUND

2.1. Market volatility

Among those who have studied market volatility, there are Mo & Wu (2007) who on a sample from 3 January 1996 to 14 August 2002 reported that implied volatility for S&P 500 Index, the FTSE 100 Index and the Nikkei-225 Stock Average ranged from 15% to 35%. Similarly, Glasserman & Wu (2011), on a sample of currency options on EUR/USD, GBP/USD and USD/JPY, reported that implied volatility ranged from 5% to 43%.

Fassas & Siriopoulos (2021) document that "68 publicly available implied volatility indices worldwide and 25 indices proposed in an academic context" and report their descriptive statistics of realized and implied volatility levels. For example, on a sample from July 2011 to December 2019 of monthly observations, on average the stock indexes and the volatility indexes (Panel 1) had a mean, median, minimum and maximum of 18.80%, 17.22%, 43.98% and 9.75%, respectively.

Finally, the authors report the data from the VIX Index which is a financial benchmark estimating the expected volatility of the S&P 500 index, obtained by using the midpoint of the real-time S&P 500 index (SPX) option bid/ask quotes. In other terms, the "VIX index measures the level of expected volatility of the S&P 500 index over the next 30 days that is implied in the bid/ask quotations of SPX options. Thus, the VIX index is a forward-looking measure, in contrast to realized (or actual) volatility, which measures the variability of historical (or known) prices" (Fassas & Siriopoulos, 2021). Table 1 below shows the overall daily distribution of the VIX since its inception up to 2021 as displayed in Figure 1. In those years the median volatility was ~19.55%, mode ~14.98% and average was ~21.74% and big values were uncommon.

Table 1

The overall daily distributions of the VIX

Percentile	**I**		**II**		**III**		**IV**		**V**		**VI**		**VII**		**VIII**		**IX**		**X**	
Vol	11.85	14.50	-	15.38	-	16.40	-	17.87	-	19.59	-	21.18	-	23.51	-	26.41	-	31.72	-	72.98

Source: Chicago Board Options Exchange (CBOE) and authors' elaboration.

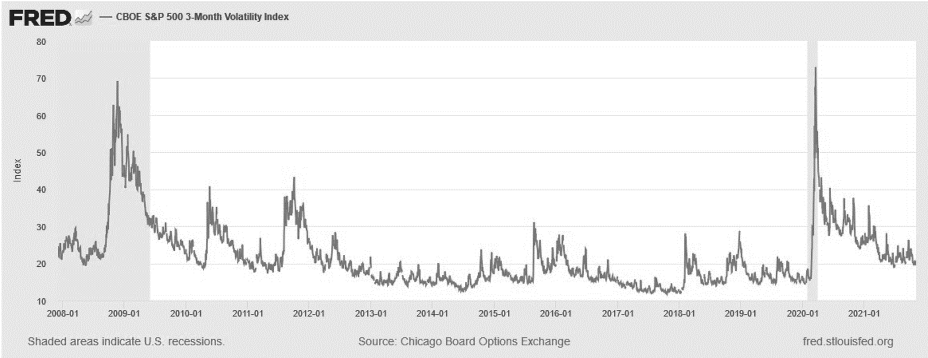


Fig. 1. Chicago Board Options Exchange, CBOE S&P 500 3-Month Volatility Index [VXVCLS]; the grey stripes indicate recessions.

Source: Chicago Board Options Exchange (CBOE) and Federal Reserve Economic Data (FRED), St. Louis Fed.

Table 2

Stock Options Volume Leaders (Top20)

Symbol	Price	Type	Strike	ExpDate	DTE	Bid	Ask	Last	Volume	OpenInt	Vol/OI	IV
AAPL	163.17	Put	150	04/14/22	40	3.15	3.3	3.25	59,136	29,364	2.01	39.61%
AAPL	163.17	Call	165	03/11/22	7	1.97	2.09	2.09	50,025	17,399	2.88	32.22%
AAPL	163.17	Put	165	03/04/22	0	1.13	2.01	1.71	49,677	26,785	1.85	22.67%
AAPL	163.17	Call	170	03/11/22	7	0.45	0.47	0.45	45,580	40,925	1.11	27.43%
AAPL	163.17	Call	162.5	03/04/22	0	0.5	1.22	1	45,440	7,611	5.97	18.03%
TSLA	838.29	Put	840	03/04/22	0	0.73	2.5	1.02	41,096	3,249	12.65	8.58%
SWN	5.38	Call	6	09/16/22	195	0.75	0.88	0.8	40,133	2,977	13.48	65.90%
ARCC	21.92	Put	20	04/14/22	40	0.4	0.45	0.43	37,979	2,038	18.64	39.53%
AAPL	163.17	Call	170	03/18/22	13	1.32	1.41	1.38	36,925	76,937	0.48	30.97%
TSLA	838.29	Call	840	03/04/22	0	0.11	0.48	0.13	34,847	1,523	22.88	3.80%
CEI	0.78	Call	1	03/11/22	7	0.1	0.11	0.11	34,347	7,260	4.73	416.92%
AAPL	163.17	Put	162.5	03/11/22	7	2.52	2.75	2.71	30,137	7,896	3.82	33.64%
AAPL	163.17	Call	167.5	03/11/22	7	0.98	1.04	1.01	29,622	19,969	1.48	28.81%
NVDA	229.36	Put	230	03/04/22	0	0.25	1.57	0.35	26,647	4,111	6.48	25.69%
AAL	14.59	Put	10	01/20/23	322	1.32	1.4	1.36	25,719	168,181	0.15	70.10%
AAPL	163.17	Put	160	03/11/22	7	1.84	1.91	1.8	25,245	6,866	3.68	35.95%
AAPL	163.17	Call	175	03/11/22	7	0.1	0.11	0.11	24,856	38,791	0.64	28.94%
TSLA	838.29	Put	835	03/04/22	0	0.03	0.2	0.19	24,518	1,856	13.21	6.72%
AAL	14.59	Put	12	04/14/22	40	0.54	0.55	0.54	23,759	334	71.13	85.45%
AMD	108.41	Put	110	03/04/22	0	0.75	1.83	1.47	23,178	6,375	3.64	35.09%

Note: American Airlines Group (AAL), Apple Inc (AAPL), Adv Micro Devices (AMD), Ares Capital Corp (ARCC), Camber Energy Inc (CEI), Nvidia Corp (NVDA), Southwestern Energy Company (SWN), Tesla Inc (TSLA).

Source: Barchart (2022) and authors' elaboration.

To complement the analysis at the level of single securities, Table 2 shows the highest option volume strikes of the most bought and sold options of the day (5 March 2022). In terms of days to expiration (DTE), the median was 7% (average 35%) and in terms of implied volatility (IV) the median was 32% (average 53%).

2.2. Literature review

To calculate both the call price and the implied volatility, numerical algorithms can be found in Dura & Moşneagu (2010), Orlando & Taglialatela (2017), Liu et al. (2019), and Kim et al. (2022). Still, as mentioned by Manaster & Koehler (1982), numerical approaches come at a cost and bring drawbacks that motivate, whenever possible, closed-form approximations; see e.g. Manaster & Koehler (1982). An example of the aforementioned problems is provided by Estrella (1995) who demonstrated that the Taylor series for the BS formula may diverge and "even when the series converges, finite approximations of very large order are generally necessary to achieve acceptable levels of accuracy". The alternative of using the exact formula while it provides more accurate results, presents some inconvenience in terms of calculations and reduced flexibility Estrella (1995). Therefore "simple solutions are desirous because they have two very attractive properties. They are easy to implement and provide very fast computational algorithms" Hofstetter & Selby (2001). Similarly, "despite recent advances in modelling option prices, traders and other practitioners have resisted using more complex theoretical formulas. The most widely used valuation procedure among practitioners is a variant of the Black-Scholes framework with ad hoc adjustments and frequent updating" Berkowitz (2001). For this reason, the BS model "can be cast as a functional approximation to the true but unknown option pricing formula. When fitted to a large number of observed prices, the implied volatility surface will exactly price the full cross-section. If the implied volatility surface is re-calibrated sufficiently frequently, then out of sample forecasts of option prices become arbitrarily accurate" Berkowitz (2001). For details on more advanced approaches such as stochastic local volatility (SLV) models, see Guerrero & Orlando (2022).

Concerning the delta, finite Taylor expansions of the first and second order of the BS formula are generally used for the approximations. In fact, from the price, the delta and gamma of the option, it is possible to obtain a quadratic approximation that is easily computable even in the case of exotic options whose derivation can be unwieldy. Even more importantly, the said approximation "is linear in the underlying price change and in its squared value and is thus readily aggregated across instruments, portfolios and business units of the firm" Estrella (1995).

Regarding the vega, between the approximations in closed form based on Taylor approximations or on the expansion of the power series of the cumulative normal distribution function (cndf), one can mention Brenner & Subrahmanyam (1988), Bharadia et al. (1995), Chance (1996), Corrado & Miller (1996a, 1996b), Liang & Tahara (2009), and Li (2005). For a review on the subject see Orlando & Taglialatela (2017). Closed-form approximations applicable only in the At-The-Money case were reported by the Pólya approximation (Polya (1949), Matic (2017) & Pianca (2005)). More recently, the topic of closed form solutions was investigated by Mininni et al. (2021) who adopted a single function to approximate the call function, and by Orlando & Taglialatela (2021) who, instead, split the approximating function into two parts: above and below the inflection point.

