OPERATIONS RESEARCH AND DECISIONS

No. 1 2017

DOI: 10.5277/ord170103

Nasrullah KHAN[1]
Muhammad ASLAM[2]
Kyung-Jun KIM[3]
Chi-Hyuck JUN[4]

# A MIXED CONTROL CHART ADAPTED TO THE TRUNCATED LIFE TEST BASED ON THE WEIBULL DISTRIBUTION

The design of a new mixed attribute control chart adapted to a truncated life test has been presented. It was assumed that the lifetime of a product follows the Weibull distribution and the number of failures was observed using a truncated life test, where the test duration was specified as a fraction of the mean lifespan. The proposed control chart consists of two pairs of control limits based on a binomial distribution and one lower bound. The average run length of the chart was determined for various levels of shift constants and specified parameters. The efficiency of the chart is compared with an existing control chart in terms of the average run length. The application of the proposed chart is discussed with the aid of a simulation study.

**Keywords:** *life test, Weibull distribution, attribute chart, binomial distribution, average run length*

## 1. Introduction

A control chart is one of the most important tools for monitoring a manufacturing process in industry. Time series of statistics of interest are plotted on the control chart to see whether the process is under control or out of control. If a plotted statistic is above

[1]Department of Statistics, Jhang Campus, University of Veterinary and Animal Sciences, Lahore 54000, Pakistan, e-mail address: nas_shan1@hotmail.com

[2]Department of Statistics, Faculty of Sciences, King Abdulaziz University, Jeddah 21551, Saudi Arabia, e-mail address: aslam_ravian@hotmail.com

[3]Department of Industrial and Management Engineering, POSTECH, Pohang 37673, Republic of Korea, e-mail address: k_kim@postech.ac.kr

[4] Department of Industrial and Management Engineering, POSTECH, Pohang 37673, Republic of Korea, e-mail address: chjun@postech.ac.kr

the upper control limit ($UCL$) or below the lower control limit ($LCL$), the process is said to be out-of-control. If the statistic lies within the control limits, the process is said to be under control. There are many factors which might cause the process to shift from the under-control state to the out-of-control state. In this situation, quick indication is needed to bring the process back under control. Delay in indicating a problem may cause defective products.

Usually, control charts are designed under the assumption that the quantity of interest follows the normal distribution. In industry, however, it may be possible that the quantity of interest follows some non-normal distribution. In such situations, a control chart designed under the assumption of normality may mislead an engineer when making a decision about whether the process is under control or not. Detailed studies about some control charts developed for non-normal distributions can be found in [1–8].

Further, the selection of control charts according to the type of data collected is an important issue. If the data are obtained from a counting process, they are called attribute data. On the other hand, data obtained from a measurement process are called variable data. For attribute data, an attribute control chart, such as an np chart is used, while for variable data, a variable chart such as an X-bar chart is used to monitor the process. An attribute control chart has the advantage of simplicity in designing the control limits because it usually involves a binomial or Poisson distribution. In general, however, variable data are more informative, so a variable control chart may be more effective. However a variable control chart requires knowledge about the probability distribution of the plotting statistic which cannot be derived for some non-normal distributions.

Sometimes, there is a need to use a mixture of attribute and variable control charts to monitor a manufacturing process. The use of such a mixed control chart may enjoy the combined advantages of both attribute and variable control charts. Recently, [9] designed a mixed control chart to monitor manufacturing processes. [10] designed a mixed chart using an exponentially weighted moving average statistic. [10] proposed a mixed control chart for attribute data. By exploring the literature on control charts, we note that there is no work on designing a mixed control chart adapted to the truncated life test. In this paper, we focus on designing a new mixed control chart adapted to the time truncated life test based on a Weibull distribution. The application of the proposed chart is discussed with the aid of a simulation study. The efficiency of the proposed chart is compared with the control chart proposed by [11].

## 2. Design of the proposed control chart

It is assumed that the lifetime of a product (denoted by the random variable $X$) follows the Weibull distribution with the scale parameter $\lambda$ and the shape parameter $\beta$ and thus has the following cumulative distribution function (cdf):

$$F(t;\lambda,\beta) = 1 - \exp\left(-\left(\frac{t}{\lambda}\right)^{\beta}\right),\ t \geq 0 \tag{1}$$

The average lifetime based on this Weibull distribution is given as follows:

$$\mu = \frac{\lambda}{\beta}\Gamma\left(\frac{1}{\beta}\right) \tag{2}$$

If the shape parameter is known, then the transformed variable $Y = X^{\beta}$ follows an exponential distribution with mean $\lambda^{\beta}$.

Let $\mu_0$ be the average lifetime when the process is under control, which corresponds to the scale parameter $\lambda_0$. It is assumed that the scale parameter changes to $\lambda_1 = c\lambda_0$, while the shape parameter remains unchanged, when the process is shifted.

In this section, we will propose a mixed type of control chart utilizing the failure data from a truncated life test.

**Step 1.** Take a sample of size $n$ from the production process. Test them until the specified time $t_0$.

**Step 2.** Count the number of failures ($d$, say) in Step 1. Declare the process to be out-of-control if $d \geq UCL_1$ or $d \leq LCL_1$. Declare the process to be under-control if $LCL_2 \leq d \leq UCL_2$. Otherwise, go to Step 3.

**Step 3.** For the sample described in Step 1, obtain the time to failure of item $i$ (denoted by $X_i$). Set $X_i = t_0$ if item $i$ has not failed by time $t_0$. Calculate $Y_i = X_i^{\beta}$ and $\bar{Y}$. Declare the process as out-of-control if $\bar{Y} \leq L_3$. Declare the process as under-control if $\bar{Y} \geq L_3$.

We call the proposed control chart a mixed control chart because Step 2 is based on attribute data and Step 3 is based on variable data. There are two pairs of control limits in Step 2 which will be described later, while there is one cutoff value in Step 3. Using the two pairs of control limits, a quick decision can be made as to whether the number of failures is small or large and the second decision will be made based on the failure times when the number of failures is moderate.

### 2.1. Control limits and in-control *ARL*

We will first derive the necessary measures used for the proposed control chart when the process is under control. It would be convenient to select the specified test time $t_0$ as a fraction of the in-control mean $\mu_0$, i.e. $t_0 = a\mu_0$, where $a$ is a constant. For example, when $a$ = 0.5, the test time is the half of the mean lifetime of a product when the process is under control. Thus, the number of failures by time $t_0$ (denoted by $d$) follows a binomial distribution with parameters $n$ and $p_0$, where

$$p_0 = 1 - \exp\left(-\left(\frac{t_0}{\lambda_0}\right)^{\beta}\right) = 1 - \exp\left(-a^{\beta}\left(\frac{\Gamma(1/\beta)}{\beta}\right)^{\beta}\right) \tag{3}$$

Therefore, we propose two pairs of chart limits to be used in Step 2 as follows:

$$UCL_1 = np_0 + k_1\sqrt{np_0(1-p_0)} \tag{4a}$$

$$LCL_1 = \max\left[0, np_0 - k_1\sqrt{np_0(1-p_0)}\right] \tag{4b}$$

$$UCL_2 = np_0 + k_2\sqrt{np_0(1-p_0)} \tag{5a}$$

$$LCL_2 = \max\left[0, np_0 - k_2\sqrt{np_0(1-p_0)}\right] \tag{5b}$$

where $k_1$ and $k_2$ are control coefficients to be determined.

The distribution of $\bar{Y}$ in Step 3 can be approximated by a normal distribution according to the central limit theorem. The mean and the variance are derived as follows:

$$E\left[\bar{Y}\right] = E\left[Y\right] = E\left[Y \mid Y \le t_0^{\beta}\right] P\left\{Y \le t_0^{\beta}\right\} + E\left[Y \mid Y > t_0^{\beta}\right] P\left\{Y > t_0^{\beta}\right\} \tag{6}$$

It should be noted that for the Weibull distribution we take a power transform because the transformed variable follows an exponential distribution and the sum of $Y$'s (or $\bar{Y}$) follows a gamma distribution. Thus, Eq. (4) can be written as

$$\begin{aligned} E\left[\bar{Y}\right] &= \int_0^{t_0^{\beta}} \frac{y}{\lambda_0^{\beta}} \mathrm{e}^{-y/\lambda_0^{\beta}} dy + t_0^{\beta} \int_{t_0^{\beta}}^{\infty} \frac{1}{\lambda^{\beta}} \mathrm{e}^{-y/\lambda_0^{\beta}} dy = \lambda_0^{\beta}\left(1 - \mathrm{e}^{-(t_0/\lambda_0)^{\beta}}\right) \\ &= \lambda_0^{\beta}\left(1 - \mathrm{e}^{-\left(\frac{a\Gamma(1/\beta)}{\beta}\right)^{\beta}}\right) \end{aligned} \tag{7}$$