Among the hybrid methods one can mention Hofstetter & Selby (2001) who replaced the cndf with the logistic distribution, Li (2008) who used rational functions in combination with the Newton-Raphson algorithm, and Orlando & Taglialatela (2017) who took the inflection point of the call function as a starting point for the Newton-Raphson algorithm. Regarding the pros and cons of the two latter solutions, one may refer to Orlando & Taglialatela (2021). Furthermore, Georgiev & Vulkov (2021) proposed a robust algorithm determining time-dependent volatility, whereas Tae-Kyoung et al. (2022) constructed a network for estimating the Black–Scholes implied volatilities through large-scale online learning. Last but not least, in case of turbulent markets the reader may refer to the analytical approximation of the critical stock price as well as of the implied volatility by Bufalo & Orlando (2022).

3. A GENERALIZED APPROXIMATION OF THE BLACK-SCHOLES FORMULA

3.1. The Black-Scholes formula

The well-known BS formula for the price of a European call option is

$$C := SN(d_1) - XN(d_2),$$

where C is the price of a call option, S the value of the underlying, $N(x)$ the cumulative distribution function of the standard normal, i.e.

$$N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

r is the interest rate, T the time to maturity in terms of a year, σ the volatility, and K the strike price. $X = Ke^{-rT}$ is the present value of the strike price,

$d_1 := \frac{\log(S/X)}{\sigma\sqrt{T}} + \frac{\sigma}{2}\sqrt{T}$ the first parameter of probability, i.e. "the factor by which the present value of contingent receipt of the stock, contingent on exercise, exceeds the current value of the stock" Nielsen (1993), $d_2 := \frac{\log(S/X)}{\sigma\sqrt{T}} - \frac{\sigma}{2}\sqrt{T}$ the second parameter of probability which represents the risk-adjusted probability of exercise.

Note that, given the values S, X and T, price C of the call implied volatility is obtained by inverting function $C = C(\sigma)$.

The objective of the following sections is to obtain a suitable approximation $\tilde{C}$ of C so that the implied volatility is obtained from the inversion of $\tilde{C}$.

3.2. The standardized call function

Mininni et al. (2021) defined the *standardized call function*, which is a one-parameter family of functions that allows to describe the whole family of call functions in the case $S \neq X$.

Let us recall here the definition and the basic properties (the proof can be found in Mininni et al. (2021)).

For $\alpha > 0$ let

$$\chi_\alpha(x) := N\left(\frac{\alpha}{2}\left(x - \frac{1}{x}\right)\right) - e^{\alpha^2/2} N\left(\frac{-\alpha}{2}\left(x + \frac{1}{x}\right)\right), \qquad x > 0.$$

The relation between the call function $C = C(\sigma)$ and the function χ_α is given for fixed $S > 0$, $X > 0$ and $T > 0$, with $X \neq S$, by the formula

$$C(\sigma) = \begin{cases} S\chi_\alpha\left(\frac{\sigma\sqrt{T}}{\alpha}\right) & \text{if } X > S, \\ S - X + X\chi_\alpha\left(\frac{\sigma\sqrt{T}}{\alpha}\right) & \text{if } X < S, \end{cases} \tag{3.1}$$

where

$$\alpha := \sqrt{2\left|\log(S/X)\right|}.$$

As

$$\tilde{\chi}_\alpha{}'(x) = \frac{\alpha}{\sqrt{2\pi}} \exp\left[\frac{-\alpha^2}{8}\left(x - \frac{1}{x}\right)^2\right]$$

and $\alpha > 0$, one can see that the function χ_α is strictly increasing in]0, $+\infty$ [with

$$\lim_{x \downarrow 0} \tilde{\chi}_\alpha(x) = 0$$

$$\lim_{x \uparrow +\infty} \tilde{\chi}_\alpha(x) = 1.$$

Moreover, as

$$\tilde{\chi}_\alpha''(x) = \frac{\alpha^3}{4\sqrt{2\pi}} \exp\left[\frac{-\alpha^2}{8} \left(x - \frac{1}{x} \right)^2 \right] \frac{1-x^4}{x^3}$$

one can see that χ_α has a sigmoidal shape: it is strictly convex in]0,1] and strictly concave in [1, $+\infty$ [, with a single inflection point at $x = 1$.

For further reference, recall also that the equation of the tangent line to χ_α at $x = 1$ is

$$y = \chi_\alpha'(1)(x-1) + \chi_\alpha(1) = \frac{\alpha}{\sqrt{2\pi}}(x-1) + \frac{1}{2} - e^{\alpha^2/2} N(-\alpha). \qquad (3.2)$$

Next, we construct an approximation of the function χ_α, which gives, thanks to Eq. (3.1), an approximation of the call functions $C(\sigma)$. To simplify the notation, set $\alpha > 0$ so that one can drop index α and denote with χ and $\tilde{\chi}$ the functions χ_α and $\tilde{\chi}_\alpha$.

3.3. The hyperbolic tangent model

Since χ has a sigmoidal form, let us consider the hyperbolic tangent for its approximation

$$\tanh(x) := \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad (3.3)$$

which has a similar shape and moreover has an inverse function expressed in terms of elementary functions:

$$\operatorname{arctanh}(x) = \frac{1}{2} \log\left(\frac{1+x}{1-x} \right). \qquad (3.4)$$

A possible candidate for approximating call function χ may be the following nonlinear regression model

$$\tilde{\chi}(x) := \frac{1}{2} + \frac{1}{2}\tanh\left(\Phi(x) + c_0\right) \tag{3.5}$$

where $\Phi :]\,0,+\infty\,[\to \mathbf{R}$ is a strictly increasing function with $\Phi(1) = 0$, and moreover

$$\lim_{x\downarrow 0}\Phi(x) = -\infty \quad \text{and} \quad \lim_{x\uparrow +\infty}\Phi(x) = +\infty,$$

so that

$$\lim_{x\downarrow 0}\tilde{\chi}(x) = 0 \quad \text{and} \quad \lim_{x\uparrow +\infty}\tilde{\chi}(x) = 1.$$

3.4. Constraints on the approximating function

It is required that $\tilde{\chi}$ has an inflection point coinciding with the inflection point of χ, and that in such a point the tangents are equal. More precisely, seeking for conditions on

$$c_0, \quad \varphi_1 := \Phi'(1), \quad \varphi_2 := \Phi''(1), \tag{3.6}$$

so that (cf. (3.2))

$$\tilde{\chi}(1) = \chi(1) = \frac{1}{2} - e^{\alpha^2/2}N(-\alpha), \tag{3.7}$$

$$\tilde{\chi}'(1) = \chi'(1) = \frac{\alpha}{\sqrt{2\pi}}, \tag{3.8}$$

$$\tilde{\chi}''(1) = \chi''(1) = 0. \tag{3.9}$$

Comparing (3.5) and (3.7) one obtains

$$\tanh(c_0) = 2\chi(1) - 1, \tag{3.10}$$

which gives

$$c_0 := \operatorname{arctanh}\left(2\chi(1) - 1\right) = \frac{1}{2}\log\left(\frac{\chi(1)}{1-\chi(1)}\right). \tag{3.11}$$

As $\tanh'(x) = 1 - \tanh^2(x)$, we have

$$\tilde{\chi}'(x) = \frac{1}{2}\left[1 - \tanh^2\left(\Phi(x) + c_0\right)\right]\Phi'(x),$$

then, using (3.10), condition (3.8) becomes

$$\frac{1}{2}\left[1-\left(2\chi(1)-1\right)^2\right]\varphi_1=\tilde{\chi}'(1)$$

hence

$$\varphi_1 := \frac{2\tilde{\chi}'(1)}{1-\left(2\chi(1)-1\right)^2}. \tag{3.12}$$

Finally, as

$$\tilde{\chi}''(x)=\frac{1}{2}\left[1-\tanh^2\left(\Phi(x)+c_0\right)\right]\left[\Phi''(x)-2\tanh\left(\Phi(x)+c_0\right)\left[\Phi'(x)\right]^2\right],$$

one can see that (3.9) holds true if

$$\varphi_2 = 2\left(2\chi(1)-1\right)\varphi_1^2,$$

that is

$$\varphi_2 := \frac{8\left(2\chi(1)-1\right)\left[\chi'(1)\right]^2}{\left[1-\left(2\chi(1)-1\right)^2\right]^2}. \tag{3.13}$$

3.5. Alternative specifications of the approximating formula

Function $\Phi(x)$ can be defined in many ways, in the following the authors show previous approximations and a general formula.

3.5.1. Previous approximations

Mininni et al. (2021) considered

$$\Phi_A(x) := c_1(x-1)-c_2\left(\frac{1}{x}-1\right). \tag{3.14}$$

As

$$\Phi_A'(x)=c_1+\frac{c_2}{x^2} \quad \text{and} \quad \Phi_A''(x)=\frac{-2c_2}{x^2},$$

one finds

$$c_1=\varphi_1+\frac{1}{2}\varphi_2, \quad \text{and} \quad c_2=\frac{-1}{2}\varphi_2, \tag{3.15}$$

hence one can rewrite (3.14) as

$$\Phi_A(x) := \left(\varphi_1 + \frac{1}{2}\varphi_2\right)(x-1) + \frac{1}{2}\varphi_2\left(\frac{1}{x} - 1\right). \tag{3.16}$$

The constant φ_2 is clearly negative (cf. (3.7) and (3.13)), whereas $\varphi_1 + \frac{1}{2}\varphi_2 > 0$ is a consequence of the Komatsu (1995) – Pollak (1956) estimate (see Mininni et al. (2021) for details).

3.5.2. A general approximating formula

In this work the authors also consider the following new specification

$$\Phi_B(x) := f_1\left(x^2 - 1\right) - f_2\left(\frac{1}{x^2} - 1\right). \tag{3.17}$$

As

$$\Phi_B{}'(x) = 2f_1 x + \frac{2f_2}{x^3}$$

$$\Phi_B{}''(x) = 2f_1 - \frac{6f_2}{x^4}$$

in order to satisfy (3.6), one needs to solve the system

$$\begin{cases} 2f_1 + 2f_2 = \varphi_1 \\ 2f_1 - 6f_2 = \varphi_2 \end{cases}$$

which gives

$$\begin{cases} f_1 := \dfrac{3\varphi_1 + \varphi_2}{8}, \\ f_2 := \dfrac{\varphi_1 - \varphi_2}{8}. \end{cases} \tag{3.18}$$

Since $\varphi_1 > 0$ and $\varphi_2 < 0$, one immediately obtains that $f_2 > 0$. Moreover

$$f_1 = \frac{c_1}{4} + \frac{\varphi_1}{8},$$

where c_1 is given in (3.15); since $c_1 > 0$, one also finds that $f_1 > 0$. Thus, we can rewrite (3.17) as

$$\Phi_B(x) := \frac{3\varphi_1 + \varphi_2}{8}\left(x^2 - 1\right) - \frac{\varphi_1 - \varphi_2}{8}\left(\frac{1}{x^2} - 1\right). \tag{3.19}$$

3.6. The piecewise hyperbolic tangent model

Instead of Eq. (3.5), one can consider a slightly more general model. As the function χ has a single inflection at $x = 1$, the interval $]0, +\infty[$ can be divided into two intervals: $]0,1]$ and $[1, +\infty[$, using two different approximating functions for each interval

$$\tilde{\chi}(x) = \begin{cases} \tilde{\chi}^+(x) := a^+ + b^+ \tanh\left(\Phi^+(x)\right), & \text{if } x \in]1, +\infty[, \\ \tilde{\chi}^-(x) := a^- + b^- \tanh\left(\Phi^-(x)\right), & \text{if } x \in]0,1]. \end{cases} \tag{3.20}$$

As before, it is required that $\tilde{\chi}$ belongs to $C^2(]0, +\infty[)$ and has the same sigmoidal shape as χ : Φ^- is an increasing and convex function in $]0,1]$, whereas Φ^+ is an increasing and concave function in $[1, +\infty[$. In order to obtain

$$\lim_{x \uparrow +\infty} \tilde{\chi}(x) = 1.$$

assume

$$\lim_{x \uparrow +\infty} \Phi^+(x) = +\infty$$

and

$$a^+ + b^+ = 1; \tag{3.21}$$

whereas, in order to obtain

$$\lim_{x \downarrow 0} \tilde{\chi}(x) = 0,$$

assume

$$\lim_{x \downarrow 0} \Phi^-(x) = -\infty$$

and

$$a^- - b^- = 0. \tag{3.22}$$

It is required as before that $\tilde{\chi}$ has an inflection point coinciding with the inflection point of χ, and that in such point the tangents lines are equal, thus assume

$$a^{+} + b^{+}\tanh\left(\Phi^{+}(1)\right) = a^{-} + b^{-}\tanh\left(\Phi^{-}(1)\right) = \chi(1), \tag{3.23}$$

$$b^{+}\frac{d}{dx}\tanh\left(\Phi^{+}(1)\right) = b^{-}\frac{d}{dx}\tanh\left(\Phi^{-}(1)\right) = \chi'(1), \tag{3.24}$$

$$b^{+}\frac{d^{2}}{dx^{2}}\tanh\left(\Phi^{+}(1)\right) = b^{-}\frac{d^{2}}{dx^{2}}\tanh\left(\Phi^{-}(1)\right) = \chi''(1) = 0. \tag{3.25}$$

Note that the piecewise tanh-model (3.20) contains ten parameters to be chosen:

$$a^{+},\quad b^{+},\quad c_0^{+},\quad \varphi_1^{+} := \Phi^{+}{}'(1),\quad \varphi_2^{+} := \Phi^{+}{}''(1),$$

$$a^{-},\quad b^{-},\quad c_0^{-},\quad \varphi_1^{-} := \Phi^{-}{}'(1),\quad \varphi_2^{-} := \Phi^{-}{}''(1),$$

whereas conditions (3.21)–(3.25) form a system of eight equations, thus one has 2 degrees of freedom in the choice of the parameters and many choices are possible.

3.7. Derivation of a new piecewise approximation

We repeat the computations in Section 2.4 for the following function

$$\tilde{\chi}(x) := a + b\tanh\left(\Phi(x) + c_0\right), \tag{3.26}$$

with $a, b \in \mathbf{R}$. Comparing (3.26) and (3.7), one obtains

$$\tanh(c_0) = \frac{\chi(1) - a}{b},$$

which gives:

$$c_0 = \operatorname{arctanh}\left(\frac{\chi(1) - a}{b}\right),$$

hence, according to Eq. (3.4)

$$c_0 = \frac{1}{2}\log\left|\frac{1 + \dfrac{\chi(1) - a}{b}}{1 - \dfrac{\chi(1) - a}{b}}\right| = \frac{1}{2}\log\left[\frac{b - a + \chi(1)}{a + b - \chi(1)}\right]. \tag{3.27}$$

As $\tanh'(x) = 1 - \tanh^{2}(x)$, one obtains

$$\tilde{\chi}'(x) = b\left[1 - \tanh^{2}\left(\Phi(x) + c_0\right)\right]\Phi'(x),$$

then (3.8) becomes

$$b\left[1-\left(\frac{\chi(1)-a}{b}\right)^2\right]\varphi_1=\chi'(1),$$

hence

$$\varphi_1=\frac{1}{b}\frac{\chi'(1)}{1-\left(\frac{\chi(1)-a}{b}\right)^2}=\frac{b\chi'(1)}{b^2-(\chi(1)-a)^2}. \tag{3.28}$$

Finally, as

$$\tilde{\chi}''(x)=b\left[1-\tanh^2(\Phi(x)+c_0)\right]\left[\Phi''(x)-2\tanh(\Phi(x)+c_0)\left[\Phi'(x)\right]^2\right],$$

one can see that (3.9) holds true if

$$\varphi_2=2\tanh(c_0)\varphi_1^2$$

that is

$$\varphi_2=\frac{2b(\chi(1)-a)\left[\chi'(1)\right]^2}{\left[b^2-(\chi(1)-a)^2\right]^2}. \tag{3.29}$$

If $a=b=\frac{1}{2}$, (3.27), (3.28) and (3.29) reduce respectively to (3.11), (3.12) and (3.13).

3.8. A simplified approximating formula

In order to have simpler and easy to use formulas, using Eq. (3.3), one can transform Eq. (3.26) as follows. As $b^- = a^-$

$$\tilde{\chi}^-(x)=a^-+a^-\frac{e^{(\Phi(x)+c_0)}-e^{-(\Phi(x)+c_0)}}{e^{(\Phi(x)+c_0)}+e^{-(\Phi(x)+c_0)}}=\frac{2a^-e^{(\Phi(x)+c_0)}}{e^{(\Phi(x)+c_0)}+e^{-(\Phi(x)+c_0)}}=\frac{2a^-}{1+e^{-2c_0}e^{-2\Phi(x)}}.$$

From (3.27) one obtains

$$e^{-2c_0}=\frac{2a^- -\chi(1)}{\chi(1)}$$

hence

$$\tilde{\chi}^{-}(x)=\frac{2a^{-}\chi(1)}{\chi(1)+\left[2a^{-}-\chi(1)\right]\exp\left[-2\Phi(x)\right]}. \tag{3.30}$$

Analogously, as $a^{+}+b^{+}=1$, one obtains

$$\tilde{\chi}^{+}(x)=a^{+}+\left(1-a^{+}\right)\frac{e^{(\Phi(x)+c_0)}-e^{-(\Phi(x)+c_0)}}{e^{(\Phi(x)+c_0)}+e^{-(\Phi(x)+c_0)}}$$

$$=1-\left(1-a^{+}\right)\left[1-\frac{e^{(\Phi(x)+c_0)}-e^{-(\Phi(x)+c_0)}}{e^{(\Phi(x)+c_0)}+e^{-(\Phi(x)+c_0)}}\right]$$

$$=1-\frac{2\left(1-a^{+}\right)e^{-(\Phi(x)+c_0)}}{e^{(\Phi(x)+c_0)}+e^{-(\Phi(x)+c_0)}}=1-\frac{2\left(1-a^{+}\right)}{e^{2c_0}e^{2\Phi(x)}+1},$$