The variance of $\bar{Y}$ can be written as follows:

$$\mathrm{Var}\left[\bar{Y}\right]=\frac{1}{n}\mathrm{Var}\left[Y\right]=\frac{1}{n}\left(E\left[Y^2\right]-\left(E\left[Y\right]\right)^2\right) \tag{8}$$

$$\mathrm{Var}\left[\bar{\mathrm{Y}}\right]=\frac{1}{\mathrm{n}}\left(\int_0^{t_0^\beta}\frac{y^2}{\lambda_0^\beta}\mathrm{e}^{-y/\lambda_0^\beta}dy+t_0^{2\beta}\int_{t_0^\beta}^{\infty}\frac{1}{\lambda_0^\beta}\mathrm{e}^{-y/\lambda_0^\beta}dy-\lambda_0^{2\beta}\left(1-\mathrm{e}^{-(t_0/\lambda_0)^\beta}\right)^2\right) \tag{9}$$

After simplification, $\mathrm{Var}\left[\bar{Y}\right]$ can be rewritten as

$$\mathrm{Var}\left[\bar{Y}\right]=\frac{1}{n}\left[\lambda_0^{2\beta}\left(1-\mathrm{e}^{-2(t_0/\lambda_0)^\beta}\right)-2\left(\lambda_0 t_0\right)^\beta\,\mathrm{e}^{-(t_0/\lambda_0)^\beta}\right] \tag{10}$$

Therefore, the probability of the process being declared under control using the proposed chart when the process is actually under control is given as follows:

$$P_{in,0}=f_{10}+\left\{f_{20}\right\}f_{30} \tag{11}$$

where

$$f_{10}=P\left(LCL_2\le d\le UCL_2\mid p_0\right)=\sum_{d=LCL_2+1}^{UCL_2}\binom{n}{d}p_0^d\left(1-p_0\right)^{n-d}$$

$$f_{20}=P\left(UCL_2\le d\le UCL_1\mid p_0\right)+P\left(LCL_1\le d\le LCL_2\mid p_0\right)$$

$$=\sum_{d=UCL_2+1}^{UCL_1}\binom{n}{d}p_0^d\left(1-p_0\right)^{n-d}+\sum_{d=LCL_1+1}^{LCL_2}\binom{n}{d}p_0^d\left(1-p_0\right)^{n-d}$$

$$f_{30}=P\left(\bar{Y}\ge L_3\right)=1-P\left(Z\le\frac{L_3-E\left[\bar{Y}\right]}{\sqrt{\mathrm{Var}\left[\bar{Y}\right]}}\right)$$

$$=1-\Phi\left(\frac{L_3-\left(\frac{\beta\mu_0}{\Gamma(1/\beta)}\right)^\beta\left(1-\mathrm{e}^{-((a/\beta)\Gamma(1/\beta))^\beta}\right)}{\sqrt{\frac{1}{n}\left[\left(\frac{\beta\mu_0}{\Gamma(1/\beta)}\right)^{2\beta}\left(1-\mathrm{e}^{-2((a/\beta)\Gamma(1/\beta))^\beta}\right)-2\left(\frac{a\beta\mu_0^2}{\Gamma(1/\beta)}\right)^\beta\mathrm{e}^{-((a/\beta)\Gamma(1/\beta))^\beta}\right]}}\right)$$