From (3.27) one obtains

$$e^{2c_0}=\frac{1-2a^{+}+\chi(1)}{1-\chi(1)},$$

hence

$$\tilde{\chi}^{+}(x)=1-\frac{2\left(1-a^{+}\right)\left[1-\chi(1)\right]}{\left(1-2a^{+}+\chi(1)\right)\exp\left[2\Phi(x)\right]+1-\chi(1)}. \tag{3.31}$$

If $a=b=\frac{1}{2}$, (3.30) and (3.31) reduce to

$$\tilde{\chi}^{-}(x)=\frac{\chi(1)}{\chi(1)+\left(1-\chi(1)\right)\exp\left[-2\Phi(x)\right]}$$

$$\tilde{\chi}^{+}(x)=1-\frac{1-\chi(1)}{\chi(1)\exp\left[2\Phi(x)\right]+1-\chi(1)}.$$

3.9. Benchmark formula

In Orlando & Taglialatela (2021), aiming for simplicity, the authors chose

$$\Phi^{+}(x)=c^{+}(x-1) \text{ and } c_0^{+}=0 \text{ for } x\geq 1,$$

$$\Phi^{-}(x)=c^{-}\left(\frac{1}{x}-1\right) \text{ for } x\in\,]0,1[\,.$$

From (3.21) and (3.27) derive

$$a^{+}=\chi(1) \text{ and } b^{+}=1-\chi(1),$$

hence, from (3.28) and (3.29) one obtains

$$\varphi_1=\frac{\chi'(1)}{1-\chi(1)} \text{ and } \varphi_2=0\,.$$

Since

$$\Phi^{-}{}'(x)=-\frac{c^{-}}{x^2} \text{ and } \Phi^{-}{}''(x)=\frac{2c^{-}}{x^3}$$

one obtains

$$\varphi_1^{-}=-c^{-} \text{ and } \varphi_2^{-}=2c^{-}\,,$$

hence

$$\varphi_2^{-}=-2\varphi_1^{-}. \tag{3.32}$$

Combining (3.32) with (3.28), (3.29) and (3.22), one arrives at

$$\frac{2a^{-}\left(\chi(1)-a^{-}\right)\left[\chi'(1)\right]^2}{\left[a^{-2}-\left(\chi(1)-a^{-}\right)^2\right]^2}=-2\frac{a^{-}\chi'(1)}{a^{-2}-\left(\chi(1)-a^{-}\right)^2}$$

which gives, after some calculations

$$a^{-}=\chi(1)\frac{\chi'(1)-\chi(1)}{\chi'(1)-2\chi(1)}.$$

Thus, one obtains

$$\varphi_1^{-}=\frac{\chi'(1)-\chi(1)}{\chi(1)}.$$

Finally, using (3.30) for $x\in\,]0,1[$ and (3.31) for $x\in\,]1,+\infty\,[$, one obtains

$$\tilde{\chi}(x)=\begin{cases}\dfrac{2\chi(1)\left(\chi'(1)-\chi(1)\right)}{\chi'(1)\exp\left[\dfrac{2\left(\chi'(1)-\chi(1)\right)}{\chi(1)}\left(\dfrac{1}{x}-1\right)\right]+\chi'(1)-2\chi(1)} & \text{if } x\in\,]0,1[,\\[2ex] 1-\dfrac{2\left(1-\chi(1)\right)}{\exp\left(\dfrac{2\chi'(1)}{1-\chi(1)}(x-1)\right)+1} & \text{if } x\in[1,+\infty[.\end{cases}$$

3.10. Approximating the call function when $S = X$

In the special case $S = X$, $d_1(\sigma)=\frac{\sigma}{2}\sqrt{T}$ and $d_2(\sigma)=\frac{-\sigma}{2}\sqrt{T}=-d_1$, hence one can write $C(\sigma)$ as

$$C(\sigma)=S\,\mathrm{erf}\left(\sigma\sqrt{\frac{T}{8}}\right),$$

where erf is the *error function* defined by

$$\mathrm{erf}(z):=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt.$$

Several numerical approximations of erf(x) are known. In particular, Ingber (1982) considered the function

$$\Theta_0(z):=\tanh\left(\frac{2}{\sqrt{\pi}}z\right)$$

which has the same limit at infinity and the same derivative in 0.

A more precise approximation of function erf(z), which in any case is still constructed starting from the hyperbolic tangent (3.3), can be obtained by considering functions of the form

$$\Theta(z):=\tanh\left(az+bz^3\right).$$

with suitable a, b . In particular, one can consider

$$\Theta_1(z):=\tanh\left(\frac{2}{\sqrt{\pi}}z+\frac{8-2\pi}{3\sqrt{\pi^3}}z^3\right),$$

which has the property of possessing the same Taylor expansion of order 3 in 0 of the error function. Alternatively, Fairclough (2000) proposed

$$\Theta_2(z) := \tanh\left(1.129324z + 0.100303z^3\right).$$

Section 4.2 shows that $\Theta_1(z)$ and $\Theta_2(z)$ provide approximations of the same order of accuracy. For this reason, $C(\sigma)$ can be approximated by

$$\tilde{C}_0(\sigma) = S\tanh\left(\sigma\sqrt{\frac{T}{2\pi}}\right) \tag{3.33}$$

or

$$\tilde{C}_1(\sigma) = S\tanh\left(\left(\sigma\sqrt{\frac{T}{2\pi}}\right) + \frac{4-\pi}{12}\left(\sigma\sqrt{\frac{T}{2\pi}}\right)^3\right), \tag{3.34}$$

or

$$\tilde{C}_2(\sigma) = S\tanh\left(1.129324\left(\sigma\sqrt{\frac{T}{8}}\right) + 0.100303\left(\sigma\sqrt{\frac{T}{8}}\right)^3\right). \tag{3.35}$$

4. APPROXIMATION OF THE IMPLIED VOLATILITY

To obtain the implied volatility, it is necessary to solve the equation

$$C(\sigma) = C. \tag{4.1}$$

If $\tilde{C}$ is an approximation of C, an approximate solution to (4.1) is found by solving the equation

$$\tilde{C}(\sigma) = C. \tag{4.2}$$

4.1. Inverting the hyperbolic tangent model in the case $S \neq X$

According to (3.1), equation (4.2) is equivalent to

$$\tilde{\chi}_\alpha\left(\frac{\sigma\sqrt{T}}{\alpha}\right) = C^*, \tag{4.3}$$

where

$$C^* := \begin{cases} C/S & \text{if } X > S, \\ (C - S + X)/X & \text{if } X < S. \end{cases}$$

Note that one can write

$$C^* := \frac{C - [S - X]^+}{S - [S - X]^+},$$

with $[S - X]^+ = \max(S - X, 0)$ denoting the pay-off. According to the tanh-model (3.5), Eq. (4.3) is equivalent to

$$\Phi\left(\frac{\sigma\sqrt{T}}{\alpha}\right) = \Lambda, \tag{4.4}$$

where

$$\Lambda = \frac{1}{2}\log\left(\frac{C^*}{1 - C^*}\right) - c_0.$$

Note that, using (3.11), one obtains

$$\Lambda = \frac{1}{2}\log\left(\frac{\left[1 - \chi(1)\right]C^*}{\chi(1)\left(1 - C^*\right)}\right). \tag{4.5}$$

If $\Phi(x) = \Phi_A(x)$ (cf. (3.14)), in order to invert (4.4), one has to solve the following equation with respect to x

$$c_1(x - 1) - c_2\left(\frac{1}{x} - 1\right) = \Lambda, \tag{4.6}$$

Hence Eq. (4.6) is equivalent to

$$c_1 x^2 - (\Lambda + c_1 - c_2)x - c_2 = 0,$$

which has a single positive solution of the form

$$x = \frac{\Lambda + c_1 - c_2 + \sqrt{(\Lambda + c_1 - c_2)^2 + 4c_1c_2}}{2c_1},$$

and one obtains an approximation of the implied volatility by

$$\sigma^A = \alpha \frac{\Lambda + c_1 - c_2 + \sqrt{(\Lambda + c_1 - c_2)^2 + 4c_1c_2}}{2\sqrt{T}c_1}. \tag{4.7}$$

Analogously, if $\Phi(x) = \Phi_B(x)$ (cf. (3.17)), in order to invert (4.4), one has to solve the following equation with respect to x

$$f_1(x^2 - 1) - f_2\left(\frac{1}{x^2} - 1\right) = \Lambda, \tag{4.8}$$

where Λ is defined in (4.5), and f_1 and f_2 are defined in (3.18). Thus Eq. (4.8) is equivalent to

$$f_1x^4 - (\Lambda + f_1 - f_2)x^2 - f_2 = 0, \tag{4.9}$$

which has a single positive solution of the form

$$x = \sqrt{\frac{\Lambda + f_1 - f_2 + \sqrt{(\Lambda + f_1 - f_2)^2 + 4f_1f_2}}{2f_1}},$$

hence one obtains an approximation of the implied volatility by

$$\sigma^B = \alpha\sqrt{\frac{\Lambda + f_1 - f_2 + \sqrt{(\Lambda + f_1 - f_2)^2 + 4f_1f_2}}{2Tf_1}}. \tag{4.10}$$

This formula, as seen in Section 4.3, provides a correction when the formula underestimates the implied volatility.