The performance of the proposed control will be evaluated using the average run length (*ARL*), which is used to indicate the mean length of time that passes before the process is declared out-of-control. When the process is under control, the *ARL* (called the under-control *ARL*) should be large. However, when the process has shifted, the *ARL* (called the out-of-control *ARL*) should be small as possible. The *ARL* for a process under control is denoted by $ARL_0$ and given as follows:

$$ARL_0 = \frac{1}{1-\left( f_{10} + f_{20} f_{30} \right)} \tag{12}$$

## 2.2. Out-of-control *ARL*

Next, we derive some necessary measures under the assumption that the process has shifted. Suppose that there has been a change in the scale parameter of the Weibull distribution, while the shape parameter remains unchanged. The scale parameter of the Weibull distribution is assumed to be shifted to $\lambda_1 = c\lambda_0$, where $c$ is a shift constant smaller than 1. For the shifted process, the probability of being declared under control using the proposed chart is given as follows:

$$P_{in,1} = f_{11} + \left\{ f_{21} \right\} f_{31} \tag{13}$$

where

$$f_{11} = P\left( LCL_2 \le d \le UCL_2 \mid p_1 \right) = \sum_{d=LCL_2+1}^{UCL_2} \binom{n}{d} p_1^d \left(1-p_1\right)^{n-d}$$

$$f_{21} = P\left( UCL_2 \le d \le UCL_1 \mid p_1 \right) + P\left( LCL_1 \le d \le LCL_2 \mid p_1 \right)$$

$$= \sum_{d=UCL_2+1}^{UCL_1} \binom{n}{d} p_1^d \left(1-p_1\right)^{n-d} + \sum_{d=LCL_1+1}^{LCL_2} \binom{n}{d} p_1^d \left(1-p_1\right)^{n-d}$$

$$f_{31} = P\left( \bar{Y} \ge L_3 \right) \tag{14}$$

where

$$p_1 = 1 - \exp\left( -\left( \frac{t_0}{\lambda_1} \right)^{\beta} \right) = 1 - \exp\left( -a^{\beta} \left( \frac{\Gamma\left(1/\beta\right)}{c\beta} \right)^{\beta} \right)$$

Table 1. The values of *ARL*s when $\beta = 1$ and $\mu_0 = 50$

| | *a* | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.4 | 0.5 | 0.7 | 0.9 | 1 |
| $k_1$ | 3.5559 | 3.3293 | 3.6768 | 3.0176 | 3.8526 | 3.9595 | 3.4987 |
| $k_2$ | 1.0175 | 1.0871 | 1.3476 | 1.3078 | 1.2570 | 1.1387 | 1.2709 |
| $L_3$ | 4.3314 | 7.9410 | 14.0947 | 14.5024 | 20.2793 | 22.8383 | 24.3463 |
| *c* | *ARL* | | | | | | |
| 1.0 | 370.75 | 370.78 | 370.11 | 370.01 | 370.04 | 370.04 | 370.15 |
| 0.9 | 174.40 | 147.07 | 134.59 | 140.67 | 113.68 | 104.73 | 119.41 |
| 0.8 | 78.02 | 54.92 | 40.83 | 45.69 | 29.21 | 25.92 | 28.12 |
| 0.7 | 33.11 | 19.68 | 12.03 | 14.47 | 7.98 | 7.14 | 7.14 |
| 0.6 | 13.49 | 7.13 | 4.02 | 4.78 | 2.76 | 2.56 | 2.42 |
| 0.5 | 5.47 | 2.85 | 1.74 | 1.89 | 1.37 | 1.33 | 1.26 |
| 0.4 | 2.39 | 1.43 | 1.10 | 1.11 | 1.03 | 1.03 | 1.01 |
| 0.3 | 1.29 | 1.04 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.2 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.1 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Table 2. The values of *ARL*s when $\beta = 1$ and $\mu_0 = 100$

| | *a* | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.4 | 0.5 | 0.7 | 0.9 | 1 |
| $k_1$ | 3.4517 | 3.4202 | 3.0933 | 3.0369 | 3.2420 | 3.1057 | 3.6698 |
| $k_2$ | 1.1968 | 1.0491 | 1.3807 | 1.2698 | 1.3105 | 1.2148 | 1.4681 |
| $L_3$ | 8.7254 | 15.8817 | 26.8782 | 29.0040 | 39.3605 | 44.9779 | 48.6938 |
| Shift | *ARL* | | | | | | |
| 1.0 | 371.77 | 370.94 | 370.04 | 370.01 | 370.03 | 370.05 | 370.00 |
| 0.9 | 168.06 | 147.13 | 129.77 | 140.67 | 130.45 | 168.86 | 119.37 |
| 0.8 | 71.99 | 54.94 | 41.37 | 45.69 | 32.88 | 39.54 | 28.11 |
| 0.7 | 29.58 | 19.68 | 12.81 | 14.47 | 8.61 | 9.31 | 7.14 |
| 0.6 | 11.93 | 7.13 | 4.26 | 4.78 | 2.81 | 2.84 | 2.42 |
| 0.5 | 4.92 | 2.85 | 1.78 | 1.89 | 1.34 | 1.33 | 1.26 |
| 0.4 | 2.23 | 1.43 | 1.10 | 1.11 | 1.02 | 1.02 | 1.01 |
| 0.3 | 1.25 | 1.04 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.2 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.1 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

The distribution of $\bar{Y}$ for the shifted process can be approximated by a normal distribution with mean and variance of

$$E\left[\bar{Y} \mid \lambda_1\right] = \lambda_1^{\beta}\left(1 - \mathrm{e}^{-(t_0/\lambda_1)^{\beta}}\right) \tag{15}$$

Table 3. The values of *ARL*s when $\beta = 1.5$ and $\mu_0 = 50$

| | *a* | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.4 | 0.5 | 0.7 | 0.9 | 1 |
| $k_1$ | 3.9668 | 3.5200 | 3.2525 | 3.3891 | 3.7946 | 3.2213 | 3.2230 |
| $k_2$ | 1.3801 | 1.3652 | 1.2562 | 1.0732 | 1.4216 | 1.1535 | 1.4034 |
| $L_3$ | 10.5079 | 28.3449 | 68.1557 | 92.5725 | 136.1281 | 167.3669 | 187.2779 |
| Shift | *ARL* | | | | | | |
| 1.0 | 373.74 | 370.14 | 370.03 | 370.01 | 370.08 | 370.02 | 370.00 |
| 0.9 | 170.18 | 139.02 | 99.09 | 86.87 | 74.66 | 64.00 | 77.30 |
| 0.8 | 77.52 | 48.25 | 25.45 | 18.23 | 12.85 | 9.89 | 10.35 |
| 0.7 | 35.07 | 15.93 | 6.76 | 4.52 | 3.09 | 2.41 | 2.33 |
| 0.6 | 15.43 | 5.44 | 2.24 | 1.67 | 1.32 | 1.16 | 1.13 |
| 0.5 | 6.50 | 2.18 | 1.17 | 1.06 | 1.01 | 1.00 | 1.00 |
| 0.4 | 2.73 | 1.21 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.3 | 1.35 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.2 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.1 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Table 4. The values of *ARL*s when $\beta = 1.5$ and $\mu_0 = 100$

| | *a* | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.4 | 0.5 | 0.7 | 0.9 | 1 |
| $k_1$ | 3.6693 | 3.6363 | 3.2196 | 3.4942 | 3.6762 | 3.2922 | 3.1810 |
| $k_2$ | 1.4652 | 1.2119 | 1.4407 | 1.1742 | 1.0897 | 1.0674 | 1.0022 |
| $L_3$ | 29.7235 | 79.7077 | 194.1979 | 261.8349 | 379.8089 | 473.3809 | 499.9637 |
| Shift | *ARL* | | | | | | |
| 1.0 | 370.55 | 370.06 | 370.09 | 370.00 | 370.17 | 370.08 | 371.48 |
| 0.9 | 168.88 | 124.66 | 96.96 | 86.87 | 64.57 | 64.01 | 64.53 |
| 0.8 | 77.03 | 40.62 | 24.25 | 18.23 | 11.42 | 9.89 | 9.80 |
| 0.7 | 34.90 | 13.36 | 6.40 | 4.52 | 2.91 | 2.41 | 2.38 |
| 0.6 | 15.37 | 4.75 | 2.15 | 1.67 | 1.29 | 1.16 | 1.15 |
| 0.5 | 6.49 | 2.02 | 1.16 | 1.06 | 1.01 | 1.00 | 1.00 |
| 0.4 | 2.73 | 1.18 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.3 | 1.35 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.2 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.1 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

$$\operatorname{Var}\left[\bar{Y}\right] = \frac{1}{n}\left[\lambda_1^{2\beta}\left(1 - \mathrm{e}^{-2\left(t_0/\lambda_1\right)^{\beta}}\right) - 2\left(\lambda_1 t_0\right)^{\beta}\,\mathrm{e}^{-\left(t_0/\lambda_1\right)^{\beta}}\right] \tag{16}$$