4.2. Case $S = X$

One can obtain an approximation of σ by solving equations $\tilde{C}_0(\sigma) = C$, $\tilde{C}_1(\sigma) = C$ or $\tilde{C}_2(\sigma) = C$, where $\tilde{C}_0$, $\tilde{C}_1$ and $\tilde{C}_2$ are defined by (3.33), (3.34) and (3.35). Equation $\tilde{C}_0(\sigma) = C$ has the solution

$$\tilde{\sigma}_0 = \sqrt{\frac{\pi}{2T}}\log\left(\frac{S + C}{S - C}\right), \tag{4.11}$$

which provides a first approximation of σ. Better approximations of σ are obtained by solving equations $\tilde{C}_1(\sigma) = C$ and $\tilde{C}_2(\sigma) = C$, that is, by solving the equations

$$\left(\sigma\sqrt{\frac{T}{2\pi}}\right)+\frac{4-\pi}{12}\left(\sigma\sqrt{\frac{T}{2\pi}}\right)^3=\frac{1}{2}\log\left(\frac{S+C}{S-C}\right) \tag{4.12}$$

and

$$a\left(\sigma\sqrt{\frac{T}{8}}\right)+b\left(\sigma\sqrt{\frac{T}{8}}\right)^3=\frac{1}{2}\log\left(\frac{S+C}{S-C}\right), \tag{4.13}$$

respectively. Recall that if $p>0$, equation

$$x^3+3px=2q$$

has a single *real* solution of the form

$$x=\sqrt[3]{\sqrt{p^3+q^2}+q}-\sqrt[3]{\sqrt{p^3+q^2}-q}.$$

Hence Eq. (4.12) has a single real solution of the form

$$\tilde{\sigma}_1:=\sqrt{\frac{2\pi}{T}}\left[\sqrt[3]{\sqrt{p^3+q^2}+q}-\sqrt[3]{\sqrt{p^3+q^2}-q}\right], \tag{4.14}$$

with

$$p:=\frac{4}{4-\pi}\quad\text{and}\quad q:=\frac{3}{4-\pi}\log\left(\frac{S+C}{S-C}\right),$$

whereas the unique real solution of Eq. (4.13) is given by

$$\tilde{\sigma}_2:=\sqrt{\frac{8}{T}}\left[\sqrt[3]{\sqrt{p^3+q^2}+q}-\sqrt[3]{\sqrt{p^3+q^2}-q}\right], \tag{4.15}$$

where

$$p:=3.753041\quad\text{and}\quad q:=2.492448\,\log\left(\frac{S+C}{S-C}\right).$$

Therefore, it can be concluded that $\tilde{\sigma}_0$, $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ given by Eq. (4.11), (4.14) and (4.15) provide approximations of the implied volatility.

5. NUMERICAL FINDINGS

5.1. Case when $S \neq X$

Figure 2 shows that approximation function $\hat{\chi}_\alpha(x)$ is very close to function $\chi_\alpha(x)$, with the exception of the far right part of the inflection where the bending is greatest and the volatility is unrealistic.

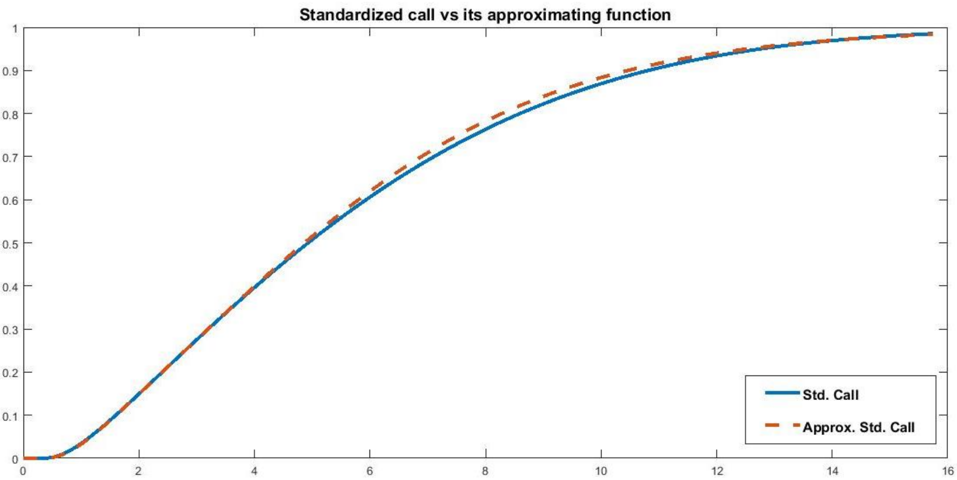


Fig. 2. Plot of $\chi_\alpha(x)$ and its corresponding approximation $\hat{\chi}_\alpha(x)$ for $\alpha = 0.3124$ i.e. $S/X = 1.05$.

Source: authors' elaboration.

To better see the differences and their magnitude, Tables 3 and 4 present a comparison between $\chi_\alpha(x)$ and its approximation $\hat{\chi}_\alpha(x)$ in (3.5), with $\Phi = \Phi_A$ defined in (3.14) and T set to 0.25 being one of the most popular maturities.

Table 3

$\chi_\alpha(x)$ versus $\hat{\chi}_\alpha(x)$, $x = \sigma\sqrt{T} / \alpha$, case $S < X$

***T*=0.25**	***S/X* = 0.75, *α* = 0.7585**				***S/X* = 0.95, *α* = 0.3203**			
σ	x	$\chi_\alpha(x)$	$\chi r_\alpha(x)$	Error	x	$\chi_\alpha(x)$	$\chi r_\alpha(x)$	Error
0.04	0.026366871	5.17768E-50	0	5.17768E-50	0.062443135	0.000033596	0.000001375	0.000032221
0.20	0.131834356	0.000067657	0.000115501	-0.000047844	0.312215677	0.019874350	0.015549652	0.004324698
0.36	0.237301841	0.004836445	0.004452570	0.000383875	0.561988218	0.050243597	0.048846840	0.001396757
0.52	0.342769326	0.020226530	0.018889540	0.001336990	0.811760759	0.081886706	0.081781602	0.000105103
0.68	0.448236811	0.043100732	0.041630853	0.001469879	1.061533301	0.113784506	0.113787822	-0.000003316
0.84	0.553704296	0.070202095	0.069165163	0.001036932	1.311305842	0.145646264	0.146030979	-0.000384715
1.00	0.659171781	0.099676155	0.099145585	0.000530569	1.561078383	0.177334827	0.179354876	-0.002020048
MSE				0.000000779				0.000003556
Std. Dev				0.000571311				0.001820267

Source: authors' elaboration.

Table 4

$\chi_\alpha(x)$ versus $\hat{\chi}_\alpha(x)$, $x = \sigma\sqrt{T} / \alpha$, case $S > X$

***T*=0.25**	***S/X* = 1.05, *α* = 0.3124**				***S/X* = 1.25, *α* = 0.6680**			
σ	x	$\chi_\alpha(x)$	$\chi r_\alpha(x)$	Error	x	$\chi_\alpha(x)$	$\chi r_\alpha(x)$	Error
0.04	0.064024893	0.000049460	0.000001949	0.000047511	0.029938003	6.51663E-32	1.44329E-15	-1.44329E-15
0.20	0.320124463	0.020640191	0.016384349	0.004255843	0.149690015	0.000498929	0.000433098	0.000065831
0.36	0.576224033	0.051181578	0.049930179	0.001251400	0.269442027	0.010373018	0.008784508	0.001588510
0.52	0.832323603	0.082869729	0.082796995	0.000072734	0.389194039	0.031336390	0.029102580	0.002233810
0.68	1.088423173	0.114772802	0.114782390	-0.000009588	0.508946051	0.057981031	0.056351416	0.001629616
0.84	1.344522743	0.146622638	0.147129355	-0.000506718	0.628698063	0.087421802	0.086610435	0.000811367
1.00	1.600622313	0.178290453	0.180694418	-0.002403965	0.748450075	0.118287072	0.118021973	0.000265100
MSE				0.000003675				0.000001557
Std. Dev.				0.001877485				0.000818523

Source: authors' elaboration.

5.2. Case when $S = X$

Figure 3 shows the error function versus its approximations Θ_0, Θ_1 and Θ_2 as defined in Section 2.10 and Table 5 compares them for some significant values.

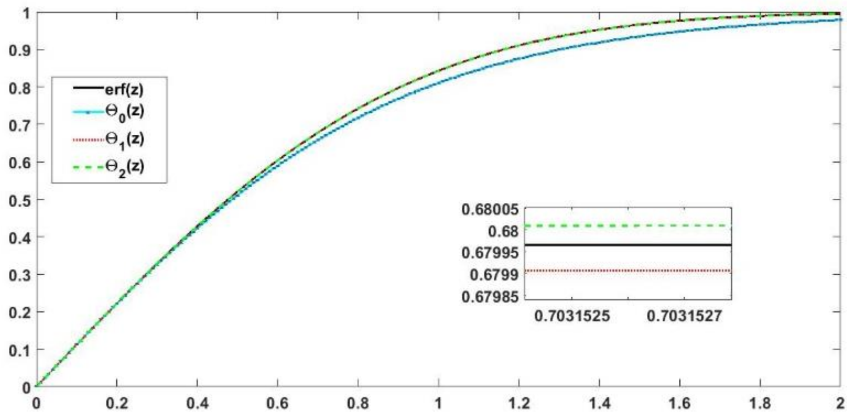


Fig. 3. A comparison between the error function and different functional forms of the hyperbolic tangent

Note that erf(z) is indistinguishable from both $\Theta_1(z)$ and $\Theta_2(z)$, therefore the picture was zoomed to show the differences.

Source: authors' elaboration.

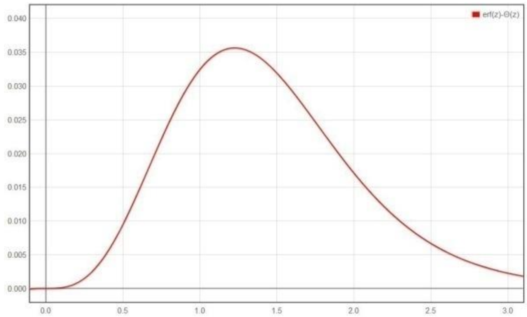
Table 5

Comparison between the error function and s functions Θ_0, Θ_1 and Θ_2 for some values of z

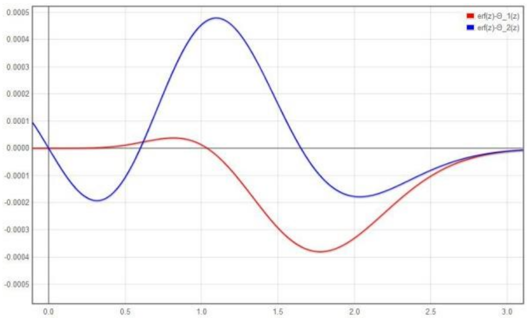
z	erf(z)	$\Theta_0(z)$	$\Theta_1(z)$	$\Theta_2(z)$
0	0.0	0.0	0.0	0.0
0.01	0.0112834	0.0112833	0.0112834	0.0112929
0.10	0.1124629	0.1123614	0.1124629	0.1125538
0.25	0.2763264	0.2748427	0.2763266	0.2765091
0.50	0.5204999	0.5110793	0.5205079	0.5206273
0.60	0.6038561	0.5895836	0.6038765	0.6038978
0.62	0.6194115	0.6041107	0.6194355	0.6194338
0.64	0.6345858	0.6182471	0.6346140	0.6345885
0.66	0.6493767	0.6319946	0.6494096	0.6493596
0.68	0.6637822	0.6453554	0.6638203	0.6637453
0.70	0.6778012	0.6583328	0.6778450	0.6777447
0.80	0.7421010	0.7176116	0.7421834	0.7419549
0.90	0.7969082	0.7680442	0.7970469	0.7967001
1.00	0.8427008	0.8104638	0.8429131	0.8424711
1.02	0.8508380	0.8180675	0.8510668	0.8506094
1.04	0.8586499	0.8253955	0.8588958	0.8584241
1.10	0.8802051	0.8458024	0.8805043	0.8799982
1.50	0.9661051	0.9344736	0.9667097	0.9662538
2.00	0.9953223	0.9783179	0.9957755	0.9956221
2.50	0.9995930	0.9929328	0.9997143	0.9996928
∞	1.0	1.0	1.0	1.0

Source: authors' elaboration.

Figures 4(a) and 4(b) display the differences between erf(z) and Θ_0, Θ_1 and $\Theta_{2,}$while the numerical values over 10,000 simulations are reported in Tables 6, 7, and 8. display the differences between erf(z) and Θ_0, Θ_1 and $\Theta_{2,}$while the numerical values over 10,000 simulations are reported in Tables 6, 7, and 8.



(a) erf(z)−Θ_0(z).



(b) erf(z)−Θ_1(z) (red) and erf(z)−Θ_2(z) (blue).

Fig. 4. Differences between erf(z) and Θ_0, Θ_1 and Θ_2.

Source: authors' elaboration.

Table 6

Comparison between the erf(z) function and Θ_0

Interval of z	Sample	Mean	Median	Maximum	Minimum	Standard deviation
[0.0,2.5]	1:10000	0.00038089	0.00019677	0.00148364	–	0.00042519
[0.0,0.5]	1:2000	0.00000321	0.00000160	0.00001281	–	0.00000363
[0.5,1.0]	2001:4000	0.00004775	0.00004305	0.00010147	0.00001281	0.00002561
[1.0,1.5]	4001:6000	0.00020429	0.00019677	0.00033707	0.00010147	0.00006808
[1.5,2.0]	6001:8000	0.00053962	0.00052977	0.00078148	0.00033707	0.00012842
[2.0,2.5]	8001:10000	0.00110945	0.00109787	0.00148364	0.00078148	0.00020290

Source: authors' elaboration.

Table 7

Comparison between the erf(z) function and Θ_1

Interval of z	Sample	Mean	Median	Maximum	Minimum	Standard deviation
[0.0,2.5]	1:10000	-0.00000004	-0.00000001	1.0842E-19	-0.00000022	0.00000006
[0.0,0.5]	1:2000	-1.0938E-11	-2.0449E-12	1.0842E-19	-6.5625E-11	1.65014E-11
[0.5,1.0]	2001:4000	-6.94556E-10	-5.00703E-10	-6.5625E-11	-2.12369E-09	5.7812E-10
[1.0,1.5]	4001:6000	-0.00000001	-0.00000001	-2.12369E-09	-0.00000002	4.08234E-09
[1.5,2.0]	6001:8000	-0.00000004	-0.00000004	-0.00000002	-0.00000007	0.00000002
[2.0,2.5]	8001:10000	-0.00000013	-0.00000013	-0.00000007	-0.00000022	0.00000004

Source: authors' elaboration.

Table 8

Comparison between the erf(z) function and Θ_2

Interval of *z*	Sample	Mean	Median	Maximum	Minimum	Standard deviation
[0.0,2.5]	1:10000	-0.00010441	-0.00011106	0.00000000	-0.00018274	0.00005442
[0.0,0.5]	1:2000	-0.00002351	-0.00002356	0.00000000	-0.00004678	0.00001353
[0.5,1.0]	2001:4000	-0.00006915	-0.00006932	-0.00004678	-0.00009085	0.00001274
[1.0,1.5]	4001:6000	-0.00011078	-0.00011106	-0.00009085	-0.00012964	0.00001122
[1.5,2.0]	6001:8000	-0.00014597	-0.00014633	-0.00012964	-0.00016088	0.00000904
[2.0,2.5]	8001:10000	-0.00017266	-0.00017308	-0.00016088	-0.00018274	0.00000633

Source: authors' elaboration.

Monte Carlo results

Let us consider lattice L of pairs (σ, T) where $\sigma = 10^{-4} j$, $j = 1,\ldots,10^4$, $T = k/12$, $k = 1,\ldots,24$, and extract 10,000 samples randomly in each interval to derive *z*. Tables 9, 10 and 11 report the error, thus confirming the goodness of the approximation.

Table 9

Comparison between erf(z) and Θ_0, 10,000 simulations for each interval of σ

Interval of σ	Mean	Median	Maximum	Minimum	Standard deviation	Root mean square error
[0.0,1.25]	0.00211435	0.00069446	0.01534277	1.62971E-14	0.00299523	0.00366632
[0.0,0.25]	2.10572E-05	5.81107E-06	0.00019290	1.30375E-16	3.2778E-05	3.8959E-05
[0.25,0.5]	0.00030507	0.00020905	0.00146955	1.71298E-06	0.00029395	0.00042365
[0.5,0.75]	0.00124517	0.00104924	0.00451579	1.36411E-05	0.00100080	0.00159751
[0.75,1.0]	0.00307912	0.00278205	0.00938279	4.58159E-05	0.00225672	0.00381756
[1.0,1.25]	0.00582032	0.00549259	0.01551613	0.00010793	0.00398289	0.00705262

Source: authors' elaboration.

Table 10

Comparison between erf(z) and Θ_1 , 10,000 simulations

Interval of σ	**Mean**	**Median**	**Maximum**	**Minimum**	**Standard deviation**	**Root mean square error**
[0.0,1.25]	-1.23405E-06	-5.75226E-08	5.42101E-20	-0.00002423	2.95917E-06	3.20618E-06
[0.0,0.25]	-3.31189E-10	-1.75199E-11	1.35525E-20	-6.31596E-09	7.7678E-10	8.44437E-10
[0.25,0.5]	-2.17191E-08	-7.24392E-09	-2.2806E-12	-2.17943E-07	3.36263E-08	4.00306E-08
[0.5,1.75]	-2.41594E-07	-1.19167E-07	-7.29428E-11	-1.78407E-06	3.10627E-07	3.93518E-07
[0.75,1.0]	-1.29135E-06	-7.05183E-07	-5.55995E-10	-7.96528E-06	1.51968E-06	1.99425E-06
[1.0,1.25]	-4.603E-06	-2.62831E-06	-2.35745E-09	-2.48933E-05	5.13655E-06	6.89723E-06

Source: authors' elaboration.