Table 5. The values of *ARL*s when $\beta = 2$ and $\mu_0 = 50$, $n = 100$

| | *a* | | | | | | |
|---|---|---|---|---|---|---|---|
| | 0.1 | 0.2 | 0.4 | 0.5 | 0.7 | 0.9 | 1 |
| $k_1$ | 5.2816 | 5.0374 | 4.1635 | 2.9966 | 3.0321 | 4.3334 | 3.0878 |
| $k_2$ | 1.0075 | 1.2602 | 1.2409 | 1.3252 | 1.4420 | 1.0335 | 1.2614 |
| $L_3$ | 24.5640 | 96.3703 | 358.6161 | 521.8815 | 925.2349 | 1338.6727 | 1464.6992 |
| Shift | *ARL* | | | | | | |
| 1.0 | 370.36 | 371.18 | 370.14 | 370.04 | 370.00 | 370.00 | 370.03 |
| 0.9 | 116.33 | 77.23 | 31.74 | 27.54 | 12.80 | 8.21 | 10.16 |
| 0.8 | 39.57 | 16.40 | 4.10 | 2.96 | 1.48 | 1.26 | 1.19 |
| 0.7 | 14.38 | 4.34 | 1.33 | 1.10 | 1.00 | 1.00 | 1.00 |
| 0.6 | 5.57 | 1.68 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.5 | 2.39 | 1.07 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.4 | 1.28 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.3 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.2 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.1 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

Thus,

$$f_{31} = 1 - \Phi\left(\frac{L_3 - E\left[\bar{Y}\right]}{\operatorname{Var}\left[\bar{Y}\right]}\right) \tag{17}$$

or equivalently

$$f_{31} = 1 - \Phi\left(\frac{L_3 - \left(\dfrac{c\beta\mu_0}{\Gamma(1/\beta)}\right)^{\beta}\left(1 - \mathrm{e}^{-\left(\frac{a}{c\beta}\Gamma(1/\beta)\right)^{\beta}}\right)}{\sqrt{\dfrac{1}{n}\left[\left(\dfrac{c\beta\mu_0}{\Gamma(1/\beta)}\right)^{2\beta}\left(1 - \mathrm{e}^{-2\left(\frac{a}{c\beta}\Gamma(1/\beta)\right)^{\beta}}\right) - 2\left(\dfrac{ac\beta\mu_0^2}{\Gamma(1/\beta)}\right)^{\beta}\mathrm{e}^{-\left(\frac{a}{c\beta}\Gamma(1/\beta)\right)^{\beta}}\right]}}\right) \tag{18}$$

Hence, the out-of-control *ARL* for the shifted process is given from Eq. (19).

$$ARL_1 = \frac{1}{1 - \left(f_{11} + f_{21} f_{31}\right)} \tag{19}$$

The values of the *ARL*s for various values of the target mean $\mu_0$, $\beta$ and sample size $n$ = 30 are reported in Tables 1–5.

From the tables, we note the following trend in the *ARL* values.

- When all other parameters are fixed, the *ARL* increases as the target $ARL_0$ increases.
- When all other parameters are fixed, the *ARL* increases as $\mu_0$ increases.
- When all other parameters are fixed, the *ARL* increases as $\beta$ increases.

## 3. Comparative study

In this section, a comparison of the proposed chart with the one proposed by Aslam et al. [11] is given. The efficiency of the proposed chart has been compared in terms of the *ARL*s. A control chart is said to be more efficient than another if it provides smaller values of the *ARL*s for the same values of all the parameters specified. To save space, we will present the *ARL*s by Aslam et al. [11] for $r_0$ = 370, $\beta$ = 0.5, $a$ = 0.5, 0.7, 0.9, and $\mu_0$ = 100 (Table 6).