Table 11

Comparison between erf(z) and Θ_2, 10,000 simulations

Interval of σ	**Sample**	**Mean**	**Median**	**Maximum**	**Minimum**	**Standard deviation**
[0.0,1.25]	-0.00012639	-0.00014109	-0.00000005	-0.00019525	0.00005936	0.00013964
[0.0,0.25]	-3.99204E-05	-3.60795E-05	-1.02281E-08	-0.00011040	2.69685E-05	4.81761E-05
[0.25,0.5]	-0.00011047	-0.00011305	-2.40804E-05	-0.00018246	3.74627E-05	0.00011665
[0.5,0.75]	-0.00015722	-0.00017151	-4.77618E-05	-0.00019525	3.86779E-05	0.00016191
[0.75,1.0]	-0.00017106	-0.00018247	-7.07184E-05	-0.00019525	2.91314E-05	0.00017352
[1.0,1.25]	-0.00015093	-0.00016262	-1.82918E-05	-0.00019525	4.06153E-05	0.00015630

Source: authors' elaboration.

5.3. Findings about the implied volatility

As illustrated, several methods are available in the literature to derive implied volatility through an approximate formula. In Orlando & Taglialatela (2017) the results obtained with the Brenner & Subrahmanyam (1998), Corrado & Miller (1996a, 1996b), and Li (2005) formulae are compared. All of them, for broad ranges of the parameters, do not provide any value. Moreover, Mininni et al. (2021) suggested a more accurate formula that, however, in some instances may underestimate the implied volatility. Thanks to Eq. (4.10) this can be corrected by averaging out the result of Eq. (4.7).

5.3.1. Case when $S \neq X$

Tables 12 and 13 compare the results obtained by Eq. (4.7) with those obtained with formula (4.10) as well as the average between the two. The option prices were generated with the BS model and, therefore, the implied volatility was derived using the inversion formulas. Each column provides the results of the said formulae for maturities T from 0.1 to 1.5 versus the true volatility. Contrary to some solutions provided in the literature, e.g. Li (2005), both σ_A and σ_B are always available for all maturities, moneyness and level of σ. However, in some occasions, σ_A may underestimate the implied volatility, therefore σ_B could be used as a correction. Specifically, from Table 1it is known that the mode of the volatility is $\sim 14.98\%$ and that the average is $\sim 21.74\%$. At the level of single securities, Table 2 shows that most traded options are short-term, and that the mode of the implied volatility is 32%. Consequently, by focusing on the first row of Table 12 (respectively Table 13), the closest value to the true implied volatility is σ_A up to 60% (respectively up to 75%). Beyond that level, the average between σ_A and σ_B should be preferred.

Table 12

Comparison of estimated implied volatilities for Out-The-Money calls

Time to expiration	True volatility								
	15%			30%			45%		
	σ^A	σ^B	Avg	σ_A	σ_B	Avg	σ_A	σ_B	Avg
0.1	12.22%	35.40%	23.81%	29.03%	56.09%	42.56%	43.64%	70.76%	57.20%
0.2	13.85%	31.90%	22.88%	29.20%	48.26%	38.73%	42.04%	60.39%	51.21%
0.3	14.46%	29.59%	22.03%	28.69%	43.89%	36.29%	40.66%	55.16%	47.91%
0.4	14.71%	27.87%	21.29%	28.11%	40.96%	34.54%	39.57%	51.93%	45.75%
0.5	14.78%	26.50%	20.64%	27.55%	38.82%	33.19%	38.68%	49.75%	44.22%
0.6	14.76%	25.37%	20.07%	27.03%	37.17%	32.10%	37.93%	48.22%	43.08%
0.7	14.69%	24.40%	19.55%	26.55%	35.85%	31.20%	37.29%	47.12%	42.21%
0.8	14.59%	23.56%	19.08%	26.10%	34.77%	30.44%	36.72%	46.34%	41.53%
0.9	14.46%	22.82%	18.64%	25.69%	33.87%	29.78%	36.21%	45.80%	41.01%
1	14.32%	22.15%	18.23%	25.29%	33.12%	29.20%	35.74%	45.44%	40.59%
1.1	14.16%	21.54%	17.85%	24.91%	32.49%	28.70%	35.31%	45.21%	40.26%
1.2	14.00%	20.99%	17.49%	24.55%	31.96%	28.25%	34.90%	45.08%	39.99%
1.3	13.83%	20.48%	17.15%	24.20%	31.51%	27.85%	34.51%	45.02%	39.77%
1.4	13.65%	20.01%	16.83%	23.86%	31.14%	27.50%	34.14%	45.00%	39.57%
1.5	13.47%	19.57%	16.52%	23.53%	30.83%	27.18%	33.78%	45.00%	39.39%
Average volatility	14.13%	24.81%	19.47%	26.29%	37.38%	31.83%	37.41%	49.75%	43.58%
Std. dev. vol.	0.66%	4.51%	2.19%	1.85%	7.00%	4.35%	2.89%	7.09%	4.91%
Time to expiration	**True volatility**								
	60%			**75%**			**90%**		
	σ^A	σ^B	Avg	σ^A	σ^B	Avg	σ^A	σ^B	Avg
0.1	56.61%	82.98%	69.80%	68.65%	94.19%	81.42%	80.17%	105.11%	92.64%
0.2	53.71%	71.39%	62.55%	64.91%	82.43%	73.67%	75.96%	94.10%	85.03%

0.3	51.84%	66.14%	58.99%	62.82%	77.88%	70.35%	73.87%	90.86%	82.37%
0.4	50.52%	63.27%	56.89%	61.47%	75.93%	68.70%	72.62%	90.07%	81.34%
0.5	49.52%	61.63%	55.57%	60.50%	75.20%	67.85%	71.77%	90.00%	80.88%
0.6	48.72%	60.72%	54.72%	59.77%	75.01%	67.39%	71.13%	89.92%	80.53%
0.7	48.07%	60.25%	54.16%	59.17%	75.00%	67.08%	70.62%	89.58%	80.10%
0.8	47.50%	60.06%	53.78%	58.66%	74.95%	66.81%	70.17%	88.94%	79.56%
0.9	47.00%	60.00%	53.50%	58.21%	74.78%	66.50%	69.76%	88.03%	78.90%
1	46.55%	60.00%	53.28%	57.80%	74.44%	66.12%	69.37%	86.93%	78.15%
1.1	46.13%	59.98%	53.06%	57.41%	73.93%	65.67%	68.98%	85.67%	77.32%
1.2	45.74%	59.91%	52.82%	57.03%	73.28%	65.15%	68.57%	84.32%	76.45%
1.3	45.36%	59.75%	52.56%	56.65%	72.51%	64.58%	68.16%	82.89%	75.52%
1.4	44.99%	59.51%	52.25%	56.26%	71.64%	63.95%	67.73%	81.41%	74.57%
1.5	44.62%	59.17%	51.90%	55.87%	70.69%	63.28%	67.27%	79.91%	73.59%
Average volatility	48.46%	62.98%	55.72%	59.68%	76.12%	67.90%	71.08%	88.52%	79.80%
Std. dev. vol.	3.34%	6.20%	4.67%	3.43%	5.50%	4.40%	3.34%	5.80%	4.53%

Notes: S = \$100, X = \$125, risk-free rate 5% p.a.
Source: authors' elaboration.

Table 13

Comparison of estimated implied volatilities for In-The-Money calls

Time to expiration	**True volatility**								
	15%			**30%**			**45%**		
	σ^A	σ^B	Avg	σ_A	σ_B	Avg	σ_A	σ_B	Avg
0.1	9.99%	34.12%	22.05%	26.70%	56.86%	41.78%	42.61%	73.43%	58.02%
0.2	11.78%	31.50%	21.64%	28.19%	50.18%	39.18%	42.77%	63.82%	53.29%
0.3	12.62%	29.75%	21.18%	28.49%	46.36%	37.43%	42.33%	58.83%	50.58%
0.4	13.09%	28.44%	20.77%	28.52%	43.80%	36.16%	41.91%	55.69%	48.80%
0.5	13.39%	27.41%	20.40%	28.45%	41.93%	35.19%	41.57%	53.50%	47.53%
0.6	13.58%	26.57%	20.08%	28.36%	40.48%	34.42%	41.29%	51.89%	46.59%
0.7	13.71%	25.87%	19.79%	28.26%	39.33%	33.80%	41.07%	50.67%	45.87%
0.8	13.80%	25.27%	19.54%	28.18%	38.38%	33.28%	40.90%	49.71%	45.31%
0.9	13.87%	24.75%	19.31%	28.10%	37.59%	32.84%	40.77%	48.94%	44.85%
1	13.92%	24.29%	19.11%	28.02%	36.92%	32.47%	40.67%	48.31%	44.49%
1.1	13.96%	23.89%	18.92%	27.96%	36.34%	32.15%	40.59%	47.80%	44.19%
1.2	13.98%	23.52%	18.75%	27.91%	35.84%	31.87%	40.53%	47.37%	43.95%
1.3	14.00%	23.20%	18.60%	27.86%	35.40%	31.63%	40.49%	47.02%	43.75%
1.4	14.02%	22.90%	18.46%	27.82%	35.01%	31.41%	40.46%	46.72%	43.59%
1.5	14.03%	22.63%	18.33%	27.79%	34.66%	31.22%	40.44%	46.47%	43.45%
Average volatility	13.32%	26.27%	19.80%	28.04%	40.61%	34.32%	41.23%	52.68%	46.95%
Std. dev. vol.	1.08%	3.28%	1.15%	0.43%	6.14%	3.01%	0.79%	7.33%	4.04%