Table 6. Comparison of the present method with the one proposed by Aslam et al. [11]

| *a* | 0.1 | | 0.5 | | 1 | |
|---|---|---|---|---|---|---|
| | Present | Aslam et al. [9] | Present | Aslam et al. [9] | Present | Aslam et al. [9] |
| | | $k_1$ = 3.9668 | | $k_1$ = 3.3891 | | $k_1$ = 3.2230 |
| | $k$ = 3.266246 | $k_2$ = 1.3800 | $k$ = 2.9755 | $k_2$ = 1.0731 | $k$ = 2.957346 | $k_2$ = 1.4033 |
| Shift | $n$ = 21 | $L_3$ = 10.50 | $n$ = 30 | $L_3$ = 92.57 | $n$ = 49 | $L_3$ = 187.27 |
| | *ARL* | | | | | |
| 1.0 | 405.82 | 373.74 | 375.23 | 370.01 | 373.93 | 370.00 |
| 0.9 | 229.89 | 170.18 | 144.44 | 86.87 | 129.96 | 77.30 |
| 0.8 | 124.42 | 77.52 | 48.17 | 18.23 | 28.54 | 10.35 |
| 0.7 | 63.95 | 35.07 | 16.07 | 4.52 | 7.28 | 2.33 |
| 0.6 | 31.02 | 15.43 | 5.70 | 1.67 | 2.47 | 1.13 |
| 0.5 | 14.16 | 6.50 | 2.35 | 1.06 | 1.27 | 1.00 |
| 0.4 | 6.12 | 2.73 | 1.28 | 1.00 | 1.02 | 1.00 |
| 0.3 | 2.61 | 1.35 | 1.02 | 1.00 | 1.00 | 1.00 |
| 0.2 | 1.28 | 1.01 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.1 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| 0.01 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |

From Table 6, we note that the proposed control chart provides smaller values of the *ARL*s as compared to the one proposed by Aslam et al. [11]. For example, when $c$ = 0.9 and $a$ = 0.5, the *ARL* for the proposed chart is 87, while it is 144 for the existing control chart. The performance of the proposed chart is better for all values of $c$ when $a$ > 0.5.

## 4. Simulation study

In this section, we discuss the implementation of the proposed mixed chart for a time truncated life test using simulated data. The first 20 observations of subgroups of size n =30 are generated from the Weibull distribution with $\beta = 1.5$ and $\lambda_0 = 55.0$ so that $\mu_0 = 50$. The next 20 observations are generated from the Weibull distribution with $\beta = 1.5$, and $\lambda_1 = 0.6 \times 55.0$.

Let us choose $a = 0.10$. Thus, the test time will be $t_0 = a\mu_0 = 5.0$. Hence, we declare a failure if the lifespan generated is smaller than 5. We counted the number of failures in each subgroup, which is listed here: $d$'s: 0, 1, 1, 0, 2, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 2, 2, 1, 0, 0, 5, 2, 0, 1, 1, 0. From Table 3, the two control chart constants and $L_3$ are given by: $k_1 = 3.9668$, $k_2 = 1.3801$, $L_3 = 10.51$.

Also, we have

$$p_0 = 1 - \exp\left( -a^\beta \left( \frac{\Gamma(1/\beta)}{\beta} \right)^\beta \right) = 0.027$$

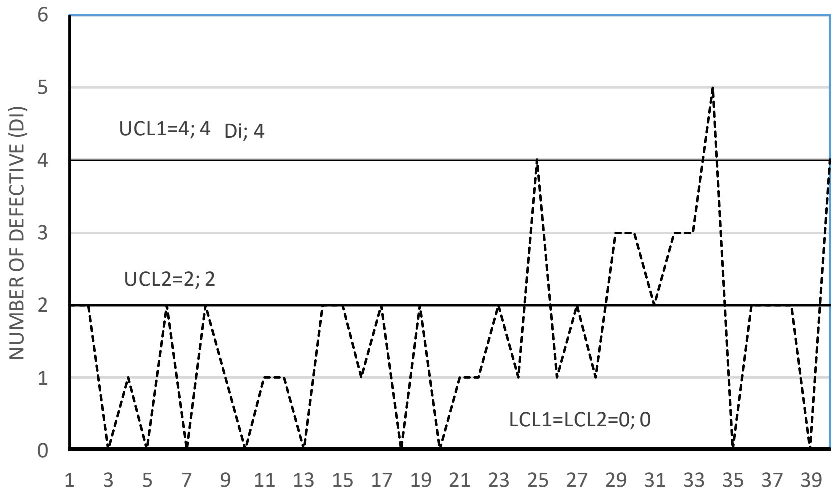


Fig. 1. The proposed control chart for simulated data

Therefore, the two pairs of control limits using the proposed chart for these simulated data are given as follows:

$$UCL_1 = np_0 + k_1\sqrt{np_0\left(1 - p_0\right)} = 30 \times 0.027 + 3.9668\sqrt{30 \times 0.027\left(1 - 0.027\right)} = 4$$

$$LCL_1 = \max[0, np_0 - k_1\sqrt{np_0\left(1-p_0\right)}]$$
$$= \max\left[0, 30\times 0.027 - 3.9668\sqrt{30\times 0.027\left(1-0.027\right)}\right] = 0$$

$$UCL_2 = np_0 + k_2\sqrt{np_0\left(1-p_0\right)}$$
$$= 30\times 0.027 + 1.3801\sqrt{30\times 0.027\left(1-0.027\right)}] = 2$$

$$LCL_2 = LCL_1 = \max\left[0, np_0 - k_2\sqrt{np_0\left(1-p_0\right)}\right]$$
$$= \max\left[0, 30\times 0.027 - 1.3801\sqrt{30\times 0.027\left(1-0.027\right)}\right] = 0$$

We plotted the number of defects in Fig. 1 along with the two pairs of control limits. The proposed chart requires Step 3 (calculating $\bar{Y}$) only when the number of failures is 3 (between 2 and 4) but there are no such cases for these data. We note that the proposed chart detects the process shift based on the 14th set of observations after the actual shift.

## 5. Concluding remarks

A mixed control chart for a life test is proposed by assuming that the lifetime of a product follows the Weibull distribution. Extensive tables are provided for industrial use. The application of the proposed chart is discussed with the help of simulated data. The efficiency of the proposed chart is compared with an existing chart and it is concluded that the proposed chart is more efficient in detecting a shift in the manufacturing process compared to the existing control chart. Proposals for charts based on other non-normal lifespan distributions should be considered in future research.

### Acknowledgements

The authors are deeply thankful to the Editor and Reviewers for their valuable suggestions to improve the quality of this manuscript. This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The author, Muhammad Aslam, therefore, acknowledges with thanks the technical and financial support from the DSR. The work by Chi-Hyuck Jun and Kyung-Jun Kim was a part of the project titled *Development of TCS System on ECO-Ship Technology*, funded by the Ministry of Oceans and Fisheries, Korea.

### References

[1] AL-ORAINI H.A., RAHIM M., *Economic statistical design of X control charts for systems with Gamma (λ, 2) in-control times*, Comp. Ind. Eng., 2002, 43 (3), 645–654.

[2] AMIN R.W., REYNOLDS M.R. Jr., SAAD B., *Nonparametric quality control charts based on the sign statistic*, Comm. Stat. Theory Meth., 1995, 24 (6), 1597–1623.

[3] WANG H., *Comparison of p control charts for low defective rate*, Comp. Stat. Data Anal., 2009, 53 (12), 4210–4220.

[4] CHANG Y.S., BAI D.S., *Control charts for positively-skewed populations with weighted standard deviations*, Quality and Reliability Engineering International, 2001, 17 (5), 397–406.

[5] CHEN F., YEH C., *Economic statistical design of non-uniform sampling scheme X bar control charts under non-normality and Gamma shock using genetic algorithm*, Expert Systems with Applications, 2009, 36 (5), 9488–9497.

[6] ISMAIL A.A., *Estimating the parameters of Weibull distribution and the acceleration factor from hybrid partially accelerated life test*, Applied Mathematical Modelling, 2012, 36 (7), 2920–2925.

[7] MCCRACKEN A., CHAKRABORTI S., *Control charts for joint monitoring of mean and variance: an overview*, Qual. Technol. Quant. Manage., 2013, 10, 17–35.

[8] AHMAD S., RIAZ M., ABBASI S.A., LIN Z., *On efficient median control charting*, J. Chin. Inst. Eng., 2014, 37 (3), 358–375.

[9] ASLAM M., AZAM M., KHAN N., JUN C.H., *A mixed control chart to monitor the process*, Int. J. Prod. Res., 2015, 53 (15), 4684–4693.

[10] ASLAM M., KHAN N., ALDOSARI M.S., JUN C.H., *Mixed control charts using EWMA statistics*, IEEE Access, 2016, 4, 8286–8293.

[11] ASLAM M., JUN C.-H., *Attribute control charts for the Weibull distribution under truncated life tests*, Qual. Eng., 2015, 27 (3), 283–288.

*Received 2 December 2016*
*Accepted 12 March 2017*