Time to expiration	True volatility								
	60%			75%			90%		
	σ^A	σB	Avg	σA	σB	Avg	σA	σB	Avg
0.1	57.14%	86.99%	72.07%	70.63%	99.08%	84.85%	83.45%	110.46%	96.96%
0.2	56.02%	75.59%	65.80%	68.54%	86.79%	77.66%	80.71%	98.09%	89.40%
0.3	55.06%	70.13%	62.59%	67.29%	81.44%	74.36%	79.36%	93.40%	86.38%
0.4	54.39%	66.91%	60.65%	66.54%	78.61%	72.57%	78.66%	91.34%	85.00%
0.5	53.92%	64.83%	59.38%	66.08%	77.00%	71.54%	78.30%	90.45%	84.38%
0.6	53.60%	63.42%	58.51%	65.80%	76.07%	70.94%	78.13%	90.11%	84.12%
0.7	53.37%	62.42%	57.90%	65.64%	75.54%	70.59%	78.08%	90.01%	84.05%
0.8	53.21%	61.71%	57.46%	65.56%	75.25%	70.40%	78.10%	90.00%	84.05%
0.9	53.11%	61.21%	57.16%	65.53%	75.10%	70.31%	78.16%	90.00%	84.08%
1	53.04%	60.84%	56.94%	65.54%	75.03%	70.28%	78.26%	89.96%	84.11%
1.1	53.01%	60.57%	56.79%	65.58%	75.01%	70.29%	78.38%	89.89%	84.13%
1.2	52.99%	60.39%	56.69%	65.63%	75.00%	70.32%	78.51%	89.78%	84.14%
1.3	53.00%	60.25%	56.62%	65.70%	75.00%	70.35%	78.65%	89.63%	84.14%
1.4	53.01%	60.16%	56.59%	65.78%	75.00%	70.39%	78.79%	89.45%	84.12%
1.5	53.04%	60.10%	56.57%	65.87%	74.98%	70.43%	78.94%	89.25%	84.09%
Average volatility	53.86%	65.03%	59.45%	66.38%	78.33%	72.35%	78.97%	92.12%	85.54%
Std. dev. vol.	1.23%	7.25%	4.23%	1.39%	6.39%	3.89%	1.36%	5.36%	3.35%

Note: S = \$100, X = \$75, risk-free rate 5% p.a.

Source: authors' elaboration.

5.3.2. Case when $S = X$

For the At-The-Money case, Table 14 provides a comparison between the implied volatilities $\hat{\sigma}_L$, $\hat{\sigma}_1$ and $\hat{\sigma}_2$ as derived with the Li formula (2008), Eq. (4.14) and Eq. (4.15), respectively. The prices of all options were generated with the BS model for volatilities between 15% and 125% and maturity between 0.1 and 1.5 years.

Table 14

Estimation error for At-The-Money options

	True volatility					
	15%	**35%**	**55%**	**75%**	**95%**	**125%**
Statistics	**Implied volatility estimation error for $\hat{\sigma}_L$,**					
Mean	0.00000983%	0.00068779%	0.00672967%	0.03275487%	0.11169245%	0.48689149%
Median	0.00000761%	0.00053019%	0.00515269%	0.02481224%	0.08326543%	0.34881310%
Minimum	0.00000012%	0.00000822%	0.00007887%	0.00037279%	0.00121955%	0.00484083%
Maximum	0.00002678%	0.00187959%	0.01850733%	0.09098208%	0.31488411%	1.42410046%
StdDev	0.00000846%	0.00059359%	0.00584033%	0.02867370%	0.09902358%	0.44499502%

	Implied volatility estimation error for $\hat{\sigma}_1$					
Mean	-0.00000002%	-0.00000131%	-0.00001338%	-0.00006825%	-0.00024386%	-0.00111527%
Median	-0.00000001%	-0.00000100%	-0.00001004%	-0.00005016%	-0.00017529%	-0.00077766%
Minimum	-0.00000005%	-0.00000362%	-0.00003744%	-0.00019455%	-0.00070786%	-0.00331424%
Maximum	0.00000000%	-0.00000002%	-0.00000015%	-0.00000070%	-0.00000231%	-0.00000928%
StdDev	0.00000002%	0.00000114%	0.00001180%	0.00006121%	0.00022249%	0.00104102%
	Implied volatility estimation error for $\hat{\sigma}_2$					
Mean	-0.01246830%	-0.02825481%	-0.04206067%	-0.05284303%	-0.05968480%	-0.06101872%
Median	-0.01246828%	-0.02825352%	-0.04204815%	-0.05278248%	-0.05948135%	-0.06017472%
Minimum	-0.01253937%	-0.02915319%	-0.04551451%	-0.06148010%	-0.07690809%	-0.09873650%
Maximum	-0.01239728%	-0.02736060%	-0.03864751%	-0.04440268%	-0.04312244%	-0.02604073%
StdDev	0.00004385%	0.00055321%	0.00211929%	0.00527096%	0.01043041%	0.02246108%

Note: S = \$100, X = \$100, time to expiration T = 0.1, 0.2, ...,1.5; risk-free rate 5% p.a.

Source: authors' elaboration.

6. APPROXIMATIONS OF THE IMPLIED VOLATILITY AND IMPLICATIONS

As mentioned by Malliaris & Salchenberger (1996), “the implied volatility, calculated using the Black-Scholes model, is currently the most popular method of estimating volatility and is considered by traders to be a significant factor in signalling price movements in the underlying market”.

In terms of the use of implied volatility, Jeon & Taylor (2013) applied it into the CAViaR models and found that the results are “comparable or superior to individual methods, such as the standard CAViaR models and quantiles constructed from implied volatility and the empirical distribution of standardised residuals”. Moreover, they provide evidence that the “implied volatility has more explanatory power as the focus moves further out into the left tail of the conditional distribution of S&P 500 daily returns”. According to Han (2008), investor sentiment affects the implied volatility smile on S&P 500 options. Ramos & Righi (2020) and Bufalo & Orlando (2022) showed that increases in implied volatility are positively linked to illiquidity and turmoil.

Park et al. (2017) affirmed that the implied volatility “captures investor sentiments, market expectations, and future prospects”. For this reason, it can be considered as a proxy of the asset price dynamics, and thus of the market risk. In fact, as reported by Giot (2005), “there is a negative and statistically significant relationship between the returns of the S&P 100 and the Nasdaq 100 stock indexes and their corresponding implied volatility indexes, VIX and VXN”.

Bekiros et al. (2017) investigated the asymmetric relationship between returns and implied volatility for twenty developed and emerging international markets, using quantile regression and found “evidence of an asymmetric and reverse return-volatility relationship”.

Concerning the decision-making process, “even though software is available for computing the implied volatility numerically, among practitioners it is common to use spreadsheets with approximated solutions which are good enough”, observe Orlando & Taglialatela (2021). Such approximations also avoid “the inconvenience of setting up iterating routines and macros that should run permanently to keep up with price changes in the market”, as noticed by Orlando & Taglialatela (2021). In terms of computational performance, as reported in Mininni et al. (2021), the suggested approximation is about 300% better performing than one of the fastest algorithms available in the literature as reported by Orlando & Taglialatela (2017).

CONCLUSION

First of all, recall the importance of calculating the value of the call for pricing as well as for inferring the implied volatility. “In many respects the story of the establishment of the Black-Scholes-Merton model simply marks the emergence of contemporary financial risk management” (Millo & MacKenzie, 2009).

In a thorough review of potential alternatives, such as the deterministic volatility function (DVF) option valuation proposed by Derman & Kani (1994), Dupire (1994), Rubinstein (1994), Dumas et al. (1998) concluded that the predictive and hedging performance of these was no better than an ad hoc procedure that merely smooths BS-implied volatilities across exercise prices and times to expiration.

This study expanded the work of Mininni et al. (2021), in which a standardized option function (representing the whole family of calls) was introduced to simplify the calculations. The authors complemented this by suggesting that a correction for those cases in which the formula by Mininni et al. (2021) underestimates the implied volatility. Hence, it is shown how the approximation of the above-mentioned standardized call can be performed by means of hyperbolic tangents instead of the commonly used Taylor truncation. The higher precision for extreme values of σ makes this approach specifically suitable for hedging and stress testing. Hence, the study derived some closed-form formulas for the approximation of the implied volatility that seems to be more accurate than those proposed in the literature so far, and which are valid regardless of moneyness. This was proved by additional simulations, graphical and numerical evidence. Finally, the authors provided further literature and the rationale for the reasons why implied volatility is used in decision making.

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